PROVING HYPERGEOMETRIC IDENTITIES

A Thesis
Presented to the
Faculty of
San Diego State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Lynda Marie Wynn
Summer 2014
SAN DIEGO STATE UNIVERSITY

The Undersigned Faculty Committee Approves the

Thesis of Lynda Marie Wynn:

Proving Hypergeometric Identities

Vadim Ponomarenko, Chair
Department of Mathematics and Statistics

J. Carmelo Interlando
Department of Mathematics and Statistics

William Root
Department of Computer Science

6/30/14
Approval Date
Copyright © 2014
by
Lynda Marie Wynn
DEDICATION

Dedicated to my husband and our families for the support and encouragement they have given.
ABSTRACT OF THE THESIS

Proving Hypergeometric Identities
by
Lynda Marie Wynn
Master of Arts in Mathematics
San Diego State University, 2014

This paper is a study of the development of four algorithms for proving hypergeometric identities, beginning with Sister Celine’s method and concluding with the WZ method. Each section includes a discussion of the algorithm as well as examples done both by hand and using the EKHAD package for Maple.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>.....................................................................................................................</td>
<td>v</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>........................................................................................................</td>
<td>vii</td>
</tr>
<tr>
<td>1</td>
<td>INTRODUCTION ..........................................................................................</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>A Brief History ...................................................................................</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Hypergeometric Sums .........................................................................</td>
<td>2</td>
</tr>
<tr>
<td>1.3</td>
<td>Definitions .........................................................................................</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>SISTER CELINE’S METHOD ......................................................................</td>
<td>7</td>
</tr>
<tr>
<td>2.1</td>
<td>Sister Celine’s General Algorithm ..................................................</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>The Fundamental Theorem ...................................................................</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>GOSPER’S ALGORITHM ...........................................................................</td>
<td>16</td>
</tr>
<tr>
<td>3.1</td>
<td>Gosper’s General Algorithm ..........................................................</td>
<td>16</td>
</tr>
<tr>
<td>3.2</td>
<td>Step 2: Factoring the Rational Function ........................................</td>
<td>19</td>
</tr>
<tr>
<td>3.3</td>
<td>Step 3: Solving for a Nonzero Polynomial Solution ...........................</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>ZEILBERGER’S ALGORITHM .....................................................................</td>
<td>26</td>
</tr>
<tr>
<td>4.1</td>
<td>How Zeilberger’s Algorithm Works ....................................................</td>
<td>30</td>
</tr>
<tr>
<td>4.2</td>
<td>Examples of Zeilberger’s Algorithm ...................................................</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>THE STANDARD WZ PROOF ALGORITHM .................................................</td>
<td>38</td>
</tr>
<tr>
<td>5.1</td>
<td>The WZ Algorithm ................................................................................</td>
<td>38</td>
</tr>
<tr>
<td>5.2</td>
<td>The WZ Proof Certificate ....................................................................</td>
<td>42</td>
</tr>
<tr>
<td>5.3</td>
<td>Additional Benefits of the WZ Method ..............................................</td>
<td>44</td>
</tr>
<tr>
<td>6</td>
<td>CONCLUSION ..........................................................................................</td>
<td>48</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>........................................................................................................</td>
<td>50</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

I would like to thank Dr. Vadim Ponomarenko for his willingness to chair my thesis committee and his support of my efforts while researching and writing this thesis. I appreciate the time you have given and the patience you have shown when I was overwhelmed or bogged down in minutia.

I would also like to thank Dr. J. Carmelo Interlando and Dr. William Root for serving on my thesis committee. Thank you for your time and your feedback.
CHAPTER 1
INTRODUCTION

Throughout history, people have observed relations in the natural world and many of these relations were quantifiable, leading to mathematical identities. One such observation was made by Archimedes as he noticed that his body was displacing his bathwater. Archimedes stated “any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.” Over time many such identities were observed, but mathematicians eventually desired more than just knowledge of identities; they also wanted proof that the identities were in fact true. The idea of creating a “universal proof machine” for identities of the form $A = B$ was fostered, but unfortunately doesn’t exist. Petkovšek, Wilf, and Zeilberger [7] challenge their readers to consider and prove the identity

$$\sin^2\left(\left|\ln 2 + \pi x\right|^2\right) + \cos^2\left(\left|\ln 2 + \pi x\right|^2\right) = 1.$$ 

While this identity is certainly true, it does not easily fit into an universal approach to proving identities of the form $A = B$. However, much progress has been made in the area of automated proving of hypergeometric identities.

1.1 A BRIEF HISTORY

Euler was the first to study hypergeometric sums, followed by many others including Gauss, Riemann, and Kummer. Hypergeometric series are a generalization of geometric series.

**Definition 1.1.** The geometric series

$$\sum_{k \geq 0} t_k$$

has a constant ratio between consecutive terms. That is,

$$\frac{t_{k+1}}{t_k}$$

is a constant.

**Definition 1.2.** The hypergeometric series

$$\sum_{k \geq 0} t_k$$

has a rational function of the summand index $k$ as the ratio between consecutive terms. That is,

$$\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)},$$

where $P, Q$ are polynomials and $Q \neq 0$.

In the early 19th century, C.F. Gauss initiated the theory of hypergeometric functions, but it wasn’t until 1974 that many combinatorial identities were actually recognized to be
special cases of a few general hypergeometric identities. The concept of an automatic summation machine for discovering recurrence relations for hypergeometric sums has been evolving since the 1940s. In 1945, Sister Mary Celine Fasenmeyer’s Ph.D. dissertation showed it was possible to algorithmically find recurrences for certain polynomial sequences. In 1978, R.W. Gosper, Jr. applied Sister Celine’s method to the problem of indefinite hypergeometric summation. Four years later, Doron Zeilberger recognized that he could use Sister Celine’s method as a basis for proving combinatorial identities. His general method involves proving an identity of the form

$$\sum_k \text{summand}(n, k) = \text{answer}(n),$$

then one finds a recurrence relation satisfied by the sum on the left hand side, shows by substitution that the right hand side satisfies this same recurrence, and checks that a sufficient number of the corresponding initial values are the same. In 1990-1991, Zeilberger introduced a “creative telescoping” algorithm for finding recurrences for combinatorial summands. Also in 1990, Herbert S. Wilf and Zeilberger considered a special case of recurrences for combinatorial summands which helped find new identities from old and enabled very short, elegant proofs. Marco Petkovšek, in his 1991 Ph.D. thesis discovered an algorithm for deciding whether a recurrence relation with polynomial coefficients has a “simple” solution. In 1992, Wilf and Zeilberger generalized their methods to multisums, $q$-sums, etc., in addition to giving proofs of the fundamental theorems and explicit estimates for the orders of the recurrences involved. In 1998, Chyzak and Salvy generalized Zeilberger’s algorithm to holonomic functions. In 2001 and 2002, Abramov and Le sought to improve Zeilberger’s algorithm in two ways: by improving the efficiency of the algorithm and by decreasing the limitations in its domain of applicability. Research of hypergeometric series continues today and we have a much better algorithmic understanding of these series thanks to the ongoing efforts of those mentioned above and many others [2, 5, 7].

### 1.2 Hypergeometric Sums

Hypergeometric sums are those whose summands only involve factorials, binomial coefficients, polynomials and exponential functions of the summand variable. Many hypergeometric sums have a closed form, while others can be shown to be equivalent to another hypergeometric sum. Some examples of hypergeometric terms are $x^k$, $k!$, and $\frac{(3k-6)!}{(k+2)!}$. The exponential, logarithmic, trigonometric, binomial and Bessel functions are also hypergeometric [7]. Once one recognizes that a series is hypergeometric, one can then compare to known results in the “hypergeometric database” or use software such as Maple or Mathematica to find a closed form. While this idea of a hypergeometric database exists, there
is not an exhaustive listing of all possible hypergeometric identities, but partial listings can be found.

In order to use a hypergeometric database, one must rewrite the series in the standard form $p{F}_q[\cdots]$. Petkovšek, Wilf, and Zeilberger [7] give the following algorithm to transform a hypergeometric series into standard form.

Step 1. Given a series $\sum_k t_k$, shift the index $k$ so that it starts at $k = 0$ with a nonzero term. Factor this term out so that $t_0 = 1$.

Step 2. Next find the ratio of two consecutive terms $\frac{t_{k+1}}{t_k}$. If this results in a rational function of the form $\frac{P(k)}{Q(k)}$, where $P(k), Q(k)$ are polynomials, then you have a hypergeometric series.

Step 3. Factor $P$ and $Q$ and write in the form

$$\frac{P(k)}{Q(k)} = \frac{(k + a_1)(k + a_2) \cdots (k + a_p)}{(k + b_1)(k + b_2) \cdots (k + b_q)(k + 1)} x.$$

If $Q(k)$ does not have a factor of $(k + 1)$, then one needs to add it to both the numerator and the denominator.

Step 4. Now we have identified the series and can write it as

$$p{F}_q\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{array} ; x \right].$$

**Example 1.1.** Using this algorithm, one can determine if the sum $\sum_k (-1)^k \binom{n}{k}^2 \binom{3n+k}{2n}$ is hypergeometric, and if so, transform it into standard hypergeometric form.

The first nonzero term occurs at $k = 0$ and $t_0 = \binom{3n}{2n}$. The ratio of consecutive terms is

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} \binom{n}{k+1}^2 \binom{3n+k+1}{2n}}{(-1)^k \binom{n}{k}^2 \binom{3n+k}{2n}} = (-1)^k \frac{(k - n)(k - n)(k + 3n + 1)}{(k + n + 1)(k + 1)(k + 1)} ,$$

a rational function of $k$. Thus the sum is hypergeometric and can be written as

$$\sum_k (-1)^k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{2n} 3{F}_2\left[\begin{array}{cccc} -n & -n & 3n+1 \\ 1 & 1 \\ n + 1 \end{array} ; -1 \right].$$
Since there is no hypergeometric database for all possible sums and series, other methods are required to find a closed form for this example. These methods will be covered in subsequent sections. The following exercise will demonstrate the process of checking whether the sum is hypergeometric and if so, transforming it into standard notation, then using some known identities in the “hypergeometric database” provided in Chapter 3 of \( A = B \) [7] to discover a simple form. Another such listing can be found in Section 5.3 of The Concrete Tetrahedron, as well as examples of applications of hypergeometric series such as hypergeometric probability distributions and elliptic arc length [5].

**Example 1.2.** Can we write the sum

\[
\sum_k (-1)^k \binom{n}{k} \frac{(k + 3a)!}{(k + a)!}
\]

in standard notation and evaluate it using a known identity in the hypergeometric database?

The first nonzero term occurs at \( k = 0 \) and \( t_0 = \frac{(3a)!}{a!} \). The ratio of consecutive terms is

\[
\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} n!(k + 3a + 1)!k!(n - k)!(k + a)!}{(-1)^k(k + 1)!(n - k - 1)!(k + a + 1)!n!(k + 3a)!} = \frac{(k - n)(k + 3a + 1)}{(k + a + 1)(k + 1)},
\]

which is a rational function of the index variable \( k \). In standard form,

\[
\sum_k (-1)^k \binom{n}{k} \frac{(k + 3a)!}{(k + a)!} = \frac{(3a)!}{a!} \; _2F_1 \left[ \frac{-n}{a + 1}, \frac{3a + 1}{a + 1} ; 1 \right].
\]

One possibility in the hypergeometric database is Kummer’s \( _2F_1 \) identity: if \( a - b + c = 1 \), then

\[
_2F_1 \left[ \frac{a}{c}, \frac{b}{c} ; 1 \right] = \frac{\Gamma \left( \frac{b}{c} + 1 \right) \Gamma \left( b - a + 1 \right)}{\Gamma \left( b + 1 \right) \Gamma \left( \frac{b}{c} - a + 1 \right)}.
\]

Since \( -n - (3a + 1) + (a + 1) = -2a - n \neq 1 \), we see that this hypergeometric series is not an identity of the same form as Kummer’s. Another possibility is Gauss’s \( _2F_1 \) identity: if \( b \) is a nonpositive integer or \( \Re(c - a - b) \) is positive, then

\[
_2F_1 \left[ \frac{a}{c}, \frac{b}{c} ; 1 \right] = \frac{\Gamma \left( c - a - b \right) \Gamma \left( c \right)}{\Gamma \left( c - a \right) \Gamma \left( c - b \right)}.
\]
As long as $3a + 1$ is a nonpositive integer or $\text{Re}(n - 2a) > 0$, the following are equivalent:

\[
\sum_{k} (-1)^k \binom{n}{k} \frac{(k + 3a)!}{(k + a)!} = \frac{(3a)!}{a!} {}_2F_1 \left[ -n, 3a + 1 \mid \frac{a + 1}{a + 1} \right] = \frac{(3a)!}{a!} \frac{\Gamma(n - 2a)\Gamma(a + 1)}{\Gamma(a + 1 + n)\Gamma(-2a)} = \frac{(3a)!}{a!} \frac{(n - 2a - 1)!a!}{(a + n)!(-2a - 1)!} = (3a)! \frac{(n - 2a - 1)!(2a)! \sin (2a + 1)\pi}{(a + n)!\pi}.
\]

One can check a few values to verify that one has found a closed form, provided that the hypotheses are met. For example, if $a = 0$ and $n = 2$, then $\text{Re}(2 - 2(0)) = 2 > 0$ and the original sum is

\[
\sum_{k} (-1)^k \binom{2}{k} \frac{(k + 3(0))!}{(k + 0)!} = \sum_{k=0}^{2} (-1)^k \binom{2}{k} = 1 - 2 + 1 = 0.
\]

Using the closed form found above, one obtains

\[
(3(0))! \frac{(2 - 2(0) - 1)!(2(0))! \sin (2(0) + 1)\pi}{(0 + 2)!\pi} = 1 \cdot 1 \cdot 1 \cdot \sin \frac{\pi}{2} = 0.
\]

What if the sum wasn’t in the list of identities in the “hypergeometric database”? This is quite possible since such a database doesn’t really exist. One has not been created due to the fact that once a list of such identities is compiled, someone would soon find an identity that isn’t listed. Supposing that such an extensive database actually existed, if one discovered that his sum was not in the database, he could try some hypergeometric transformation rules to re-express his sum. One of these transformation rules is Euler’s hypergeometric transformation

\[
{}_2F_1 \left[ a, b \mid c, z \right] = (1 - z)^{c-a-b} {}_2F_1 \left[ c-a, c-b \mid c, z \right].
\]

This is just one example of the many transformation rules one could use. For further reading on Euler’s transformation, Maier [6] has published an article in which he shows that there is an analogue for Euler’s transformation at all higher levels. If this transformation didn’t work, one could try another transformation rule, or a combination of two or more rules, and so on. Another possibility is to make substitutions for the parameters to see if the sum is a special case of another identity in the database. In many cases, such substitutions would be difficult to discover. Clearly, using substitutions and transformation rules haphazardly would not be an efficient way to find a closed form. One needs an algorithm for doing hypergeometric sums in closed form which works with much broader conditions than what one would use in a
database lookup, and the algorithm needs to exhaust all possibilities, returning either a closed form for the sum or confidently asserting that no such simple form exists. The following chapters will look at four such algorithms.

1.3 Definitions

The following definitions will be useful in studying hypergeometric sums and series.

**Definition 1.3.** [7] The rising factorial function, denoted by $(a)_n$, is defined as

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

**Definition 1.4.** The falling factorial function, denoted by $(a)^j$, is defined as

$$(a)^j = a(a-1)(a-2)\cdots(a-j+1), \text{ for } j \geq 0.$$  

**Definition 1.5.** [7] The gamma function, denoted by $\Gamma(z)$, is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt,$$

if $\text{Re}(z) > 0$, and elsewhere by analytic continuation. If $z$ is a nonnegative integer, then

$$\Gamma(z+1) = z!.$$  

Below are a few of the ways that these three notations are related. [7]

$$\Gamma(n+1) = n! = (a)_n$$

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)}$$

One can express the general hypergeometric series in terms of the rising factorial function.

**Definition 1.6.** [7] The general hypergeometric series

$$pF_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q \end{array} ; z \right] = \sum_{k \geq 0} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k} \frac{z^k}{k!},$$

where the $b_i$ are not negative integers or zero. If any one of the $a_j$ is a nonpositive integer, the series terminates.
CHAPTER 2
SISTER CELINE’S METHOD

In 1945, Sister Mary Celine Fasenmayer completed her Ph.D. thesis at the University of Michigan. In her thesis, Sister Celine developed an algorithm for finding recurrence relations for hypergeometric polynomials from the series expansions of the polynomials. She further developed her method in subsequent publications. Sister Celine’s adviser, Earl Rainville, included Sister Celine’s algorithm in his 1960 book, Special Functions. Her algorithm paved the way for future computerized methods of proving identities, in addition to providing general existence theorems for recurrence relations that are satisfied by hypergeometric sums [7]. In 1982, Doron Zeilberger [10] furthered Sister Celine’s ideas by showing that one can evaluate any sum involving products of binomial coefficients. Zeilberger also generalized her method to verifying any identity involving sums and integrals of products of special functions.

2.1 SISTER CELINE’S GENERAL ALGORITHM

Suppose we have the sum $f(n) = \sum_k F(n, k)$, where $F$ is doubly hypergeometric.

**Definition 2.1.** [7, 8] A function $F$ is doubly hypergeometric if both

\[
\frac{F(n+1, k)}{F(n, k)} \text{ and } \frac{F(n, k+1)}{F(n, k)}
\]

are rational functions of $n$ and $k$.

Our goal is to find a recurrence relation for the sum $f(n)$. In order to do so, one first finds a recurrence relation for the summand $F(n, k)$ that has the form

\[
\sum_{i=0}^I \sum_{j=0}^J a_{i,j}(n) F(n-j, k-i) = 0.
\]

(2.1)

First, choose values for $I$ and $J$, such as $I = J = 1$. Next, assume the recurrence formula (2.1). The coefficients $a_{i,j}$ will be determined later, if possible. Third, divide (2.1) through by $F(n, k)$, reducing each fraction $\frac{F(n-j, k-i)}{F(n, k)}$, so that only a rational function in $n$ and $k$ is left. Next, find a common denominator and express the numerator as a polynomial in $k$. Finally, solve the resulting system of linear equations when each coefficient of the powers of $k$ are set equal to zero. If there is no solution, then start over again with larger values of $I$ and/or $J$.

**Example 2.1.** One can use Sister Celine’s algorithm to verify the well known identity

\[
\sum_k \binom{n}{k} = 2^n
\]

by hand.
Consider the function \( f(n) = \sum_k \binom{n}{k}, n = 0, 1, 2, \ldots \) Applying Sister Celine’s algorithm with \( I = J = 1 \), assume
\[
\sum_{i=0}^{1} \sum_{j=0}^{1} a_{i,j}(n) F(n - j, k - i) = 0, \quad \text{where } F(n, k) = \binom{n}{k}.
\]

Expanding the sum gives the following equation
\[
a_{0,0}(n) F(n, k) + a_{0,1}(n) F(n - 1, k) + a_{1,0}(n) F(n, k - 1) + a_{1,1}(n) F(n - 1, k - 1) = 0.
\]

Dividing through by \( F(n, k) \) and setting \( a_{0,0}(n) = a, a_{0,1}(n) = b, a_{1,0}(n) = c, \) and \( a_{1,1}(n) = d \) yields
\[
a + b \frac{n-k}{n} + c \frac{k}{n-k+1} + d \frac{k}{n} = 0.
\]

Rewriting over the common denominator produces
\[
\frac{an(n-k+1) + b(n-k)(n-k+1) + cnk + dk(n-k+1)}{n(n-k+1)} = 0.
\]

Next, collecting the powers of \( k \) gives
\[
\frac{(a+b)n^2 + (a+b)n + ((-a-2b+c+d)n + (d-b))k + (b-d)k^2}{n(n-k+1)} = 0.
\]

Since each coefficient of the powers of \( k \) must vanish, one can solve the following system:
\[
\begin{bmatrix}
n(n+1) & n(n+1) & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d 
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 
\end{bmatrix}.
\]

One is guaranteed to find a nontrivial solution to this system since there are three equations in four unknowns. Using elementary row operations to find the reduced row echelons form, the solution is
\[
\begin{bmatrix}
a \\
b \\
c \\
d 
\end{bmatrix}
= \begin{bmatrix}
-1 \\
1 \\
0 \\
1 
\end{bmatrix}.
\]

\( d \) is a free variable, so without loss of generality, one can assume that \( d = 1 \). We next substitute the solution into the assumed equation as follows.
\[
-F(n, k) + F(n - 1, k) + F(n - 1, k - 1) = 0
\]
Recall that this recurrence relation is only for the summand, $F(n, k) = \binom{n}{k}$. To find the recurrence for the sum, one simply sums over all $k$ and we have

$$-f(n) + f(n - 1) + f(n - 1) = 0, \text{ for all } n = 1, 2, \ldots; f(0) = 1.$$  

Now one can easily find the following recurrence for $f(n)$.

$$f(n) = 2f(n - 1) = 2^2f(n - 2) = 2^3f(n - 3) = \cdots = 2^nf(0) = 2^n.$$  

While it is possible to find closed form solutions for sums with doubly hypergeometric summands by hand, it is much more efficient to use technology to perform the calculations. Doron Zeilberger of Rutgers University wrote a Maple package entitled “EKHAD” which not only performs Sister Celine’s algorithm, but also the algorithms that appear in later chapters. Using Zeilberger’s package, one can quickly find a simple form for sums such as $\sum_k \binom{n}{k}^2$.

Load the EKHAD package, then call the Sister Celine algorithm using the following command

```
celine((n, k) → \binom{n}{k}^2, 2, 2);
```

and Maple returns the following statement, “The full recurrence is

$$(n - 1)b_8 F(n - 2, k - 2) + b_8(-2n + 2)F(n - 2, k - 1) + (-2n + 1)b_8 F(n - 1, k - 1) + (n - 1)b_8 F(n - 2, k) + (-2n + 1)b_8 F(n - 1, k) + nb_8 F(n, k), == 0.$$  

The $b_8$ is an arbitrary constant, so we can let $b_8 = 1$. This gives the recurrence for the summand. To obtain the recurrence for the sum, one would sum over all $k$ to obtain

$$(n - 1)f(n - 2) + (-2n + 2)f(n - 2) + (-2n + 1)f(n - 1) + (n - 1)f(n - 2) + (-2n + 1)f(n - 1) + nf(n) = 0.$$  

By collecting terms, this simplifies to

$$2(-2n + 1)f(n - 1) + nf(n) = 0.$$
Solving for \( f(n) \) gives

\[
\begin{align*}
f(n) &= \frac{2(2n-1)}{n} f(n-1) \\
&= \frac{2^2(2n-1)(2n-3)}{n(n-1)} f(n-2) \\
&\vdots \\
&= \frac{2^n(2n-1)(2n-3) \cdots (1)}{n(n-1) \cdots (1)} f(0) \\
&= \frac{2^n(2n)(2n-1)(2n-2) \cdots (2)(1)}{n(n-1) \cdots (1)(2n-2) \cdots (2)} \cdot 1 \\
&= \frac{2^n(2n)!}{2^n n^2 (n-1)^2 \cdots (1)^2} \\
&= \frac{(2n)!}{n!^2} \\
&= \binom{2n}{n}
\end{align*}
\]

\( \Box \)

### 2.2 The Fundamental Theorem

Informally, the Fundamental Theorem guarantees that every proper hypergeometric term satisfies a recurrence relation that can be found using Sister Celine’s method provided the span of the recurrence is large enough. The theorem also provides a method for computing the upper bounds of the span.

**Definition 2.2.** [7] If a function \( F(n, k) \) can be written in the form

\[
F(n, k) = P(n, k) \frac{\prod_{i=1}^{uu}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{vv}(u_i n + v_i k + w_i)!} x^k, \quad (2.2)
\]

where

(i) \( P \) is a polynomial;

(ii) \( x \) is an indeterminate over the complex numbers;

(iii) \( a_i, b_i, u_i, \) and \( v_i \) are specific integers for all \( i \); and

(iv) \( uu, vv \) are finite, specific, nonnegative integers,

then \( F(n, k) \) is a proper hypergeometric term.

**Definition 2.3.** [7] A function \( F \) of the form (2.2) is well-defined at a point \( (n, k) \) if none of the elements of the set \( \{a_i n + b_i k + c_i\}_{i=1}^{uu} \) is a negative integer.
Definition 2.4. [7] The function \( F(n,k) = 0 \) if \( P(n,k) = 0 \) or if \( F \) is well-defined at \((n,k)\) and at least one of the elements of the set \( \{u_in + v_ik + w_i\}_{i=1}^m \) is a negative integer.

It is easy to see that the term \((\binom{3n}{k})5^k\) is a proper hypergeometric term, since the binomial coefficient can be re-expressed in factorials:

\[
F(n,k) = \binom{3n}{k}5^k = \frac{(3n)!}{k!(3n-k)!}5^k,
\]

which is of the form of (2.2) with \( P(n,k) = 1 \) and \( x = 5 \). Other proper hypergeometric terms are not as obvious as this example. Consider the function \( G(n,k) = \frac{1}{2n+5k+1} \). \( G \) is a rational function, not a polynomial, with no factorials or binomial coefficients. As written, \( G \) is not in proper hypergeometric form. However, a creative multiplication by “one” easily transforms \( G \) into a proper hypergeometric term:

\[
G(n,k) = \frac{1}{2n+5k+1} = \frac{(2n+5k)!}{(2n+5k+1)!}.
\]

Note that this worked because the degree of \( n \) and \( k \) is one. If the degrees of \( n \) and \( k \) were two, it would be impossible to rewrite this function in the form of (2.2) and so the function would not be proper hypergeometric.

A formal statement of the Fundamental Theorem follows.

Theorem 2.1. [7] Let \( F(n,k) \) be a proper hypergeometric term. Then \( F \) satisfies a \( k \)-free recurrence relation. In other words, there exist positive integers \( I,J \) and polynomials \( a_{i,j}(n) \) for \( i = 0, \ldots, I; j = 0, \ldots, J \) which are not all zero, such that

\[
\sum_{i=0}^I \sum_{j=0}^J a_{i,j}F(n-j,k-i) = 0 \tag{2.3}
\]

holds for all \((n,k)\) at which \( F(n,k) \neq 0 \) and all of the values of \( F \) that occur in (2.3) are well-defined. Additionally, there is such a recurrence with \((I,J) = (I^*,J^*)\) where

\[
J^* = \sum_s |b_s| + \sum_s |v_s|; I^* = 1 + \deg(P) + J^* \left( \sum_s |a_s| + \sum_s |u_s| \right) - 1.
\]

So the theorem guarantees that every proper hypergeometric term satisfies a \( k \)-free recurrence relation and we can determine an upper bound for the order of this recurrence relation before we get started. This recurrence relation can be found using Sister Celine’s method.

Example 2.2. Consider the function \( f(n,k) = \sum_k^k (\frac{2n+1}{2k+1}) \). Is this function a proper hypergeometric term, and if so, what is the upper bound for the span of the recurrence relation? If it satisfies a \( k \)-free recurrence relation of the form (2.3), we can use Sister Celine’s method to find a closed form for the sum.
First, let’s rewrite the summand $F(n, k)$ to see if it can be written in proper hypergeometric form.

$$F(n, k) = k \binom{2n + 1}{2k + 1} = k \frac{(2n + 1)!}{(2k + 1)!(2n - 2k)!}$$

We see that $F(n, k)$ is a proper hypergeometric term with $P(n, k) = k$ and $x = 1$. The upper bound for the order of the recurrence relation is found as follows.

$$J^* = \sum_s |b_s| + \sum_s |v_s| = 1 + (2 + 2) = 5,$$

and

$$I^* = 1 + \text{deg}(P) + J^* \left( \left\{ \sum_s |a_s| + \sum_s |u_s| \right\} - 1 \right) = 1 + 1 + 1(2 + (0 + 2)) - 1 = 5.$$ Then, at most, $I^* = J^* = 5$. Since 5 represents the upper bound, let’s try to find a recurrence with $I = J = 1$. Using Maple, one enters

$$\text{celine} \left( (n, k) \rightarrow k \binom{2n + 1}{2k + 1}, 1, 1 \right);$$

and the output is “The full recurrence is 0, == 0. The telescoped form is 0, == $G(n, k) - G(n, k - 1)$ where $G(n, k) = R(n, k) \cdot F(n, k)$ and the rational function $R(n, k)$ is 0.” This tells us that our choice of $I, J$ was too small, so we increment either $I$ or $J$ or both until we find the recurrence. Trying $I = 2, J = 1$ and $J = 2, I = 1$ yields the same result as before, so we keep incrementing $I$ and $J$. Recall that at most, $I = J = 5$. We don’t have to go that far though. Using $I = J = 2$, we see that Maple returns the following output. “The full recurrence is

$$\begin{align*}
(4n^2 - 4n + 1)b_8 F(n - 2, k - 2) + (-8n^2 + 8n - 2)b_8 F(n - 2, k - 1) \\
+ (-8n^2 + 16n - 2)b_8 F(n - 1, k - 1) + (4n^2 - 4n + 1)b_8 F(n - 2, k) \\
+ (-8n^2 + 16n - 10)b_8 F(n - 1, k) + b_8 (4n^2 - 12n + 9)F(n, k), == 0. \quad (2.4)
\end{align*}$$

The telescoped form is

$$\frac{(2n - 3)(2F(n, k)n - 8F(n - 1, k)n - 3F(n, k) + 4F(n - 1, k))b_8}{(2n - 1)^2},$$

$$== G(n, k) - G(n, k - 1)$$

where $G(n, k) = R(n, k) \cdot F(n, k)$ and the rational function $R(n, k)$ is

$$\frac{(2n - 3)(-n + k)(4kn - 6n^2 - 2k + 9n - 2)(-2n + 1 + 2k)b_8}{(2n + 1)(2n - 1)^2n(n - 1)}.$$. 
To find a closed form for the sum $f(n, k)$, let $b_8 = 1$ and sum (2.4) over all $k$ to obtain

$$(4n^2 - 4n + 1)f(n - 2) + (-8n^2 + 8n - 2)f(n - 2) + (-8n^2 + 16n - 2)f(n - 1)$$

$$+ (4n^2 - 4n + 1)f(n - 2) + (-8n^2 + 16n - 10)f(n - 1) + (4n^2 - 12n + 9)f(n) = 0$$

Simplifying yields the following equation

$$(-16n^2 + 32n - 12)f(n - 1) + (4n^2 - 12n + 9)f(n) = 0.$$ 

We next solve for $f(n)$ and simplify to find the closed form.

$$f(n) = \frac{(16n^2 - 32n + 12)f(n - 1)}{4n^2 - 12n + 9}$$

$$= \frac{4(2n - 1)f(n - 1)}{2n - 3}$$

$$= \frac{4^2(2n - 1)f(n - 2)}{2n - 5}$$

$$= \frac{4^3(2n - 1)f(n - 3)}{2n - 7}$$

$$\vdots$$

$$= \frac{4^{n-1}(2n - 1)}{2n - 1}, \text{ for } n \geq 1.$$ 

Thus

$$f(n, k) = \sum_k k\binom{2n + 1}{2k + 1} = \begin{cases} 
0, & \text{if } n = 0; \\
4^{n-1}(2n - 1), & \text{if } n \geq 1.
\end{cases}$$

In 1992, Wilf and Zeilberger published a proof of the fundamental theorem in *Inventiones Mathematicae* [7]. In their proof, they use behavior of translates of a proper hypergeometric term. The following examples demonstrate this concept.

**Example 2.3.** Let $f(n) = (3n + 5)!$. Then what is $\frac{f(n-1)}{f(n)}$?

We have

$$\frac{f(n-1)}{f(n)} = \frac{(3(n-1) + 5)!}{(3n + 5)!}$$

$$= \frac{(3n + 2)!}{(3n + 5)!}$$

$$= \frac{1}{(3n + 5)(3n + 4)(3n + 3)}.$$ 

which is the reciprocal of some polynomial in $n$. 

□
Example 2.4. Let \( f(n) = (5 - 3n)! \). What is \( \frac{f(n-1)}{f(n)} \)?

We have

\[
\frac{f(n-1)}{f(n)} = \frac{(5 - 3(n-1))!}{(5 - 3n)!} = \frac{(8 - 3n)!}{(5 - 3n)!} = (8 - 3n)(7 - 3n)(6 - 3n),
\]

which is a polynomial in \( n \).

The examples above help demonstrate the following fact for the one-variable case [7].

If \( f(n) = (an + b)! \), then

\[
\frac{f(n-j)}{f(n)}
\]

with \( j \geq 0 \) is

(1) a polynomial in \( n \) if \( a \leq 0 \), and

(2) the reciprocal of a polynomial in \( n \) if \( a > 0 \).

One can also find the translate of a proper hypergeometric term in two-variables.

Example 2.5. Let \( f(n, k) = (3n - 5k + 2)! \). What are \( \frac{f(n-2,k-1)}{f(n,k)} \) and \( \frac{f(n-1,j-2)}{f(n,k)} \)?

We find that

\[
\frac{f(n-2,k-1)}{f(n,k)} = \frac{(3(n-2) - 5(k-1) + 2)!}{(3n - 5k + 2)!} = \frac{(3n - 5k + 1)!}{(3n - 5k + 2)!} = \frac{1}{3n - 5k + 2},
\]

which is the reciprocal of a polynomial in \( n \) and \( k \), and

\[
\frac{f(n-1,k-2)}{f(n,k)} = \frac{(3(n-1) - 5(k-2) + 2)!}{(3n - 5k + 2)!} = \frac{(3n - 5k + 9)!}{(3n - 5k + 2)!} = (3n - 5k + 9)(3n - 5k + 8) \cdots (3n - 5k + 4)(3n - 5k + 3),
\]

which is a polynomial in \( n \) and \( k \).

In the two-variable case [7], if \( f(n, k) = (an + bk + c)! \), then, for \( i, j \geq 0 \),

\[
\frac{f(n-j,k-i)}{f(n,k)}
\]

is

(1) a polynomial in \( n \) and \( k \) if \( aj + bi \geq 0 \), and

(2) the reciprocal of a polynomial in \( n \) and \( k \) if \( aj + bi < 0 \).
Wilf and Zeilberger also stated and proved a version of the Fundamental Theorem for several summation variables, as well as for $q$ and multi-$q$ identities. In each case, one can find an a priori explicit upper bound for the order of the recurrence.
CHAPTER 3
GOSPER’S ALGORITHM

R.W. Gosper, Jr. was part of the development team for Macsyma, one of the first symbolic algebra programs. His work on Macsyma led him to discovering his algorithm for finding a simple formula for an indefinite sum, if one exists. An indefinite sum is a summation whose upper bound is a variable that does not appear anywhere else in the summand [5]. Gosper’s algorithm was an important advancement toward the goal of automatic proving machines. Gosper’s work paved the way for Zeilberger’s creative telescoping algorithm and the WZ algorithm, both of which will be discussed in subsequent chapters.

3.1 Gosper’s General Algorithm

In Chapter 5 of A=B [7], we are introduced to Gosper’s general algorithm for finding a simple formula for an indefinite sum. The following is a summary of the authors’ introduction to Gosper’s algorithm.

Consider the sum
\[ s_n = \sum_{k=0}^{n-1} t_k, \]
where \( t_k \) is a hypergeometric term for which the ratio of consecutive terms \( r(k) = \frac{t_{k+1}}{t_k} \) is a rational function of \( k \). The goal is to express \( s_n \) in closed form. Note that \( t_n = s_{n+1} - s_n \). We would like to know if it is possible to find a hypergeometric term \( z_n \) such that
\[ z_{n+1} - z_n = t_n. \] (3.1)
If so, then one has expressed the sum as a single hypergeometric term plus a constant. Note also that
\[ z_n = z_{n-1} + t_{n-1} = z_{n-2} + t_{n-2} + t_{n-1} = \cdots = z_0 + \sum_{k=0}^{n-1} t_k = s_n + c, \]
where \( c = z_0 \) is a constant. If it is possible to find such a \( z_n \), then \( s_n \) can be expressed as a hypergeometric term plus a constant. Gosper’s algorithm produces this hypergeometric term. When this happens, one says that \( t_n \) is Gosper summable. Otherwise, the algorithm proves that no such solution exists.

Suppose that \( z_n \) is a hypergeometric term that satisfies (3.1). Then
\[ \frac{z_n}{t_n} = \frac{z_{n+1} - z_n}{z_n} = \frac{1}{\frac{z_{n+1}}{z_n} - 1} \]
is a rational function of \( n \). Let \( z_n = y(n)t_n \), where \( y(n) \) is some rational function of \( n \). Substituting this into (3.1) shows that
\[ y(n + 1)t_{n+1} - y(n)t_n = t_n. \]
Dividing through by $t_n$ gives

$$y(n + 1) \frac{t_{n+1}}{t_n} - y(n) = 1,$$

or equivalently,

$$y(n + 1)r(n) - y(n) = 1. \tag{3.2}$$

One observes that (3.2) is a first-order linear recurrence with rational coefficients and constant right hand side. So the problem has transformed from finding hypergeometric solutions of (3.1) to one of finding rational solutions of (3.2). But Gosper didn’t stop here. He reduced the problem even more by making it possible to find polynomial solutions of a different first-order recurrence.

Suppose that the rational function $r(n)$ can be written in the form

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n)}{c(n)}, \tag{3.3}$$

where $a, b, c$ are polynomials in $n$ and

$$\gcd (a(n), b(n + h)) = 1, \text{ where } h \text{ is a nonnegative integer}. \tag{3.4}$$

Gosper found that the $y(n)$ in (3.2) will have the form

$$y(n) = \frac{b(n - 1)x(n)}{c(n)}, \tag{3.5}$$

where $x(n)$ is a rational function. By substituting $r(n)$ (3.3) and $y(n)$ (3.5) into (3.2) one obtains the following:

$$a(n)x(n + 1) - b(n - 1)x(n) = c(n). \tag{3.6}$$

Astonishingly, the rational function $x(n)$ is actually a polynomial.

**Theorem 3.1.** [7] Let $a(n), b(n), c(n)$ be polynomials such that (3.4) holds. If $x(n)$ is a rational function that satisfies (3.6), then $x(n)$ is a polynomial in $n$.

**Proof.** [7] Since $x(n)$ is a rational function, let $x(n) = \frac{P(n)}{Q(n)}$ where $P, Q$ are relatively prime polynomials in $n$. Then (3.6) is

$$a(n) \frac{P(n + 1)}{Q(n + 1)} - b(n - 1) \frac{P(n)}{Q(n)} = c(n).$$

Multiplying through by $Q(n)Q(n + 1)$ yields

$$a(n)P(n + 1)Q(n) - b(n - 1)P(n)Q(n + 1) = c(n)Q(n)Q(n + 1). \tag{3.7}$$

Suppose that $x(n)$ is not a polynomial. Then $\deg(Q(n)) \geq 1$. Let $N$ be the largest non-negative integer so that $\gcd(Q(n), Q(n + N))$ is a polynomial of degree greater than or
equal to one; i.e. not a constant polynomial. Let $f(n)$ be a non-constant polynomial divisor of $Q(n)$ and $Q(n + N)$. Then $f(n - N) | Q(n)$ and we see from (3.7) that

$$f(n - N) | b(n - 1)P(n)Q(n + 1).$$

By assumption, we know that $f(n - N) \nmid P(n)$ since $f(n - N) | Q(n)$. Also, $f(n - N) \nmid Q(n + 1)$ since if it did, then $f(n)$ would be a non-constant polynomial divisor of $Q(n)$ and $Q(n + N + 1)$, which contradicts our choice of $N$. Then $f(n - N) | b(n - 1)$, and we can conclude that $f(n + 1) | b(n + N)$. We can also see from (3.7) that

$$f(n + 1) | a(n)P(n + 1)Q(n).$$

$f(n + 1) \nmid P(n + 1)$ by assumption, since since $f(n + 1) | Q(n + 1)$. We also see that $f(n + 1) \nmid Q(n)$, otherwise $f(n)$ would be a non-constant polynomial divisor of $Q(n - 1)$ and $Q(n + N)$ which contradicts our choice of $N$. Then, $f(n + 1) | a(n)$. But then we would have that $f(n + 1)$ is a common divisor of $b(n + N)$ and $a(n)$, which contradicts (3.4). Therefore, $Q(n)$ must be a constant polynomial and thus $x(n)$ is a polynomial in $n$. \[QED\]

Given a hypergeometric term $t_n$, we want to find a hypergeometric term $z_n$ such that $z_{n+1} - z_n = t_n$, if it exists. Otherwise, we would like to know the sum $\sum_{k=0}^{n-1} t_k$. The following steps outline Gosper’s Algorithm [7].

1. Find the term ratio $r(n) = \frac{t_{n+1}}{t_n}$. This ratio will be a rational function of $n$ since $t_n$ is a hypergeometric term.
2. We want to write $r(n)$ in the following form:

$$r(n) = \frac{a(n)}{b(n)} \cdot \frac{c(n + 1)}{c(n)},$$

where $a(n)$, $b(n)$, and $c(n)$ are polynomials and $\text{gcd}(a(n), b(n + h)) = 1$, for all $h \geq 0$.
3. Find a solution $x(n)$ of (3.6), if it exists, where $x(n)$ is a nonzero polynomial. If such a solution does not exist, return $\sum_{k=0}^{n-1} t_k$ and stop.
4. If a solution exists, find $z_n = \frac{b(n-1)x(n)}{c(n)}t_n$ and stop.

Once you have $z_n$, you desire the sum $s_n = z_n - z_0$. If our summand is undefined for certain $k \in \mathbb{Z}$, then we start the summation at some $k_0$ larger than the singularities. In this case, we would want the sum $s_n = z_n - z_{k_0}$.

**Example 3.1.** Can we use Gosper’s Algorithm to determine whether

$$S_n = \sum_{k=0}^{n} \binom{4k + 1}{k!} \frac{k!}{(2k + 1)!}$$

can be expressed in closed form? If so, what is the closed form?
Let \( t_n = \frac{(4n + 1)(n + 1)}{(2n + 1)!} \), which is clearly a proper hypergeometric term with 
\( P(n, k) = 4k + 1, b_1 = w_1 = 1, v_1 = 2, a_1 = c_1 = u_1 = 0, \) and \( x = 1 \). So the term ratio \( r(n) \)
is a rational function in \( n \).

\[
\frac{r(n)}{t_n} = \frac{(4n + 5)(n + 1)}{(2n + 3)(2n + 2)(4n + 1)} = \frac{4n + 5}{2(2n + 3)(4n + 1)}
\]
To write \( r(n) \) in canonical form, let \( a(n) = 1, b(n) = 2(2n + 3), \) and \( c(n) = 4n + 1 \). Note that \( \gcd(a(n), b(n + h)) = 1 \) for all nonnegative integers \( h \), so we have satisfied (3.3) and (3.4). Plugging these into (3.6), we have

\[
x(n + 1) - 2(2n + 1)x(n) = 4n + 1.
\]
Now we need a nonzero polynomial solution \( x(n) \), if one exists. One way to do this is by trying to find polynomial solutions of degree \( n \), where \( n = 0, 1, 2, \ldots \). Fortunately, we find that \( \deg(x) = 0 \).

\[
\alpha - 2(2n + 1)\alpha = 4n + 1
\]
\[
-4\alpha n - \alpha = 4n + 1,
\]
then we have \(-4\alpha = 4 \) and \(-\alpha = 1 \), so \( \alpha = -1 \). Thus \( x(n) = -1 \). Substituting this in to (3.5), we see that

\[
z_n = \frac{b(n - 1)x(n)}{c(n)}t_n = \frac{2(2n + 1)(-1)}{4n + 1} \cdot \frac{n!}{(2n + 1)!} = -2 \frac{n!}{(2n)!},
\]
satisfies \( z_{n+1} - z_n = t_n \). Now we want to find

\[
s_n = z_n - z_0 = -2 \frac{n!}{(2n)!} - (-2) = 2 - 2 \frac{n!}{(2n)!}.
\]
Since the upper index of our summation is \( n \), not \( n - 1 \), the sum we are looking for has the form

\[
S_n = s_{n+1} = 2 - 2 \frac{(n + 1)!}{(2n + 2)!} = 2 - \frac{n!}{(2n + 1)!}.
\]

3.2 Step 2: Factoring the Rational Function
In this last example, it was easy to find the factorization of \( r(n) \) and the polynomial solution \( x(n) \) by inspection. However, it is not usually the case that one can do either of these steps simply. Here is a more detailed description for Step 2 of Gosper’s Algorithm [7].
2.1. Let \( r(n) = Z \frac{f(n)}{g(n)} \), where \( f, g \) are monic relatively prime polynomials and \( Z \) is a constant. Let \( R(h) = \text{Resultant}_n(f(n), g(n + h)) \), where the resultant of two polynomials \( f, g \) is the product of the values of \( g \) at the zeros of \( f \). Let \( S = \{h_1, h_2, \ldots, h_N\} \) be the set of nonnegative integer zeros of \( R(h) \), where \( N \geq 0 \) and \( 0 \leq h_1 \leq \cdots \leq h_N \).

2.2. Let \( p_0(n) = f(n) \) and \( q_0(n) = g(n) \). Then for \( j = 1, 2, \ldots, N \) do

\[
\begin{align*}
    s_j(n) &= \gcd(p_{j-1}(n), q_{j-1}(n + h_j)) \\
    p_j(n) &= \frac{p_{j-1}(n)}{s_j(n)} \\
    q_j(n) &= \frac{q_{j-1}(n)}{s_j(n - h_j)}.
\end{align*}
\]

Then let

\[
\begin{align*}
    a(n) &= Zp_N(n), \\
    b(n) &= q_N(n), \text{ and} \\
    c(n) &= \prod_{i=1}^{N} \prod_{j=1}^{h_i} s_i(n - j).
\end{align*}
\]

It is clear from the above algorithm that the factorization of \( r(n) \) into the canonical form (3.3) can be quickly done by a computer. We can also easily check that the algorithm really does produce (3.3).

**Proof.**

\[
\frac{a(n)}{b(n)} \cdot \frac{c(n + 1)}{c(n)} = \frac{Zp_N(n)}{q_N(n)} \cdot \prod_{i=1}^{N} \prod_{j=1}^{h_i} \frac{s_i(n + 1 - j)}{s_i(n - j)}
\]

\[
= Z \frac{p_0(n)}{q_0(n)} \prod_{i=1}^{N} \frac{s_i(n - h_i)}{q_0(n)} \prod_{i=1}^{N} \frac{s_i(n)}{s_i(n - h_i)}
\]

\[
= Z \frac{p_0(n)}{q_0(n)}
\]

\[
= Z \frac{f(n)}{g(n)}
\]

\[= r(n)\]

Also note that by definition,

\[
\gcd(p_k(n), q_k(n + h_k)) = \gcd \left( \frac{p_{k-1}(n)}{x_k(n)}, \frac{q_{k-1}(n + h_k)}{s_k(n)} \right) = 1,
\]

for all \( 1 \leq k \leq N \).
When we construct the polynomials $a(n)$, $b(n)$, and $c(n)$ in Step 2 of Gosper’s algorithm, we saw above that equations (3.3) and (3.4) are both satisfied. The following theorem and corollaries show that we know a bit more about these polynomials. The proofs can be found in Chapter 5 of A=B [7].

**Theorem 3.2.** Let $K$ be a field of characteristic zero and let $r \in K[n]$ be a nonzero rational function. Then there are polynomials $a, b, c \in K[n]$ such that $b, c$ are monic and

$$r(n) = \frac{a(n)}{b(n)} \cdot \frac{c(n + 1)}{c(n)},$$

where

1. $\gcd(a(n), b(n + h)) = 1$ for every nonnegative integer $h$,
2. $\gcd(a(n), c(n)) = 1$, and
3. $\gcd(b(n), c(n + 1)) = 1$.

**Corollary 3.1.** This factorization of $r(n)$ is unique.

**Corollary 3.2.** Of all possible polynomials $a(n)$, $b(n)$, and $c(n)$ that satisfy equations (3.3) and (3.4), the triple produced by Gosper’s algorithm in step 2 has $c(n)$ of least degree.

### 3.3 Step 3: Solving for a Nonzero Polynomial Solution

To conclude our discussion of Gosper’s algorithm, let’s look at how to obtain the nonzero polynomial solutions $x(n)$ of equation (3.6) [7]. Suppose that $x(n)$ is a polynomial solution of (3.6) and that $\deg(x(n)) = d$. If we knew $d$, then we could substitute a generic polynomial of that degree into equation (3.6) and set the coefficients of the powers of $n$ equal, much like we did in Example 3.1. Surprisingly, there are at most two candidates for $d$. First consider the case in which the degrees of $a(n)$ and $b(n)$ are not the same, or that the leading coefficients of $a(n)$ and $b(n)$ are not equal. Since the leading terms on the left hand side of (3.6) do not cancel, the degree of the left hand side must be the degree of $x(n)$ plus the degree of $a(n)$ or the degree of $x(n)$ plus the degree of $b(n)$, whichever is larger. On the right hand side we only have $c(n)$, so the degree of the right hand side is $\deg(c(n))$. Then we have

$$d = \deg(c(n)) - \max\{\deg(a(n)), \deg(b(n))\}.$$

Now suppose that the leading coefficients of $a(n)$ and $b(n)$ are the same. Then they have the same degree. Let $\lambda$ represent the coefficient of the shared leading term. Here we consider two cases:

- **Case 1:** The second-highest degree terms on the left hand side are not equal. Then the degree of the left hand side is the degree of $x(n)$ plus the degree of $a(n)$ minus 1. So we have

$$d = \deg(c(n)) - \deg(a(n)) - 1.$$
Case 2: The second-highest degree terms on the left hand side are equal. Let
\[ a(n) = \lambda n^k + An^{k-1} + O(n^{k-2}), \]  
(3.8)
\[ b(n-1) = \lambda n^k + Bn^{k-1} + O(n^{k-2}), \]  
(3.9)
\[ x(n) = C_0 n^d + C_1 n^{d-1} + O(n^{d-2}). \]  
(3.10)

Consider the left hand side of equation (3.6). We can compute the following using the above definitions
\[ x(n+1) = C_0 n^d + (C_0 d + C_1) n^{d-1} + O(n^{d-2}) \]
\[ a(n)x(n+1) = C_0 \lambda n^k + (\lambda(C_0 d + C_1) + AC_0) n^{k+d-1} + O(n^{k+d-2}), \]
\[ b(n-1)x(n) = C_0 \lambda n^k + (BC_0 + \lambda C_1) n^{k+d-1} + O(n^{k+d-2}), \]
\[ a(n)x(n+1) - b(n-1)x(n) = C_0 (\lambda d + A - B) n^{k+d-1} + O(n^{k+d-2}). \]

By assumption, the coefficient of \( n^{k+d-1} \) vanishes, so we have \( C_0 (\lambda d + A - B) = 0 \). Then we can conclude that
\[ d = \frac{B - A}{\lambda}. \]

In Step 3 of Gosper’s algorithm, we seek a polynomial solution \( x(n) \) of equation (3.6). The following provides more details of how the algorithm computes the degree of the polynomial solution and the solution itself, if such a solution exists [7].

3.1. If \( \deg a(n) \neq \deg b(n) \) or \( \text{lc} a(n) \neq \text{lc} b(n) \), then
\[ D = \{ \deg c(n) - \max\{\deg a(n), \deg b(n)\} \} \]
else
\[ \text{let } A, B \text{ be as in (3.8), (3.9), respectively; } \]
\[ D = \left\{ \deg c(n) - \deg a(n) + 1, \frac{B - A}{\text{lc} a(n)} \right\}. \]

Let \( D = D \cap \{0, 1, 2, \ldots \} \).
If \( D = \emptyset \) then return ”no nonzero polynomial solution” and stop.
else \( d = \max D \).

3.2. Using the method of undetermined coefficients, find a nonzero polynomial solution \( x(n) \) of (3.6), of degree \( d \) or less.
If none exists return ”no nonzero polynomial solution” and stop.

The following examples demonstrate the implementation of Gosper’s algorithm.
Example 3.2. Can we find a closed form for the sum of the first \( n + 1 \) factorials; i.e., is

\[
S_n = \sum_{k=0}^{n} k!
\]

Gosper summable?

We set \( t_n = n! \) and find the ratio of consecutive terms.

\[
r(n) = \frac{t_{n+1}}{t_n} = \frac{(n + 1)!}{n!} = n + 1,
\]

which is a rational function of \( n \). If we let \( a(n) = n + 1 \) and \( b(n) = c(n) = 1 \), then \( r(n) \) can be written in canonical form

\[
r(n) = \frac{n + 1}{1} \cdot \frac{1}{1},
\]

and clearly satisfies the condition that \( \gcd(a(n), b(n+h)) = 1 \) for all nonnegative integers \( h \).

Substituting these polynomials into equation (3.6), we obtain

\[
(n + 1)x(n + 1) - x(n) = 1. \tag{3.11}
\]

Observe that \( \deg a(n) = 1 \) and \( \deg b(n) = 0 \), so the degree of \( x(n) \) is given by

\[
D = \{\deg c(n) - \max\{\deg a(n), \deg b(n)\}\}
\]

\[
= 0 - \max\{1, 0\}
\]

\[
= -1.
\]

Since \( d = -1 \), there is no nonzero polynomial solution of (3.11) and we conclude that our sum does not have a closed form.

Example 3.3. Can we find a closed form for the sum

\[
S_n = \sum_{k=1}^{n} kk! ?
\]

Let \( t_n = nn! \). Then

\[
r(n) = \frac{t_{n+1}}{t_n} = \frac{(n + 1)(n + 1)!}{nn!} = \frac{(n + 1)^2}{n},
\]

a rational function of \( n \). We can factor \( r(n) \) as follows.

\[
r(n) = \frac{n + 1}{1} \cdot \frac{n + 1}{n},
\]

so we have \( a(n) = n + 1, \ b(n) = 1, \) and \( c(n) = n \). Condition (3.4) is satisfied, so we can substitute \( a(n), b(n), \) and \( c(n) \) into (3.6), and we have

\[
(n + 1)x(n + 1) - x(n) = n.
\]
Like the previous example, we have $\deg a(n) \neq \deg b(n)$, but now we have $\deg c(n) = 1$. So the degree of $x(n)$ is

$$D = \{\deg c(n) - \max\{\deg a(n), \deg b(n)\}\}$$

$$= 1 - \max\{1, 0\}$$

$$= 0.$$

Thus, $x(n)$ is a constant. By inspection, it is easy to see that $x(n) = 1$ satisfies our equation. Then

$$z_n = \frac{b(n - 1)x(n)}{c(n)} t_n = \frac{1}{n n!} = n!$$

and

$$s_n = z_n - z_1 = n! - 1,$$

so we have

$$S_n = s_{n+1} = (n + 1)! - 1.$$

Since most hypergeometric identities are more complicated and tedious to compute by hand, the following examples illustrate the use of Maple to find hypergeometric sums.

**Example 3.4.** Using Maple, we can compute the sum

$$f(n) = \sum_{k=0}^{n} \frac{4^k}{\binom{2k}{k}}.$$

In Maple, we simply enter the sum as above, utilizing the Expression template for summation on the left hand side of the screen. We click on the summation symbol, $\sum_{i=k}^{n} f$, then replace $i, k, n, \text{ and } f$ with our indices and summand. We could also enter the sum as follows.

```maple
sum(4^k/binomial(2*k,k), k = 0..n);
```

Either way, Maple returns the following output.

$$\frac{\left(\frac{2}{3}n + \frac{1}{3}\right)4^{n+1}}{\binom{2n+2}{n+1}} + \frac{1}{3},$$

which is a hypergeometric term plus a constant.
Example 3.5. Evaluate the sum

$$\sum_{n=0}^{m} \frac{(3n)!}{n!(n+1)!(n+2)!27^n}.$$ 

Entering the sum in Maple, we quickly obtain the following result.

$$\frac{9}{40} \frac{(m + 3)(m + 2)(81(m + 1)^2 + 99m + 119)(3m + 3)!}{(m + 1)!(m + 2)!(m + 3)!27^{m+1}} = \frac{9}{2},$$

a hypergeometric term plus a constant. We notice here that Maple didn’t return a simplified form of the hypergeometric term. A little bit of algebra shows that this could be written as

$$\frac{1}{40} \frac{(81m^2 + 261m + 200)(3m + 2)!}{(m + 2)!(m + 1)!m!27^m} = \frac{9}{2}.$$
CHAPTER 4

ZEILBERGER’S ALGORITHM

For a given hypergeometric term, Gosper’s algorithm conclusively answers whether or not the term can be indefinitely summed. In this chapter, we will look at Zeilberger’s algorithm for determining whether a given hypergeometric term can be definitely summed. A definite sum is a sum in which the upper summation variable is also found in the summand [5]. Zeilberger’s algorithm is also known as the method of creative telescoping. Similarly to Sister Celine’s method, one seeks a recurrence relation for $f(n)$, which is defined as

$$f(n) = \sum_{k} F(n, k),$$

where $F(n, k)$ is doubly hypergeometric. For now, one may think of the range of $k$ to be the set of all integers, but this assumption can actually be relaxed [7]. While one may use Sister Celine’s method to find the recurrence relation for the sum $f(n)$, the method of creative telescoping provides a way to find the recurrence faster.

One can find many summands $F(n, k)$ that are not indefinitely summable, but the sum $f(n)$ when $k$ runs over all integers can be written in simple form. For example, consider the binomial coefficient $\binom{n}{k}$ as a function of $k$ in which $n$ is fixed. We have seen in Example 2.1 that the unrestricted sum $\sum_{k} \binom{n}{k} = 2^n$, but one finds that the indefinite sum $\sum_{k=0}^{M} \binom{n}{k}$ cannot be expressed in terms of $M$ and $n$. When one uses Gosper’s algorithm on this indefinite sum, Maple returns the output:

$$2^n - \binom{n}{M+1} \, _2F_1 \left[ \begin{array}{c} 1 \\ M+2 \end{array} ; -1 \right].$$

Thus the indefinite sum is not Gosper summable, since this result is not a hypergeometric term plus a constant.

One can compare the concepts of definite and indefinite summation to that of definite and indefinite integration. The indefinite integral $\int e^{-t^2/2} \, dt$ cannot be evaluated since $e^{-t^2/2}$ has no “anti-derivative”, that is, $e^{-t^2/2}$ is not the derivative of a simple elementary function. But it can be shown that the definite integral $\int_{-\infty}^{\infty} e^{-t^2/2} \, dt = \sqrt{2\pi}$.

When looking for a recurrence relation for $f(n) = \sum_{k} F(n, k)$, one first finds a recurrence for the summand $F(n, k)$. While it isn’t very often that a $G(n, k)$ can be found so that $F(n, k) = G(n, k + 1) - G(n, k)$, one can often find a $G(n, k)$ such that $F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$. When this happens, one can prove that the
definite summation \( f(n) \) is a constant. Generally speaking, one can’t always expect this to happen either. But under very general circumstances, there is something that one can expect to happen when using a more general difference operator in \( n \) on the left hand side of this equation.

**Definition 4.1.** [7] Let \( N \) be the forward shift operator in \( n \); that is, \( Ng(n, k) = g(n + 1, k) \). Let \( K \) be the forward shift operator in \( k \); that is, \( Kg(n, k) = g(n, k + 1) \).

Now one can re-express \( F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k) \) in operator terms as

\[
NF(n, k) - F(n, k) = KG(n, k) - G(n, k)
\]

or equivalently,

\[
(N - 1)F = (K - 1)G.
\]

It is almost always possible to find a difference operator

\[
p(n, N) = a_0(n) + a_1(n)N + a_2(n)N^2 + \cdots + a_J(n)N^J
\]

such that

\[
p(n, N)F(n, k) = (K - 1)G(n, k),
\]

where the coefficients \( \{a_i(n)\}_{i=0}^{J} \) are polynomials in \( n \) and such that \( \frac{G(n, k)}{F(n, k)} \) is a rational function of \( n, k \). In other words,

\[
\sum_{j=0}^{J} a_j(n)F(n + j, k) = G(n, k + 1) - G(n, k). \tag{4.1}
\]

Producing this recurrence (4.1) is the goal of Zeilberger’s algorithm, given the summand \( F(n, k) \).

Suppose that one has found the recurrence of the form (4.1) for the sum \( f(n) = \sum_k F(n, k) \), along with a rational function \( R(n, k) \) such that \( G(n, k) = R(n, k)F(n, k) \). The goal was to find the sum \( f(n) \), so we need to sum (4.1) over all \( k \in \mathbb{Z} \). Note that on the left hand side the coefficients are independent of \( k \). Summing over all integers \( k \), we have

\[
\sum_{j=0}^{J} a_j(n)f(n + j) = 0, \tag{4.2}
\]

provided that \( G(n, k) \) has compact support in \( k \) for each \( n \). Now one can consider the following possibilities.

First, one may find that \( J = 1 \). Then (4.2) is a first order recurrence with polynomial coefficients; i.e.

\[
a_0(n)f(n) + a_1(n)f(n + 1) = 0.
\]
Then we see that
\[
\frac{f(n+1)}{f(n)} = \frac{a_0(n)}{a_1(n)}.
\]
Since \(a_0(n)\) and \(a_1(n)\) are both polynomials in \(n\), we see that \(\frac{f(n+1)}{f(n)}\) is a rational function of \(n\). Then we know that \(f(n)\) is a hypergeometric term, and we can express it as
\[
f(n) = f(0) \prod_{j=0}^{n-1} \frac{-a_0(j)}{a_1(j)},
\]
and the sum is complete.

Now suppose that \(J > 1\) and the coefficients \(\{a_i(n)\}_{i=0}^{J}\) are constant. In this case, we have a linear recurrence relation with constant coefficients. Solving this homogeneous linear recurrence relation with constant coefficients yields an explicit formula for the sum \(f(n)\) \cite{7}. The following example demonstrates the process of solving a recurrence relation of this form.

**Example 4.1.** Consider the recurrence relation \(a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}\), for \(n \geq 3\), with initial conditions \(a_0 = 2\), \(a_1 = -4\), and \(a_2 = 26\).

Suppose that \(a_n = x^n\) is a solution of the recurrence. Then
\[
a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}
\]
becomes
\[
x^n = x^{n-1} + 8x^{n-2} - 12x^{n-3}.
\]
Dividing through by \(x^{n-3}\) yields the characteristic equation
\[
x^3 = x^2 + 8x - 12.
\]
This is a polynomial of degree 3, so by the Fundamental Theorem of Algebra, we know that the characteristic equation has 3 complex roots. Solving the equation, we find that the solutions are \(x = -3\) and \(x = 2\) with multiplicity two. Then \(a_n\) has the following form
\[
a_n = \alpha 2^n + \beta n 2^n + \gamma (-3)^n.
\]
By substituting in the initial conditions, we have the following system of equations:
\[
a_0 = \alpha + \gamma = 2
\]
\[
a_1 = 2\alpha + 2\beta - 3\gamma = -4
\]
\[
a_2 = 4\alpha + 8\beta + 9\gamma = 26
\]
Solving this system of equations yields \(\alpha = 0\), \(\beta = 1\), and \(\gamma = 2\), so the closed-form expression for the recurrence relation is \(a_n = n2^n + 2(-3)^n\). 
\(\square\)
The final possibility is that \( J > 1 \) in the recurrence (4.1) and the coefficients \( \{a_i(n)\}_{i=0}^L \) are non-constant polynomials of \( n \). In this case, one would not know if the recurrence formula can be solved, and if it is solvable, how one would do it. While this sounds difficult or even impossible, Petkovšek’s algorithm, Hyper, can handle this situation and either provides the solution or proves that a solution does not exist. Algorithm Hyper can be found in Chapter 8 of \( A=B \) [7].

**Example 4.2.** Using the method of creative telescoping, one can evaluate the following sum that appeared as problem 10424 in The American Mathematical Monthly [5, 7].

\[
f(n) = \sum_{0 \leq k \leq \frac{n}{2}} 2^k \frac{n}{n-k} \binom{n-k}{2k}
\]

Let \( F(n, k) = 2^k \frac{n}{n-k} \binom{n-k}{2k} \), and use the EKHAD package in Maple. By entering the following command,

\[
ct \left( 2^k \frac{n}{n-k} \binom{n-k}{2k}, 3, k, n, N \right);
\]

one obtains the third order recurrence

\[
N^3 - 2N^2 + N - 2
\]

along with the rational function

\[
-\frac{2k(2k-1)(-n+k)}{(-n-1+3k)(-n-2+3k)(-n-3+3k)}.
\]

Since \( G(n, k) = R(n, k)F(n, k) \), we see that

\[
G(n, k) = -\frac{2k(2k-1)(-n+k)}{(-n-1+3k)(-n-2+3k)(-n-3+3k)} \cdot 2^k \frac{n}{n-k} \binom{n-k}{2k}
\]

\[
= -\frac{2^k n}{n-3k+3} \binom{n-k}{2k-2}.
\]

This output gives the following recurrence

\[
(N^3 - 2N^2 + N - 2) F(n, k) = (K - 1) G(n, k).
\]

Summing over \( 0 \leq k \leq n-1 \), then for \( n \geq 2 \) the right hand side telescopes to zero, and we have the following recurrence with constant coefficients

\[
(N^3 - 2N^2 + N - 2) f(n) = 0.
\]

Since \( N^3 - 2N^2 + N - 2 = N^2(N-2) + (N-2) = (N-2)(N^2 + 1) = 0 \), we can solve for \( N \) to obtain 2, \( i \), and \( -i \). Thus the recurrence has a general solution of the form

\[
a_n = \alpha 2^n + \beta i^n + \gamma (-i)^n.
\]
Using the initial conditions, \( f(1) = f(2) = 1 \) and \( f(3) = 4 \), we find that
\[
a_n = \frac{1}{2} 2^n + \frac{1}{2} i^n + \frac{1}{2} (-i)^n = 2^{n-1} + \frac{1}{2} \left((e^{\pi/2})^n + (e^{-\pi/2})^n\right) = 2^{n-1} + \cos \frac{n\pi}{2},
\]
for \( n \geq 2 \).

The following theorem guarantees the existence of the telescoped recurrence that is central to Zeilberger’s algorithm.

**Theorem 4.1.** [7] If \( F(n, k) \) is a proper hypergeometric term, then \( F \) satisfies the nontrivial recurrence
\[
\sum_{j=0}^J a_j(n)F(n+j, k) = G(n, k+1) - G(n, k),
\]
where \( \frac{G(n,k)}{F(n,k)} \) is a rational function of \( n \) and \( k \).

### 4.1 How Zeilberger’s Algorithm Works

Since the telescoped form of the recurrence always exists for a proper hypergeometric term, let’s look at how Zeilberger’s algorithm works [7]. First, we will fix the order \( J \) of the recurrence (4.1)
\[
\sum_{j=0}^J a_j(n)F(n+j, k) = G(n, k+1) - G(n, k)
\]
and then try to find a recurrence of this order, if possible. If the recurrence doesn’t exist, we will increase \( J \) and try again. Let \( t_k \) represent the left hand side of (4.1) with our fixed \( J \):
\[
t_k = a_0 F(n, k) + a_1 F(n+1, k) + \cdots + a_J F(n+J, k). \tag{4.3}
\]

Then the ratio of successive terms is
\[
\frac{t_{k+1}}{t_k} = \frac{a_0 F(n, k+1) + a_1 F(n+1, k+1) + \cdots + a_J F(n+J, k+1)}{a_0 F(n, k) + a_1 F(n+1, k) + \cdots + a_J F(n+J, k)}
\]
\[
= \frac{\sum_{j=0}^J a_j F(n+j, k+1)}{\sum_{j=0}^J a_j F(n+j, k)}
\]
\[
= \frac{\sum_{j=0}^J a_j F(n+j, k+1)/F(n, k+1)}{\sum_{j=0}^J a_j F(n+j, k)/F(n, k)} \cdot \frac{F(n, k+1)}{F(n, k)}
\]

Since \( F(n, k) \) is a hypergeometric term, we know that \( \frac{F(n,k+1)}{F(n,k)} \) is a rational function of \( n, k \). Let’s denote this fraction by \( \frac{r_1(n,k)}{r_2(n,k)} \), where \( r_1, r_2 \) are polynomials. Similarly, let \( \frac{F(n,k)}{F(n-1,k)} = \frac{s_1(n,k)}{s_2(n,k)} \), where \( s_1, s_2 \) are polynomials. Then
\[
\frac{F(n+j, k)}{F(n, k)} = \prod_{i=0}^{j-1} \frac{F(n+j-i, k)}{F(n+j-i-1, k)} = \prod_{i=0}^{j-1} \frac{s_1(n+j-i, k)}{s_2(n+j-i, k)}.
\]
Then it follows that
\[
\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-i} s_1(n+j-i,k+1) \right\} \cdot r_1(n,k)}{\sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-i} s_1(n+j-i,k) \right\} \cdot r_2(n,k)}
= \frac{\sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i,k+1) \prod_{r=j+1}^{J} s_1(n+j-r,k+1) \right\} \cdot r_1(n,k) \prod_{r=1}^{J} s_1(n+r,k)}{\sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_1(n+r,k) \right\} \cdot r_2(n,k) \prod_{r=1}^{J} s_1(n+r,k+1)}.
\]

Let
\[
p_0(k) = \sum_{j=0}^{J} a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i,k) \prod_{r=j+1}^{J} s_2(n+r,k) \right\},
\]
\[
r(k) = r_1(n,k) \prod_{r=1}^{J} s_2(n+r,k), \text{ and}
\]
\[
s(k) = r_2(n,k) \prod_{r=1}^{J} s_2(n+r,k+1).
\]

Then we have
\[
\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \cdot \frac{r(k)}{s(k)}. \tag{4.4}
\]

Observe that the unknown coefficients \( \{a_j\}_{j=0}^{J} \) only appear in \( p_0(k) \), not in \( r(k) \) nor \( s(k) \).

Now by Theorem 3.2, we can write \( \frac{r(k)}{s(k)} \) in the canonical form
\[
\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \cdot \frac{p_2(k)}{p_3(k)}. \tag{4.5}
\]

Now let \( p(k) = p_0(k)p_1(k) \), and we have
\[
\frac{t_{k+1}}{t_k} = \frac{p_0(k+1)}{p_0(k)} \cdot \frac{r(k)}{s(k)} \tag{4.6}
= \frac{p_0(k+1)}{p_0(k)} \cdot \frac{p_1(k+1)}{p_1(k)} \cdot \frac{p_2(k)}{p_3(k)} \tag{4.7}
= \frac{p(k+1)}{p(k)} \cdot \frac{p_2(k)}{p_3(k)}, \tag{4.8}
\]

which is the standard setup for Gosper’s algorithm. Then \( t_k \) is an indefinitely summable term if and only if the recurrence
\[
p_2(k)b(k+1) - p_3(k-1)b(k) = p(k) \tag{4.9}
\]
has a polynomial solution $b(k)$. (Compare to equation (3.6) and Theorem 3.1.) Notice that the right hand side of this equation depends linearly on the unknown coefficients $\{a_j\}_{j=0}^J$, but the left hand side is independent of them. We now seek a polynomial solution to (4.9). Assume that the upper bound on $J$ is $M$ for the polynomial solution, and let

$$b_k = \sum_{i=0}^{M} \beta_i k^i.$$  

The coefficients will be determined later. We can substitute this into (4.9), and then find a system of simultaneous linear equations in the $M + J + 2$ unknowns $\{a_0, \ldots, a_J, \beta_0, \ldots, \beta_M\}$. If this system cannot be solved, then there is no recurrence of order $M$ and we would need to start over, this time assuming that the order of the recurrence is $M + 1$. If there is a solution, then we now know all of the $\{a_j\}_{j=0}^J$ in (4.1). By equation (3.5) we also have $G(n, k)$, where

$$G(n, k) = \frac{p_3(k - 1)}{p(k)} b(k) t_k.$$  

For further reading, S.A. Abramov and his colleagues have published several papers in which they have sought to improve upon Zeilberger’s algorithm, some of which are cited in this paper [1, 2, 3]. Some of their improvements deal with the efficiency of the algorithm. For example, instead of beginning the algorithm at zero, Abramov and Le [3] published an algorithm for computing a lower bound for the order of the minimal telescoper. In [1], Abramov studies when Zeilberger’s algorithm works and expands the use of Zeilberger’s algorithm beyond proper hypergeometric terms. The final paper referenced here [2] provides another look some of the common methods for computing closed forms of definite sums as well as provides an introduction to a software package SumTools for Maple which is a “symbolic summation toolbox”.

### 4.2 Examples of Zeilberger’s Algorithm

**Example 4.3.** [7] Use the method of creative telescoping to evaluate the sum $\sum_k \binom{n}{k}^2$.

Let $F(n, k) = \binom{n}{k}^2$ and we fix $J = 1$ in hopes of finding a first order recurrence. Then

$$t_k = a_0 F(n, k) + a_1 F(n + 1, k),$$
and the term ratio is
\[
\frac{t_{k+1}}{t_k} = \frac{a_0 \binom{n}{k+1}^2 + a_1 \binom{n+1}{k+1}^2}{a_0 \binom{n}{k}^2 + a_1 \binom{n+1}{k}^2}
\]
\[
= \frac{a_0 \binom{n}{k+1}! \binom{n-k-1}{k+1}!^2 + a_1 \binom{n+1}{k+1}! \binom{n-k-1}{k+1}!^2}{a_0 \binom{n}{k}! \binom{n-k}{k-1}!^2 + a_1 \binom{n+1}{k}! \binom{n-k}{k-1}!^2}
\]
\[
= \frac{a_0 \binom{n-k}{k}^2 + a_1 \binom{n+1}{k}^2}{a_0 + a_1 \binom{n-k}{k}^2}
\]
\[
= \left\{ \frac{a_0 (n-k)^2 + a_1 (n+1)^2}{a_0 (n-k+1)^2 + a_1 (n+1)^2} \right\} \left\{ \frac{(n-k+1)^2}{(k+1)^2} \right\}.
\]

Note that this is of the form (4.4) with
\[
p_0(k) = a_0(n-k+1)^2 + a_1(n+1)^2,
\]
\[
r(k) = (n-k+1)^2, \text{ and }
\]
\[
s(k) = (k+1)^2.
\]

We can write this is canonical form of (4.5) with \(p_1(k) = 1\), \(p_2(k) = (n-k+1)^2\), and \(p_3(k) = (k+1)^2\).
\[
\frac{r(k)}{s(k)} = \frac{1}{1} \cdot \frac{(n-k+1)^2}{(k+1)^2}
\]

If we set \(p(k) = p_0(k)p_1(k)\), we have
\[
p(k) = a_0(n-k+1)^2 + a_1(n+1)^2,
\]
and we know from (4.8) that
\[
\frac{t_{k+1}}{t_k} = \frac{a_0(n-k)^2 + a_1(n+1)^2}{a_0(n-k+1)^2 + a_1(n+1)^2} \cdot \frac{(n-k+1)^2}{(k+1)^2}.
\]

Now we want to solve the recurrence of the form (4.9)
\[
(n-k+1)^2b(k+1) - k^2b(k) = a_0(n-k+1)^2 + a_1(n+1)^2. \quad (4.10)
\]

To find the upper bound for the degree of the polynomial solution, if one exists, we look at the terms of \(p_2(k)\) and \(p_3(k)\). The highest degree terms, \(k^2\), are the same and the next highest degree terms, \(-2(n+1)k\) and \(-2k\) are not the same. Then the degree of our polynomial solution is
\[
d = \text{deg}(a_0(n-k+1)^2 + a_1(n+1)^2) - \text{deg}((n-k+1)^2) + 1 = 2 - 2 + 1 = 1.
\]
So we can assume that \( b(k) = \alpha + \beta k \), and we can substitute this into our expression (4.10) to obtain

\[
(n - k + 1)^2(\alpha + \beta(k + 1)) - k^2(\alpha + \beta k) = a_0(n - k + 1)^2 + a_1(n + 1)^2.
\]

Collecting powers of \( k \) on each side of the equation and setting the coefficients equal, one can find the following solution:

\[
\alpha = -3(n + 1), \beta = 2, a_0 = -2(2n + 1), a_1 = n + 1.
\]

So we know that

\[
F(n, k) = \binom{n}{k}^2
\]

satisfies the recurrence

\[
-2(2n + 1)F(n, k) + (n + 1)F(n + 1, k) = G(n, k + 1) - G(n, k), \tag{4.11}
\]

where

\[
G(n, k) = \frac{k^2}{-2(2n + 1)(n - k + 1)^2 + (n + 1)(n + 1)^2} \cdot (-3(n + 1) + 2k) \\
\cdot \left( -2(2n + 1)\binom{n}{k}^2 + (n + 1)\binom{n + 1}{k}^2 \right)
\]

\[
= \frac{k^2(2k - 3n - 3)}{-2(2n + 1)(n - k + 1)^2 + (n + 1)^3} \cdot \left( \frac{-2(2n + 1)n!^2}{k!^2(n - k)!^2} + \frac{(n + 1)(n + 1)!^2}{k!^2(n - k + 1)!^2} \right)
\]

\[
= \frac{k^2(2k - 3n - 3)}{-2(2n + 1)(n - k + 1)^2 + (n + 1)^3} \cdot \frac{-2(2n + 1)(n - k + 1)^2n!^2 + (n + 1)^3n!^2}{k!^2(n - k + 1)!^2}
\]

\[
= \frac{(2k - 3n - 3)n!^2}{(k - 1)!^2(n - k + 1)!^2}.
\]

Now that we have the telescoped form of the recurrence, we sum (4.11) over all integers \( k \) to obtain

\[
-2(2n + 1)f(n) + (n + 1)f(n + 1) = 0.
\]
Then
\[ f(n + 1) = \frac{2(2n + 1)f(n)}{n + 1} \]
\[ = \frac{2(2n + 1) \cdot 2(2n - 1)f(n - 1)}{n + 1} \cdot \frac{n}{n - 1} \]
\[ = \frac{2^2(2n + 1)(2n - 1)}{(n + 1)n} \cdot \frac{2(2n - 3)f(n - 2)}{n - 1} \]
\[ : \]
\[ = \frac{2^{n+1}(2n + 1)(2n - 1) \cdots (1)f(0)}{(n + 1)!}, \text{ where } f(0) = 1 \]
\[ = \frac{2^{n+1}(2n + 2)(2n + 1)(2n)(2n - 1)(2n - 2) \cdots (2)(1)}{(n + 1)!(2n + 2)(2n)(2n - 2) \cdots (2)} \]
\[ = \frac{2^{n+1}(2n + 2)!}{(n + 1)!2^{n+1}(n + 1)!} \]
\[ = \binom{2n + 2}{n + 1}. \]

So our desired recurrence is
\[ f(n) = \sum_k \binom{n}{k}^2 = \binom{2n}{n}, \]

Since it was possible to do the creative telescoping algorithm by hand, it is time-consuming. Since the algorithm is completely computerizable, one may desire to take advantage of the power of a computer to do the computations much more quickly. The following examples are done using the EKHAD package for Maple.

**Example 4.4.** Evaluate the sum \( f(n) = \sum_k (-1)^k \binom{n}{x+k}. \)

We let \( F(n, k) = (-1)^k \binom{n}{x+k}, \) and use the following command in Maple.

\[ ct \left( (-1)^k \binom{n}{x+k}, 1, k, n, N \right); \]

Maple returns the following output
\[ -n - x + (n + x + 1)N, \frac{k(x + k)}{-n - 1 + k}, \]

so our recurrence is
\[ (-n - x + (n + x + 1)N)F(n, k) = G(n, k + 1) - G(n, k). \]
The second part of the Maple output is the rational function $R(n, k)$, so it follows that

$$G(n, k) = (-1)^k \frac{{n \choose k}}{\frac{1}{x+k}} \cdot \frac{k(x+k)}{-n-1+k}.$$ 

When we sum the recurrence over all integers $k$, we have

$$(n-x+(n+x+1)N) f(n) = 0.$$

A little bit of algebra and the fact that $f(0) = 1$ shows the following.

$$(n-x) f(n) + (n + x + 1) f(n+1) = 0.$$

Our desired recurrence is

$$f(n) = \frac{x}{n + x}.$$

\[\square\]

**Example 4.5.** In 1891, Dixon found a closed form for the sum $\sum_k (-1)^k \binom{2n}{k}^3$. We can use the creative telescoping algorithm to evaluate Dixon’s sum.

We first enter the following command in Maple.

$$ct \left((-1)^k \binom{2n}{k}^3, 1, k, n, N \right) ;$$

Maple returns the recurrence

$$-6(3n+2)(3n+1) - 2(n+1)^2 N$$

and a very complicated rational function $R(n, k)$. We sum the recurrence over all integers $k$ to obtain

$$-6(3n+2)(3n+1) f(n) - 2(n+1)^2 f(n+1) = 0.$$
Then
\[ f(n + 1) = \frac{-3(3n + 2)(3n + 1)f(n)}{(n + 1)^2} \]
\[ = \frac{(-1)^23^2(3n + 2)(3n + 1)(3n - 1)(3n - 2)f(n - 1)}{(n + 1)^2n^2} \]
\[ = \frac{(-1)^33^3(3n + 2)(3n + 1)(3n - 1)(3n - 2)(3n - 4)(3n - 5)f(n - 2)}{(n + 1)^2n^2(n - 1)^2} \]
\[ \vdots \]
\[ = \frac{(-1)^{n+1}3^{n+1}(3n + 3)(3n + 2)(3n + 1)(3n) \cdots (3)(2)(1)f(0)}{(n + 1)^2n^2 \cdots (2)^2(1)^2(3n + 3)(3n)(3n - 3) \cdots (3)} \]
\[ = \frac{(-1)^{n+1}3^{n+1}(3n + 3)!f(0)}{3^{n+1}(n + 1)!^3} \]
\[ = \frac{(-1)^{n+1}(3n + 3)!f(0)}{(n + 1)^3} \]

Since \( f(0) = 1 \), we have
\[ f(n) = \sum_k (-1)^k \binom{2n}{k}^3 = \frac{(-1)^n(3n)!}{n!^3}. \]
CHAPTER 5

THE STANDARD WZ PROOF ALGORITHM

Wilf and Zeilberger have developed a method for providing short and elegant proofs of combinatorial identities. All that one needs is a rational function, called the proof certificate. All known identities, such as the examples we have done in previous chapters and those found in the “hypergeometric database” in Chapter 3 of $A = B$ [7], can be proved by a single rational function. Note that the WZ algorithm is different from the previous algorithms in that it does not provide the closed form for an unknown hypergeometric sum. The WZ algorithm provides short proofs of known identities, but it also enables us to discover new identities from these known identities.

5.1 THE WZ ALGORITHM

The WZ Algorithm is used to prove that an identity is true. The following steps guide us through the process of using the algorithm [7, 8, 9].

Step 1. If we want to show that

$$\sum_{k} t(n, k) = \text{rhs}(n),$$

then we assume that for all $n$, $t(n, k)$ vanishes for all $k$ outside of some finite interval.

Step 2. Divide both sides by the right hand side, resulting in a sum of the form

$$\sum_{k} F(n, k) = 1,$$

where $F(n, k) = \frac{t(n, k)}{\text{rhs}(n)}$.

Step 3. Let $R(n, k)$ be the rational function provided by Gosper’s algorithm as proof of the identity, and define a new function

$$G(n, k) = R(n, k)F(n, k).$$

Step 4. Verify that

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$$

is true, then sum this equation over all integers $k$. The right hand side telescopes to zero, so we have

$$\sum_{k} F(n + 1, k) = \sum_{k} F(n, k).$$
Then $\sum_k F(n,k)$ is independent of $n$, so it is constant. When (5.1) is true, we call the functions $F$ and $G$ a WZ pair.

Step 5. Verify that this constant is 1 by calculating $\sum_k F(0,k) = 1$.

Example 5.1. To demonstrate the standard WZ proof algorithm, consider $\sum_k \binom{n}{k} = 2^n$ and the key $R(n,k) = \frac{k}{2(n-k-1)}$.

Step 1: Begin by setting $t(n,k) = \binom{n}{k}$ and $rhs(n) = 2^n$.

Step 2: Next, one divides both sides by $2^n$ and sets $F(n,k) = \binom{n}{k} 2^{-n}$. Now we want to show that $\sum_k \binom{n}{k} 2^{-n} = 1$.

Step 3: One then defines $G(n,k)$ and simplifies.

$$G(n,k) = R(n,k)F(n,k) = \frac{k}{2(k-n-1)} \cdot \binom{n}{k} 2^{-n}$$

$$= \frac{kn! 2^{-n}}{2(k-n-1)k!(n-k)!}$$

$$= -\frac{n!}{2(n-k+1)!(n-k)!} \cdot 2^{-n-1}$$

$$= -\binom{n}{k-1} 2^{-n-1}.$$ 

Step 4: We wish to show that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$.

$$\binom{n+1}{k} 2^{-n-1} - \binom{n}{k} 2^{-n} = -\binom{n}{k} 2^{-n-1} - (-\binom{n}{k-1} 2^{-n-1})$$

We can multiply both sides by $2^n$ to eliminate the exponential terms.

$$\binom{n+1}{k} \frac{1}{2} - \binom{n}{k} \frac{1}{2} = -\binom{n}{k} \frac{1}{2} + \binom{n}{k-1} \frac{1}{2}$$

Next, we rewrite all binomial coefficients as factorials.

$$\frac{(n+1)!}{2k!(n+1-k)!} - \frac{n!}{k!(n-k)!} = -\frac{n!}{2k!(n-k)!} + \frac{n!}{2(k-1)!(n-k+1)!}$$

Now we multiply both sides by $\frac{2k!(n+1-k)!}{n!}$ in order to eliminate both the factorials and
the fractions.

\[ n + 1 - 2(n + 1 - k) = -(n + 1 - k) + k \]

\[ 2k - n - 1 = 2k - n - 1 \]

\[ 0 = 0 \]

**Step 5:** Finally, we want to show that \( \sum_k F(0, k) = 1 \).

\[ F(0, k) = \binom{0}{k} 2^{-0} = \binom{0}{k} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise}. \end{cases} \]

Thus \( \sum_k F(0, k) = 1 \).

**Example 5.2.** Write out the complete proof of the identity, using the full text of the standardized WZ proof together with the appropriate rational function certificate.

\[ \sum_k \binom{n}{k} \binom{x}{k + r} = \frac{k(k + r)}{(n + x + 1)(k - n - 1)} \]

**Step 1:** Let \( t(n, k, x, r) = \binom{n}{k} \binom{x}{k + r} \) and \( \text{rhs}(n, k, x, r) = \binom{n + x}{n + r} \).

**Step 2:** Next, we divide through by the right hand side to obtain

\[ F(n, k, x, r) = \frac{\binom{n}{k} \binom{x}{k + r}}{\binom{n + x}{n + r}} = \frac{n!x!(n + r)!(x - r)!}{k!(n - k)!(k + r)!(x - k - r)!(n + x)!} \]

**Step 3:** Let

\[ R(n, k, x, r) = \frac{k(k + r)}{(n + x + 1)(k - n - 1)} \]

(the given rational function certificate) and use it to compute

\[ G(n, k, x, r) = R(n, k, x, r) F(n, k, x, r). \]

\[ G(n, k, x, r) = \frac{k(k + r)}{(n + x + 1)(k - n - 1)} \cdot \frac{n!x!(n + r)!(x - r)!}{k!(n - k)!(k + r)!(x - k - r)!(n + x)!} \]

\[ = - \frac{(k - 1)!(n - k + 1)!(k + r - 1)!(x - k - r)!(n + x + 1)!}{n!(n + r)!(x - r)!} \]

\[ = - \binom{n}{k - 1} \frac{x!(n + r)!(x - r)!}{(k + r - 1)!(x - k - r)!(n + x + 1)!} \]
Step 4: Now we need to show whether

\[ F(n + 1, k, x, r) - F(n, k, x, r) = G(n, k + 1, x, r) - G(n, k, x, r) \]

is true. Writing out the terms, we have

\[
\frac{(n + 1)!}{k!(n - k)!(k + r)!(x - k - r)!(n + x + 1)!} \times \frac{n!}{n!(n + r)!(x - r)!} - \frac{n!}{k!(n - k)!(k + r)!(x - k - r)!(n + x)!} \times \frac{n!}{n!(n + r)!(x - r)!}
\]

Let’s clean this up by multiplying through by

\[
\frac{k!(n - k)!(k + r)!(x - k - r)!(n + x + 1)!}{n!x!(n + r)!(x - r)!}
\]

to eliminate all the factorials and we obtain

\[
\frac{(n + 1)(n + r + 1)}{n - k + 1} - (n + x + 1) = -(x - k - r) + \frac{k(k + r)}{n - k + 1}
\]

\[
(n + 1)(n + r + 1) - (n + x + 1)(n - k + 1) = -(x - k - r)(n - k + 1) + k(k + r)
\]

\[
r + k - nx + kx - x + k = -nx + kx - x + nk + k + nr + r
\]

\[ 0 = 0. \]

Thus \( F(n + 1, k, x, r) - F(n, k, x, r) = G(n, k + 1, x, r) - G(n, k, x, r) \) is true.

Step 5: In our final step, we want to show that \( \sum_k F(0, k, x, r) = 1 \).

\[
F(0, k, x, r) = \frac{\binom{0}{k} \binom{x}{k+r}}{\binom{x}{r}}
\]

\[ = \begin{cases} 
1, & \text{if } k = 0; \\
0, & \text{otherwise.} 
\end{cases} \]

Thus, \( \sum_k F(0, k, x, r) = 1 \), and the identity \( \sum_k \binom{n}{k} \frac{\binom{x}{k+r}}{\binom{n+x}{n+r}} \) is true. \qed
5.2 The WZ Proof Certificate

In the previous examples, the proof certificate was provided. Typically, we won’t know the proof certificate for the identity. In order to find the proof certificate, we use Gosper’s algorithm. If we wish to certify the sum

$$\sum_k f(n, k) = r(n)$$

is true, we first define a new function $F(n, k)$. If $r(n) \neq 0$, then

$$F(n, k) = \frac{f(n, k)}{r(n)}.$$ Otherwise, let $F(n, k) = f(n, k)$. Now we define

$$f(k) = F(n + 1, k) - F(n, k),$$

which is the input function for Gosper’s algorithm. If Gosper’s algorithm fails, then so does the WZ algorithm. Otherwise, Gosper’s algorithm returns $G(n, k)$, the WZ mate of $F$. We then find the rational function

$$R(n, k) = \frac{G(n, k)}{F(n, k)},$$

which is the WZ proof certificate for the identity $\sum_k F(n, k) = \text{constant}$.

We can use Maple and the EKHAD package to produce the WZ proof certificate for a hypergeometric identity. Instead of Gosper’s algorithm, we use the creative telescoping algorithm to find $R(n, k)$.

Example 5.3. Using Maple, find the WZ proof of

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}.$$  

Let $f(n, k) = \binom{n}{k}^2$ and rewrite the sum in the form $\sum_k F(n, k) = 1$. Then we have

$$F(n, k) = \binom{n}{k}^2 \binom{2n}{n}.$$  

Entering the following command in Maple

$$ct \left( \binom{n}{k}^2 \binom{2n}{n}, 1, k, n, N \right);$$

yields the following output.

$$N - 1, \frac{1}{2} \frac{(-3n - 3 + 2k)k^2}{(-n - 1 + k)^2(2n + 1)}$$
Proof. To prove that \( \sum_k \binom{n}{k}^2 = \binom{2n}{n} \), rewrite it as
\[
\sum_k \frac{\binom{n}{k}^2}{\binom{2n}{n}} = 1,
\]
and take
\[
R(n, k) = \frac{1}{2} \frac{(-3n - 3 + 2k)k^2}{(-n - 1 + k)^2(2n + 1)}.
\]

Example 5.4. Prove the Binomial Theorem using the WZ method.
\[
\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n
\]

Proof. Rewrite the Binomial Theorem as
\[
\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = 1,
\]
and take
\[
R(n, k) = \frac{kb}{(-n - 1 + k)(a + b)}.
\]

One may be uncomfortable with this proof at first because it seems as though the rational function certificate just appears out of nowhere. However, one can now verify the steps by hand, if desired.

Example 5.5. Prove the Binomial Theorem using the full text of the standardized WZ proof together with the rational function certificate, \( \frac{kb}{(-n - 1 + k)(a + b)} \).

Proof. Let \( t(n, k) = \binom{n}{k} a^k b^{n-k} \) and \( r.h.s(n, k) = (a + b)^n \). Divide through by the right hand side to obtain
\[
F(n, k) = \frac{\binom{n}{k} a^k b^{n-k}}{(a + b)^n}.
\]
Let \( R(n, k) \) be the rational function certificate and compute \( G(n, k) = R(n, k)F(n, k) \).
\[
G(n, k) = \frac{kb}{(-n - 1 + k)(a + b)} \cdot \frac{\binom{n}{k} a^k b^{n-k}}{(a + b)^n} = \frac{n! a^k b^{n-k} k!}{(n - k)! (a + b)^n (-n - 1 + k)(a + b)}
\]
\[
= -\frac{(k - 1)! (n - k + 1)!(a + b)^{n+1}}{n! a^k b^{n-k+1}}
\]
\[
= -\binom{n}{k-1} \frac{a^k b^{n-k+1}}{(a + b)^{n+1}}.
\]
Next we verify that the WZ equation (5.1) holds.

\[
\frac{(n + 1)!a^kb^{n-k+1}}{k!(n - k + 1)!(a + b)^{n+1}} - \frac{n!a^kb^{n-k}}{k!(n - k)!(a + b)^n} = \frac{n!a^{k+1}b^{n-k}}{(n - k)!(a + b)^{n+1}} + \frac{n!a^kb^{n+1}}{(k - 1)!(n - k + 1)!(a + b)^{n+1}}
\]

Let’s eliminate all the factorials, exponential functions, and the denominators by multiplying both sides by

\[
\frac{k!(n - k + 1)!(a + b)^{n+1}}{n!a^kb^{n-k}}.
\]

We now have

\[
(n + 1)b - (a + b)(n - k + 1) = -a(n - k + 1) + bk
\]
\[
bn + b - an + ak - a - bn + bk - b = -an + ak - a + bk
\]
\[
-an + ak - a + bk = -an + ak - a + bk
\]
\[
0 = 0.
\]

Our final step is to show that \(\sum_k F(0, k) = 1\). We have

\[
F(0, k) = \left(\begin{array}{c} 0 \\ k \end{array}\right) a^kb^{-k} = \left(\begin{array}{c} 0 \\ k \end{array}\right) a^k = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise}. \end{cases}
\]

Thus \(\sum_k F(0, k) = 1\).

\[\Box\]

### 5.3 Additional Benefits of the WZ Method

The WZ method does more than certifying known identities. For each identity proved by the WZ method, we obtain at least three more identities. These additional identities are known as the companion identity, dual identities, and the definite-sum-made-indefinite.

Before we discuss each of these new identities, let’s look at the theorem that supports the WZ method. The theorem relies on the following hypotheses [7].

(1) For each integer \(k\), the limit

\[
f_k = \lim_{n \to \infty} F(n, k)
\]

exists and is finite.

(2) For each integer \(n \geq 0\),

\[
\lim_{k \to \pm \infty} G(n, k) = 0.
\]

(3)

\[
\lim_{L \to \infty} \sum_{n \geq 0} G(n, -L) = 0.
\]
Theorem 5.1. [7] Let $F$ and $G$ satisfy the WZ equation (5.1). If (5.2) holds, then

$$\sum_{k} F(n, k) = \text{constant} \ (n = 0, 1, 2, \ldots).$$

If both (5.2) and (5.4) hold, then

$$\sum_{n \geq 0} G(n, k) = \sum_{j \leq k-1} (f_j - F(0, j)), \quad (5.5)$$

where $f$ is defined as in (5.2).

Equation (5.5) is known as the companion identity.

Example 5.6. Consider the identity

$$\sum_{k} k \binom{n}{k} = n2^{n-1}, \ n \geq 1.$$ 

What is the companion identity for this sum?

Let

$$F(n, k) = \frac{k \binom{n}{k}}{n2^{n-1}}.$$ 

Using the creative telescoping algorithm in the EKHAD package, we find the rational proof certificate to be

$$R(n, k) = \frac{1}{2} \frac{k - 1}{(-n - 1 + k)}.$$ 

Multiplying these together we obtain

$$G(n, k) = R(n, k)F(n, k) = \frac{1}{2} \frac{k - 1}{(-n - 1 + k)} \frac{kn!}{n2^{n-1}k!(n-k)!} = -\frac{1}{2^n} \binom{n-1}{k-2}.$$ 

For each $k$, we have

$$f_k = \lim_{n \to \infty} F(n, k) < \infty.$$ 

For each integer $n \geq 0$,

$$\lim_{k \to \pm\infty} G(n, k) = 0.$$ 

We also have

$$\lim_{L \to \infty} \sum_{n \geq 0} G(n, -L) = 0.$$ 

Since all the hypotheses are satisfied, by Theorem 5.1 we know that the companion identity is

$$\sum_{n \geq 1} G(n, k) = \sum_{n \geq 1} \frac{1}{2^n} \binom{n-1}{k-2} = \begin{cases} 1 & \text{if } k \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$ 

\[\square\]
If we have a summand of the form

\[ F(n, k) = x^n y^k \rho(n, k) \prod_i (a_i n + b_i k + c_i)! \prod_i (u_i n + v_i k + w_i)! \]

where \( \rho(n, k) \) is a rational function, then we are able to find a dual identity for the summation. We take our WZ pair \( F, G \) and do the following operation to turn both \( F \) and \( G \) into new hypergeometric terms. The result, say \( \tilde{F} \) and \( \tilde{G} \), is also a WZ pair! The operation involves selecting a factor \((an + bk + c)!\) from the numerator of \( F(n, k) \). Remove this factor from the numerator, place \((-1 - an - bk - c)!\) in the denominator, and multiply by \((-1)^{an+bk}\). This new function is \( \tilde{F} \). Do exactly the same operation on \( G(n, k) \) to get \( \tilde{G} \).

The third possibility for finding additional identities was discovered by Zeilberger [7]. He found that a definite sum, such as \( h(n) = \sum_{k=0}^{n} F(n, k) \), where \( F \) has a WZ mate \( G \), can be made indefinite. Since \( F \) has a WZ mate \( G \), we know that the following is true.

\[ F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k) \]

When we sum this equation over all \( k \leq n \), we have

\[ h(n + 1) - F(n + 1, n + 1) - h(n) = G(n, n + 1). \]

Now we replace \( n \) by \( j \) and sum over \( j = 0, \ldots, n - 1 \) to obtain

\[ h(n) = h(0) + \sum_{j=1}^{n} (F(j, j) + G(j - 1, j)). \]

That is,

\[ \sum_{k=0}^{n} F(n, k) = F(0, 0) + \sum_{j=1}^{n} (F(j, j) + G(j - 1, j)). \]

On the right hand side, we can see that the upper summation bound \( n \) does not appear anywhere in the summand, thus we have changed the definite sum into an indefinite sum.

In 1984, Ira Gessel [4] extended Wilf and Zeilberger’s work to create a plethora of new identities. Gessel took some familiar identities, such as Dixon’s theorem and Saalschütz’s theorem, and used Maple to produce almost ninety new identities from twenty-one known theorems. His ideas expanded Wilf and Zeilberger’s approach to finding dual identities. Gessel studied only WZ functions, where we have a summand \( F(n, k) \) for which the sum of \( F \) over all \( k \) is one and \( F \) has a WZ mate \( G \). He then restricted his efforts to terminating identities, which are those that \( F(n, k) \) has compact support which depends upon \( n \). The following are a few of Gessel’s observations [4, 7].

1. If \((F(n, k), G(n, k))\) is a WZ pair, then so is \((G(k, n), F(k, n))\). Note that \( \sum_k G(k, n) \) may not be terminating.
2. If $F(n, k)$ is a WZ function, then so is $F(n + a, k + b)$ for all $a, b$.
3. If $F(n, k)$ is a WZ function then so are $F(-n, k)$ and $F(n, -k)$.

It is worth noting that although we are calling these results “new” hypergeometric identities, it is possible that some of the almost ninety identities that appear in Gessel’s work [4] may actually be the same as other known identities but differ by applying a transformation rule to the known identity. In this case, the identities aren’t truly new, but expressed in a new way. Gessel’s work further demonstrates how impossible it would be to maintain a hypergeometric database.
CHAPTER 6

CONCLUSION

We have looked at four methods for proving hypergeometric identities; three are capable of producing an unknown sum, if it exists, and the fourth gives succinct proofs of known identities as well as providing many ways of finding additional identities. While the newer methods are more efficient, the groundbreaking work of Sister Celine made it possible for Gosper, Zeilberger, Wilf, Petkovšek, and many others to discover new ways of implementing her ideas and broadening the spectrum of functions to which they are applied.

Sister Celine’s doctorate work and subsequent publications on finding recurrence relations for hypergeometric polynomials directly from the series expansions of the polynomials was purely algorithmic. This algorithmic nature of Sister Celine’s method is the true genius of her work, since it can be completely automated. While her method is effective and easy to use, it is much slower than Zeilberger’s algorithm.

Gosper’s algorithm is for finding closed form hypergeometric identities of sums whose consecutive terms have a rational function as their common ratio. His algorithm is useful when looking for an indefinite sum of hypergeometric terms. It is possible for Gosper’s algorithm to fail to find a closed form of the indefinite sum, but one should not give up hope for finding a closed sum for this hypergeometric term. It is still possible that if we sum over all \( n \), or some particular set of values of \( n \), we can still find a closed form for this hypergeometric sum, provided that the coefficients involve another variable. For example, suppose that \( a(n, k) \) is doubly hypergeometric, then we would be able to use either Zeilberger’s algorithm or Petkovšem’s algorithm to find a closed form for

\[
\sum_k a(n, k).
\]

Zeilberger’s algorithm, or the method of creative telescoping, is also useful for finding whether a given hypergeometric term can be definitely summed. A bit of research shows that mathematicians, such as S.A. Abramov, have improved upon Zeilberger’s algorithm by increasing the speed at which it processes a sum and have expanded the use of the algorithm beyond proper hypergeometric terms.

Unlike the other methods, the WZ method isn’t capable of producing a closed form for an unknown sum. We must work with a known sum to produce the proof certificate and the WZ mate. However, Gosper’s algorithm or Zeilberger’s algorithm can provide that information, then we can proceed with finding WZ proofs or WZ pairs. We have seen that
once we have a WZ pair, Wilf, Zeilberger, and Gessel [4, 7] have demonstrated that there are several ways in which we can create new WZ pairs in addition to finding new identities.

Each of the methods we have seen can be generalized to multivariate sums and to $q$- and multi-$q$-sums. For further reading on this topic, $A = B$ [7] is a good starting point. In this paper, we have used the EKHAD package for Maple, but algorithms for these methods have also been written for Mathematica and are discussed in $A = B$ [7].
BIBLIOGRAPHY


