An Exploration Into Using
Ferromagnetic Material and Input Current
for Maximizing Uses of Electron Spin

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Katherine L. Beauvais
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The Undersigned Faculty Committee Approves the

Thesis of Katherine L. Beauvais:

An Exploration Into Using
Ferromagnetic Material and Input Current
for Maximizing Uses of Electron Spin

Antonio Palacios, Chair
Department of Mathematics and Statistics

Joseph Mahaffy
Department of Mathematics and Statistics

Mathew Anderson
Department of Physics

12/18/13
Approval Date
DEDICATION

This is dedicated to those who have been there for me. Thank you:

my parents, Dan and Evelyn,
my brothers, Josh and Jake,
my sister, Kristi,
my Uncle and Aunt, Dave and Jenne,
and my friend, Kellita.
We are not what we know but what we are willing to learn.

– Mary Catherine Bateson
The field of spintronics, the manipulation of electron spin and magnetic moment rather than the electron charge, is a growing field that can be applied to enhance electronic devices for communication and computing. In 1980, after the discovery of giant magnetoresistance by Peter Grunberg and Albert Fert, the field of spintronics was started. The layers of ferromagnetic and nonmagnetic material to control electron spin and therefore, creating magnetic precession in devices called Spin Torque Nano-Oscillators, are used to generate a microwave signal. The power output of coupled oscillators, as compared to the uncoupled oscillators increases by the amount of oscillators squared. However, for an increase in coupled oscillators, the regions of synchronization decreases. Therefore, in this paper we explore the synchronization of the system for varying values of input DC current, a coupling parameter in the system.
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CHAPTER 1
INTRODUCTION

Spintronics, an emerging technology controlling the electron spin rather than electron charge in circuits, is a new and growing field to improve upon electrical devices [26]. The electron spin is altered by applying a magnetic field to an electrical current. The electron spin is the intrinsic torque characteristics of an electron, giving the electron two directions, spin up or spin down. The spin direction of the electron can carry information, similar to the charge of the electron, through a device. This alternation of electrons can lead to new innovations using binary code. It is more efficient to control spin rather than electron charge in devices because of its easy manipulation by an applied magnetic field and its perpetual spin state [46].

![Figure 1.1. Electron spin directions of (left) spin up or (right) spin down due to the electron dipole](image)

An application, which utilizes spintronics, is the sensitive magnetic field sensor, which can be used in devices such as MRI machines and molecule tagging appliances [55]. Using electron spin in sensitive magnetic field sensors causes the devices to be more stable so that they do not need to be cooled or recalibrated [29]. In the medical field [55], spintronic sensors can be used to replace electromechanical magnetic switches because spintronic sensors are more stable and function more precisely over temperature and voltage thresholds. Thus, they can be employed in devices such as hearing aids, pacemakers, implantable cardioverter-defibrillators, and neurostimulators [5]. In the field of computer science, spintronics is changing how computers function by redesigning transistors to stop over heating by simply changing the orientation of the spin instead of creating heat-generated
resistance. This allows the transistors to be more compact, thereby increasing the amount that can be put into a chip [6]. Spintronics is also used in magnetic random access memory (MRAM), which is being researched and developed at laboratories such as IBM, Hitachi, Motorola, and Honeywell [1]. The Hitachi Deststar 7K3000 can contain 3 TB of memory storage using this technology.

A nano-scale microwave signal generator is yet another application of great interest in spintronics because microwave signals are used in cellular communication components, GPS, audio and video broadcasting, satellite communications, radar, and electronic warfare [4]. Microwaves also have a higher frequency and more bandwidth than radiowaves improving wireless communication [56, 46]. This would be very attractive, since the structure of an antenna used for wireless communications using a microwave signal can be smaller than radio frequency antennas. This is due to the small wavelengths of microwave signals caused by high frequencies which creates a small beamwidth. Also, microwave signals contain a large bandwidth creating a large range of frequency channels to transmit or receive signals. In order to transmit a microwave signal using Spintronics, magnetic precession due to the change of electron spin through thin layers of ferromagnetic material is produced. As polarized current passes through a free directional magnetic layer, the change in direction creates a torque, causing precession by both the electron and the magnetic field. This is known as spin transfer torque, which was discovered by John Slonczewski and Luc Berger [58].

Another effect used in spintronics is the giant magnetoresistance (GMR). This consists of drastic changes in resistance due to the alignment of electron spin with the magnetic direction of various layers of ferromagnetic and nonmagnetic material in a capacitor. The change in electron direction can create electron scattering, which is the deflection of the path of electrons as they pass through a solid material. The electrostatic force causing the deflection decreases the electron speed creating resistance [3]. This effect can cause a simple binary system, which is then applied to many applications as already mentioned. Both the spin transfer torque and the giant magnetoresistance effect are both effects used in Spin Transfer Nano-Oscillators (STNO) to produce a microwave frequency voltage signal.

The STNO device consists of three layers. Two of these layers contain ferromagnetic material, one consisting of a fixed magnetic direction and the other a free magnetic direction. The third layer is a thin nonmagnetic layer used to sustain polarization of the flowing current. This layer resides between the two ferromagnetic layers. The Army has created a STNO device with a length of 100nm. It is able to produce a 250pW microwave signal that can travel 1m. This power output is not sufficient for practical uses. However, since the device is very small, multiple STNOs can be coupled to increase the power exponentially using the Landau-Lifshitz-Gilbert-Slonczewski-Berger Equation to model the system. The coupled
oscillators create a power output congruent to the number of oscillators squared times greater than the uncoupled oscillators. Therefore, synchronization regions must be found in the system to utilize the power given off. Though natural synchronization known as soft coupling exists for simple systems, the STNO system is complex. The synchronization regions are initially small and decrease as the number of oscillators increases. Therefore, the system must be analyzed using bifurcation analysis, transverse Lyapunov exponents, and coherence calculations to find these regions.

The rest of the thesis is organized as follows. Chapter two contains background information about Spin Transfer Nano Oscillators and electron spin discussing the importance of GMR and spin transfer torque. Chapter three consist of the Landau-Lifshitz Gilbert equation used to analyze the system and to conduct simulations. This equation is simplified by decreasing the number of dimensions through the Complex Stereographic Projection, using real values found by the separation of real and imaginary terms of the stereographic equation, and the decreasing the amount of parameter in the system by using a center manifold reduction. Chapter four contains the models of series and parallel circuits showing the difference in the power of coupled to uncoupled oscillators is the number of oscillators squared, $N^2$, bifurcation diagrams showing the long term state of oscillations in the system based on input current, and transverse Lyapunov exponents which show local stability for a two oscillator system. The last chapter explains synchronization in the system using the Order Parameter based on different parameter values. This chapter demonstrates synchronization in the system through time with constant parameters for a two oscillator system, synchronization in the system based on varying input current, and synchronization in the system with a varying applied magnetic direction and a varying DC current.
CHAPTER 2
BACKGROUND

Giant Magnetoresistance is the only effect creating variations in the resistance of STNOs, causing a voltage output. This is due to the amount of electron scattering generated by the change in electron direction. In addition, the alignment of the electron spin and the magnetic direction of the ferromagnetic material create a torque referred to as the spin transfer torque. The spin transfer torque causes magnetic precession producing a strong microwave signal output. This can lead to multiple applications of a STNO device. Some practical uses include telecommunications and broadcasting, since microwaves have a short wavelength ranging between 1mm and 1m. This allows the use of directional antennas, which are smaller than other antennas used for radio waves [56, 46]. Figure 2.1 shows the electromagnetic spectrum, which illustrates the different frequencies for electromagnetic radiation, showing that microwaves have a wavelength slightly less than radiowaves. The spectrum contains regions of ionizing and nonionizing radiation where frequencies below the middle range of UV rays ionize. Microwaves are, as a result, nonionizing meaning the radiation has enough energy to excite an electron by causing the electron to jump to higher energy states, but not enough energy to remove an electron from a molecule. Thereby microwaves have lower health risks than ionizing radiation. Microwaves containing high frequencies are a part of the optical region of the electromagnetic spectrum, which produces heating waves. Infrared and visible light are also a part of this region. Microwaves also contain low frequencies that span to regions containing radiowaves where concepts such as voltage and current are used. Therefore, microwaves can be applied to many different applications like radios, heating, household electricity, and biological effects. To produce applicable results to a microwave signal generator, the microwave signals consist of low frequencies [39].

Figure 2.1. Electromagnetic Spectrum
2.1 Spin Transfer Torque

Electrons contain properties of angular momentum. Since electrons have two degrees of freedom, the spin of the electron is either referred to as a spin up or spin down state [27]. In 1966, J.C. Slonczewski [49] and L. Berger [10] theorized that the current flowing through magnetic layers consisting of fixed and free magnetic directions can alter the magnetic state of the electron. Electric current is generally unpolarized where the orientation of the electrons, either a spin up or a spin down state, is randomized. Since electrons have an intrinsic alignment interaction with ferromagnetic material, the spin of the electrons is altered to align with the magnetic direction. As current travels through a fixed magnetic layer, the spin of the electrons is altered to align with the magnetic direction causing a majority of electrons to have the same spin direction and, therefore, polarizing the current [46]. As the electrons pass through the free ferromagnetic layer, both the electron direction and magnetic direction are altered to align with each other causing a magnetic moment. Since the angular momentum must be conserved in the system, the change in the orientation of the electrons between layers is perceived as spin transfer torque as seen in Figure 2.2. In this figure, we can see the precession based on directions of the fixed magnetic direction, referred to as M hard FM, and the free magnetic direction, referred to as M soft FM, where the magnetic precession is created by damping and spin transfer torque. Due to the interaction between the electron spin and the easy magnetic field direction in the free ferromagnetic material layer, the electron spin influences the magnetic direction causing a magnetic precession, producing a microwave signal. [42].

Figure 2.2. The damping and spin transfer torque cause the changes in orientation creating precession about the easy axis.

2.2 Giant Magnetoresistance

Magnetoresistance can be defined as the change of resistance in a conductor due to the application of an external magnetic field. For the specific case of ferromagnets, the change in resistance is dependent on the direction of the external field relative to the direction of the current as it travels through the magnet. The discovery of magnetoresistance was due to W. Thomson, also known as Lord Kelvin, who measured the behavior of the resistance of ferromagnets, made of iron and nickel, in the presence of a magnetic field. He discovered
anisotropic magnetoresistance (AMR), which is the difference of the resistance between the parallel case, ferromagnetic layers aligned with each other, and the perpendicular cases, ferromagnetic layers opposed to each other [51]. There was little research done with magnetoresistive material until 1988 when two independent groups, one lead by Peter Grünberg and the other by Albert Fert investigated this phenomenon [2]. Interest in magnetic nanostructures lead to the discovery of antiferromagnetic coupling in STNOs, coupling in a material containing electrons of opposite spin alignments than neighboring electrons. This lead to the shared Nobel Prize in 2007 by Grünberg and Fert for the discovery of giant magnetoresistance [42].

GMR is caused by electron switching due to the electron spin direction realigning to coincide with the direction of each distinct layer of ferromagnetic material in a capacitor. When the direction of the magnetization layer is opposite (anti-parallel) to the electron spin, the electrons align with the magnetic direction. This change in direction causes some electrons to be projected away from their original path creating electron scattering. As higher numbers of electrons are scattered, fewer electrons remain on the original projected path causing a high resistance in the system. When the field of the magnetic layer is in the same direction as the spin of the electrons (in parallel), fewer electrons have to change trajectory to align to the magnetic direction, allowing the electrons to pass through the material in the shortest distance. This leads to less electron scattering causing a small resistance in the circuit, which can be seen in Figure 2.3 [46]. The layers labeled $FM$ are the ferromagnetic layers and $NM$ are the nonmagnetic layers. The arrows indicate the magnetic direction as compared to the spin direction. The green boxes below the circuit show the resistance throughout each configuration. The magnetic layers in parallel have the highest and lowest resistance, while the magnetic layers in anti-parallel vary. The dramatic change in resistance is applied in electronic devices as a binary code to increase efficiency and memory storage in computers [36]. The GMR is the only affect to cause resistance in the STNO device creating a voltage output.

### 2.3 Spin Torque Nano-Oscillators

The STNO is a device that utilizes the spin transfer torque of electrons to produce constant oscillations in the magnetization and to produce giant magnetoresistance to create an output voltage. The STNO device, which has a size less than 100nm, consists of three layers. The first layer is a fixed ferromagnetic material in the direction of the easy axis. As the electrons pass through the ferromagnetic material of the first layer, the poles of the electron align along the easy axis causing uniformity among all the electrons polarizing the current [35]. The second layer consists of a thin nonmagnetic layer to sustain polarization of the
Figure 2.3. The current flows through the different layers of ferromagnetic material (FM) and nonmagnetic material (NM).

electrons and to separate the magnetic layers. The last layer consists of a ferromagnetic material with a free magnetic direction with an external magnetic field applied to the layer. When the current, consisting of electrons with a uniformed spin direction, passes through the layer, the applied magnetic field adds a torque to the electron spin direction. Due to the ferromagnetic free direction being effected by both the applied field direction and the electron spin precession about the easy axis. This leads to precession of the free layer. Figure 2.4 shows the direction of the electrons through each layer in the STNO. The value $M$ represents the direction of the fixed magnetic direction, while $m$ symbolizes the free magnetic direction. The electron spin and magnetic direction are also affected by the applied magnetic field labeled as $H_{app}$.

The change in the orientation of the free layer causes a change in resistance in the STNOs due to the giant magnetoresistance effect [58]. This causes a fluctuation in the voltage output. The magnetic moment produced by the influence of the applied magnetic direction and the electron spin direction precession due to the spin transfer torque cause the free layer to sustain oscillations at microwave frequencies. Therefore, a STNO produces a microwave frequency voltage output. These frequencies are dependent upon the amount of input DC current inputted into the circuit. However, the power output of the device for one oscillator is very weak, about 1 nW [56]. In this case, multiple synchronized devices are needed at a coherent frequency to generate a power output great enough for practical uses [28, 31, 44, 47].

An electron has properties of spinning, but does not physically spin since an electron is a point particle and contains almost no mass. This spin is a quantum mechanical
phenomenon which can be viewed as an intrinsic spin in the electron. Since an electron contains momentum, one can use the laws of conservation of momentum to describe the STNO system since this law is a more fundamental idea and holds for cases where Newton’s third law fails like in the cases of quantum mechanics. Therefore, Newton’s third law, stating that for every action there must be a reaction, does not act upon the STNO system. Thereby, a torque is not observable in a STNO [27].

The theory that intrinsic spin in an electron is due to quantum mechanical effects has been proven experimentally through the Stern-Gerlach experiment conducted by Otto Stern and Walther Gerlach. In this experiment, illustrated in Figure 2.5, a calibrated beam of silver atoms travel in the y-direction and pass through a nonuniform magnetic field with a direction along the z-axis. Due to the torque exerted on the electron spin by the magnetic field, the magnetic moment of the electron will precess about the direction of the magnetic field. Therefore, in the experiment the beam projection shifts along the x-direction throughout time. Also in this experiment, the beam should be projected onto the screen as a line along the z-direction. The magnetic field, being nonuniform, should randomize the directions of the electrons which would cause a continuous spread of the beam onto the projection screen in the z-direction. However, as in the illustration, the beam is split into an upper and lower beam displaying two points on the projection screen. The two beam points are due to the two spin directions, either spin up or spin down, an electron contains due to the intrinsic spin.
Uhlenbach and Goudsmit proposed this effect is due to the electrons containing intrinsic spin and angular momentum [15].

Figure 2.5. Stern-Gerlach Experiment proving quantum mechanical spin of an electron

2.4 LANDAU-LIFSHITZ-GILBERT EQUATION

The Landau-Lifshitz (LLG) equation is used for differential equations to describe the precessional motion of magnetization. T. L. Gilbert added a damping term that depended on the time dependence of the magnetic field to create a more realistic model [22]. To trace the magnetic field direction of the free magnetic layer, the Landau-Lifshitz equation is used with the addition of the Slonczewski-Berger spin torque term, expanding the equation into the Landau-Lifshitz Gilbert Slonczewski-Berger equation [11, 12, 18]

\[
\frac{d\hat{m}}{dt} = \text{precession} - \gamma \hat{m} \times H_{\text{eff}} + \lambda \hat{m} \times \frac{d\hat{m}}{dt} - \gamma ag (P, \hat{m} \cdot \hat{S}_p) \hat{m} \times (\hat{m} \times \hat{S}_p). \quad (2.1)
\]

This equation contains three terms which represent precession, damping, and spin transfer torque. The first term is the precession term which is directly related to the electron orientation, where \( \gamma \) is the absolute value of the gyromagnetic ratio, the ratio of the electron’s magnetic dipole moment to its angular momentum, and \( \hat{m} \), which is the unit vector in the direction of the free layer magnetization. The effective field, \( H_{\text{eff}} \), is a combination of the demagnetization field (the easy plane field), the anisotropy field (the easy axis field), the applied field, and the exchange field [54]. The anisotropy field is the direction of the field based on the material used in the ferromagnetic field. The exchange interaction is created
through the interaction of electrons with the same spin, which can be explained through the Pauli Exclusion Principle, which states that two electrons with the same spin can not have the same position. The applied field produces the small deviation of the magnetization from the fixed axis to the easy axis. The demagnetization field is apart of the anisotropy field. The demagnetization field acts on the magnetization, reducing the total magnetic moment, since it is difficult to calculate. It is usually shaped giving rise to the shape of the anisotropy field. For ellipsoidal shapes, the demagnetization field is linearly related to the magnetization by a geometric dependent constant called the demagnetization factor [13].

The second term in the equation is the phenomenological dissipation term, which contains the damping effects on the electron precession. This term contains the damping parameter, which accounts for the loss in energy by spin-flip scattering, a process that does not conserve spin. This causes a change in the spin direction when going through magnetic material creating scattering and coupling of oscillators [58]. \( \lambda \) is the magnitude of the damping term.

The last term in the equation is the spin-transfer torque term. The spin polarized current created by the electron spin orientates the free magnetic layer causing a spin transfer torque. This term contains \( a \), which is the coupling parameters, and \( g \), which is a function of \( P \) (the polarized factor), and \( \hat{S}_\nu \), which is the fixed magnetic field direction [54].

The LLG equation describes the revolution of a magnetic field with constant magnitude \( |\hat{m}| \). The magnitude \( |\hat{m}| \) is constant because \( d\hat{m}/dt \) is a function of the cross products of \( \hat{m} \) and, therefore, is perpendicular to \( \hat{m} \). One can verify this conclusion by considering the right hand side of Eq. (2.1) in the form:

\[
c \ast (\hat{m} \times \hat{v}),
\]

where \( c \) is a constant and \( \hat{v} \) is a vector. If we take the dot product of \( \hat{m} \) with Eq. (2.2), it transforms each term into

\[
c \ast \hat{m} \cdot (\hat{m} \times \hat{v}).
\]

Using the identity \( a \cdot (b \times c) = c \cdot (a \times b) \), we can rewrite this as

\[
c \ast \hat{v} \cdot (\hat{m} \times \hat{m}).
\]

The cross product of identical vectors equal zero, so all the terms go to zero leaving

\[
\hat{m} \cdot \frac{d\hat{m}}{dt} = 0.
\]

(2.3)

Using the concept that \( \hat{m} \cdot \hat{m} = |\hat{m}|^2 \), the time derivative of \( \hat{m} \cdot \hat{m} \) gives

\[
\frac{d}{dt}(\hat{m} \cdot \hat{m}) = \hat{m} \cdot \frac{d\hat{m}}{dt} + \frac{d\hat{m}}{dt} \cdot \hat{m}.
\]

(2.4)
Using Eq. (2.3), we see that
\[ \hat{m} \cdot \frac{d\hat{m}}{dt} = \frac{d\hat{m}}{dt} \cdot \hat{m} = 0. \] (2.5)

Substituting Eq. (2.5) into Eq. (2.4) gives
\[ \frac{d}{dt}(\hat{m} \cdot \hat{m}) = 0. \]

This means that the value $|\hat{m}|^2$ is constant and therefore $|\hat{m}|$ is also constant. This proves the magnetic field $\hat{m}$ direction has an invariant magnitude for the LLG equation. Throughout the rest of the paper we will assume that $\hat{m}$ is a unit vector having the magnitude $|\hat{m}| = 1$ meaning the magnetic field direction will be traced upon a surface of a unit sphere [53].
CHAPTER 3

ARRAYS OF SPIN TRANSFER NANO-OSCILLATORS

A complex stereographic projection is a process pertaining to projecting a spherical three dimensional plot onto a complex two dimensional surface in order to simplify a model. This gives a total of $2 \times N$ dimensions in our system, where $N$ is the number of oscillators. Therefore, the center manifold reduction is used to project the system onto the center manifold in order to reduce the total amount of dimensions.

In order to create simulations of the power output of a circuit containing an array of STNOs, the value for the resistance based on either a series or parallel circuit is calculated along with solving the ordinary differential equation for the trace of the magnetic direction. The power is then calculated using a Fourier Transform. The power output ratio of coupled to uncoupled oscillators for the STNOs model is equal to the number of oscillators squared. Since the injected DC current is a part of the coupling parameter, varying the input DC current forms different oscillation cycles in the system. Therefore, a bifurcation diagram is used to determine regions of synchronization and stability in these devices to maximize the power output. The angle of the applied magnetic field is altered to explore different dynamics in the system using the values $\theta_h = 0, \pi/4$, and $3\pi/4$. Transfer Lyaponuv Exponents are used to calculate the stability and verify the stability given in the bifurcation diagrams.

3.1 COMPLEX STEREOGRAPHIC PROJECTION

When tracing the Landau-Lifshitz equation, a three dimensional spherical graph is produced with axes in unit magnetic vectors, $\vec{M}$. The Complex Stereographic Projection is used in order to simplify the Landau-Lifshitz equation by changing our three dimensional equation in Cartesian coordinates into a two dimensional equation in coordinates of complex variables, $\omega$ and $\bar{\omega}$. To preform this simplification a spherical graph is projected onto a two dimensional plane, where the North Pole of the sphere is oriented at the origin of the complex axes and the South Pole, which is the projection point, goes off to infinity. Therefore, the magnetic direction traced on the surface of a sphere can be mapped onto a two dimensional plane. This allows one to easily view the orbit of the of the electron and simplify the LLG equation. In order to view the complex stereographic projection equation, the equation from the Landau-Lifshitz equation is derived.[32]

The variable $\omega$ is used to project our Landau-Lifshitz equation into complex stereographic coordinates. This term is created by the complex number $x + iy$ divided by the
distance in the \( \vec{z} \) axis of the projected point on the sphere to the two dimensional plane [32]. This equation takes the form

\[ \omega \equiv \frac{m_1 + im_2}{1 + m_3}. \] (3.1)

Eq. (3.1) is manipulated to find \( m_1, m_2, m_3 \) using \( \bar{\omega} \) to represent the complex conjugate of \( \omega \) which, as a unit vector \( \hat{m} \), is [33, 54]

\[
\hat{m} = \begin{bmatrix}
\frac{\omega + \bar{\omega}}{1 + |\omega|^2} \\
\frac{i(\omega - \bar{\omega})}{1 + |\omega|^2} \\
\frac{1 - |\omega|^2}{1 + |\omega|^2}
\end{bmatrix}.
\] (3.2)

The differentiation of \( m_1, m_2, m_3 \) with respect to time using \( \dot{\omega} \) as the derivative of \( \omega \) and \( \ddot{\omega} \) as the derivative of \( \bar{\omega} \) gives us the following results [33]

\[
\frac{dm_1}{dt} = \frac{\dot{\omega} (1 + \omega \bar{\omega}) - \bar{\omega} (\omega + \bar{\omega})}{(1 + |\omega|^2)^2} + \frac{\dot{\bar{\omega}} (1 + \omega \bar{\omega}) - \bar{\omega} (\omega + \bar{\omega})}{(1 + |\omega|^2)^2},
\]

\[
\frac{dm_2}{dt} = -i\omega (1 + \omega \bar{\omega}) - \bar{\omega} (i\omega + i\bar{\omega}) + \frac{i\dot{\omega} (1 + \omega \bar{\omega}) - \bar{\omega} (-i\omega + i\bar{\omega})}{(1 + |\omega|^2)^2},
\]

\[
\frac{dm_3}{dt} = -\dot{\omega} \omega (1 + \omega \bar{\omega}) - \bar{\omega} (1 - \omega \bar{\omega}) + \frac{-\dot{\bar{\omega}} \omega (1 + \omega \bar{\omega}) - \bar{\omega} (1 - \omega \bar{\omega})}{(1 + |\omega|^2)^2}.
\]

These time derivatives of the directions of magnetization can be simplified down to the following:

\[
\frac{dm_1}{dt} = \frac{\dot{\omega} (1 - \bar{\omega}^2) + \dot{\bar{\omega}} (1 - \omega^2)}{(1 + |\omega|^2)^2}.
\]
\[
\frac{dm_2}{dt} = -i\omega (1 + \omega^2) + i\dot{\omega} (1 + \omega^2) \over (1 + |\omega|^2)^2, \\
\frac{dm_3}{dt} = -2 (\dot{\omega}\bar{\omega} - \ddot{\omega}\omega) \over (1 + |\omega|^2)^2.
\]

From these we get the time derivative unit vector \( \frac{d\hat{m}}{dt} \)

\[
\frac{d\hat{m}}{dt} = \begin{bmatrix}
\frac{\omega(1-\omega^2)+\dot{\omega}(1-\omega^2)}{(1+|\omega|^2)^2} \\
\frac{-i\omega(1+\omega^2)+i\dot{\omega}(1+\omega^2)}{(1+|\omega|^2)^2} \\
\frac{-2(\dot{\omega}\bar{\omega}-\ddot{\omega}\omega)}{(1+|\omega|^2)^2}
\end{bmatrix}.
\]

Differentiating Eq. (3.1) with respect to time, where \( \dot{\omega} \) is the derivative of \( \omega \), yields

\[
\dot{\omega} = \frac{(1 + m_3)(\frac{dm_1}{dt} + i\frac{dm_2}{dt}) - (m_1 + im_2)\frac{dm_3}{dt}}{(1 + m_3)^2}. (3.3)
\]

Substituting Eq. (3.2) into Eq. (3.3) produces

\[
\dot{\omega} = \frac{1}{2}\frac{dm_1}{dt} + i\frac{(1 + |\omega|^2)}{2}\frac{dm_2}{dt} - \frac{\omega(1 + |\omega|^2)}{2}\frac{dm_3}{dt}.
\]

This can be rewritten as the dot product

\[
\dot{\omega} = \begin{bmatrix}
\frac{1 + |\omega|^2}{2} \\
i\frac{(1 + |\omega|^2)}{2} \\
\frac{\omega(1 + |\omega|^2)}{2}
\end{bmatrix} \cdot \begin{bmatrix}
\frac{dm_1}{dt} \\
\frac{dm_2}{dt} \\
\frac{dm_3}{dt}
\end{bmatrix}. (3.4)
\]

The Effective Field acting on the spin vector, which is the right-hand side term used in Eq. (2.1), is:

\[
\vec{H}_{eff} = \vec{H}_{exchange} + \vec{H}_{anisotropy} + \vec{H}_{demagnetization} + \vec{H}_{applied} (3.5)
\]

The exchange field is defined as \( \vec{H}_{exch} = D \nabla^2 \hat{m} \). In this case, we are assuming the spin states are homogeneous and therefore, since the exchange field would be redundant, \( H_{exchange} = 0 \) [38]. Then, the expanded form of the Landau-Lifshitz equation becomes

\[
\frac{d\hat{m}}{dt} = -\gamma \left[ \hat{m} \times \left( \vec{H}_{anis} + \vec{H}_{demag} + \vec{H}_{app} \right) \right] + \lambda \hat{m} \times \frac{d\hat{m}}{dt} - \gamma_{ag} (P, \hat{m} \cdot \hat{S}_p) \hat{m} \times (\hat{m} \times \hat{S}_p). (3.6)
\]
The applied field is kept fixed in the yz-plane, so that

\[
\vec{H}_{\text{applied}} = \begin{bmatrix} 0 \\ h_{a2} \\ h_{a3} \end{bmatrix}.
\]

The anisotropy field is defined as \(\vec{H}_{\text{anisotropy}} = \kappa (\hat{m} \cdot \hat{e}_\parallel) \hat{e}_\parallel\). The direction of uniaxial anisotropy, referred to as \(\hat{e}_\parallel\), is the radius in spherical coordinates in relation to Cartesian coordinates, \(\hat{r} = \sin(\theta) \cos(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + \cos(\theta) \hat{z}\):

\[
\hat{e}_\parallel = \begin{bmatrix} \sin \theta \parallel \cos \phi \parallel \\ \sin \theta \parallel \sin \phi \parallel \\ \cos \theta \parallel \end{bmatrix}.
\]

Using the substitution of \(m_\parallel = \hat{m} \cdot \hat{e}_\parallel\) and \(\hat{e}_\parallel\) into the dot product of the anisotropy term gives the new anisotropy term [54]:

\[
\vec{H}_{\text{anis}} = \kappa (m_\parallel) \hat{e}_\parallel = \kappa m_\parallel \begin{bmatrix} \sin \theta_\parallel \cos \phi_\parallel \\ \sin \theta_\parallel \sin \phi_\parallel \\ \cos \theta_\parallel \end{bmatrix}.
\]

The demagnetization term is defined as \(\vec{H}_{\text{demag}} = -4\pi S_0 \nabla \cdot \vec{N} \hat{m}\). It has the predisposition to affect the total magnetic field to reduce the total magnetic moment. The demagnetization field does not have a particular shape and therefore for spherical fields, the demagnetization vectors are assumed to be linear by using the demagnetization factor, \(\vec{N}\), which is limited by the regions of a sphere and held constant as \(N_1 + N_2 + N_3 = 1\) [30]:

\[
\vec{H}_{\text{demag}} = -4\pi S_0 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \cdot \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 4\pi S_0 \begin{bmatrix} -N_1 (\omega + \bar{\omega}) \\ i N_2 (\omega - \bar{\omega}) \\ -N_3 (1 - |\omega|^2) \\ \frac{1}{1 + |\omega|^2} \end{bmatrix}.
\]

The first term of the right hand side of the LL equation, Eq. (2.1), is \(-\gamma \hat{m} \times H_{\text{eff}}\). Using Eq. (3.5), we can expand Eq. (3.6) to solve for each cross product separately for the effective field

\[-\gamma \hat{m} \times H_{\text{eff}} = -\gamma \left( \hat{m} \times H_{\text{aniso}} \right) + \left( \hat{m} \times H_{\text{demag}} \right) + \left( \hat{m} \times H_{\text{app}} \right).\]

We take the cross product of \(\hat{m}\) with the anisotropy term:
\[-\gamma \dot{m} \times \vec{H}_{\text{anis}} = -\gamma \dot{m} \times \kappa m_{\parallel} \begin{bmatrix} \sin \theta_{\parallel} \cos \phi_{\parallel} \\ \sin \theta_{\parallel} \sin \phi_{\parallel} \\ \cos \theta_{\parallel} \end{bmatrix},\]

\[-\gamma \dot{m} \times \vec{H}_{\text{anis}} = -\gamma \kappa m_{\parallel} \begin{bmatrix} -i(\omega - \bar{\omega}) \cos \theta_{\parallel} - (1 - \omega \bar{\omega}) \sin \theta_{\parallel} \sin \phi_{\parallel} \\ \frac{1 + \omega^2}{1 + \omega^2}(1 - \omega \bar{\omega}) \sin \phi_{\parallel} - (\omega + \bar{\omega}) \cos \theta_{\parallel} \\ \frac{(\omega + \bar{\omega}) \sin \theta_{\parallel} \sin \phi_{\parallel} + i(\omega - \bar{\omega}) \sin \theta_{\parallel} \cos \phi_{\parallel}}{1 + \omega^2} \end{bmatrix}. \tag{3.7}\]

We take the cross product of \(-\gamma \dot{m}\) with the demagnetization term:

\[-\gamma \dot{m} \times \vec{H}_{\text{demag}} = -\gamma \begin{bmatrix} 4i(\omega - \bar{\omega})\pi S_o N_3(1 - \omega \bar{\omega}) - 4i(1 - \omega \bar{\omega})\pi S_o N_2(\omega - \bar{\omega}) \\ \frac{1 + \omega^2}{1 + \omega^2}(-4(1 - \omega \bar{\omega})\pi S_o N_1(\omega + \bar{\omega}) + 4(\omega + \bar{\omega})\pi S_o N_3(1 - \omega \bar{\omega})) \\ 4i(\omega + \bar{\omega})\pi S_o N_2(\omega - \bar{\omega}) - 4i(\omega - \bar{\omega})\pi S_o N_1(\omega + \bar{\omega}) \end{bmatrix}. \tag{3.8}\]

We take the cross product of \(-\gamma \dot{m}\) with the applied field term:

\[-\gamma \dot{m} \times \vec{H}_{\text{applied}} = \gamma \begin{bmatrix} \frac{i(\omega - \bar{\omega})h_{a3}}{1 + \omega^2} + \frac{(1 - \omega^2)h_{a2}}{1 + \omega^2} \\ \frac{(\omega + \bar{\omega})h_{a3}}{1 + \omega^2} \\ -\frac{(\omega + \bar{\omega})h_{a2}}{1 + \omega^2} \end{bmatrix}. \tag{3.9}\]

The second term of the right-hand side of Equation (2.1) consists of the phenomenological dissipation term. This term is found by the cross product of \(\lambda \dot{m}\) and the time derivative of \(\dot{m}\).

The variable \(\lambda\) is the phenomenological constant used for mathematical purposes giving:

\[\lambda \dot{m} \times \frac{d\dot{m}}{dt} = \begin{bmatrix} i\gamma(-\bar{\omega}^2 + \omega + \dot{\omega}\omega^2 - \dot{\omega}) \\ \frac{(1 + \omega^2)^2}{(1 + \omega^2)^2}(\gamma(\omega^2 + \dot{\omega} + \dot{\omega}\omega^2)) \\ \frac{2i\gamma(-\bar{\omega}^2 + \omega\dot{\omega} + \dot{\omega}\omega^2)}{(1 + \omega^2)^2} \end{bmatrix}. \tag{3.10}\]

The third term of the right-hand side of the LL equation, (2.1), consists of the spin transfer torque. This term \(-\gamma ag(P, \dot{m} \cdot \dot{S}_P)\dot{m} \times (\dot{m} \times S_P)\) is found by the cross product of the current
density multiplied by \( \hat{m} \) with the spin polarization crossed with \( \hat{m} \). We are assuming spin polarization \( S_p \) is in the z-direction \([58]\):

\[
S_p = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

To simplify the Landau-Lifshitz equation, \( a \), which is the spin torque magnitude, is associated to the electrical current by using

\[
a \equiv \frac{\hbar A j}{2 S_o V e} \tag{3.11}
\]

where \( A \) is the area of the cross section, \( j \) is the current density, \( S_o \) is the saturation magnetization, and \( V \) is the volume of the pinned layer \([41]\) and \( g(P, \hat{m} \cdot \hat{S}_p) \) is substituted for \( 1/f(P)(3 + \hat{m} \cdot \hat{S}_p) - 4 \) where \( f(P) \) is the polarization factor. We will assume the variable \( g(P, \hat{m} \cdot \hat{S}_p) \) is a constant equal to 1 \([54, 36]\), therefore:

\[
-\gamma a \hat{m} \times (\hat{m} \times S_p) = -\gamma a \hat{m} \times (\hat{m} \times S_p) = -\gamma \left[ \begin{array}{c}
\frac{a(1 - \omega \bar{\omega})(\omega + \bar{\omega})}{(1 + |\omega|^2)^2} \\
-ia(1 - \omega \bar{\omega})(\omega - \bar{\omega}) \\
-a((\omega + \bar{\omega})^2 + (\omega - \bar{\omega})^2))
\end{array} \right].
\]

(3.12)

The equality derived in Eq. (3.4) is applied term-by-term to the right hand side of Eq. (2.1) to produce an expression for \( \dot{\omega} \). Using Eq. (3.7) in conjunction with Eq. (3.4), we take the cross product of \( \hat{m} \) with the anisotropy term:

\[
-\gamma \hat{m} \times \vec{H}_{anis} = -\gamma k m_{||} \left[ \begin{array}{c}
\frac{1 + |\omega|^2}{2} \\
\frac{i(1 + |\omega|^2)}{2} \\
-\omega(1 + |\omega|^2)
\end{array} \right] = -\gamma k m_{||} \left[ \begin{array}{c}
-\frac{i(\omega - \bar{\omega}) \cos(\theta_{||}) - (1 - \omega \bar{\omega}) \sin(\theta_{||}) \sin(\phi_{||})}{1 + |\omega|^2} \\
\frac{(1 - \omega \bar{\omega}) \sin(\theta_{||}) \cos(\phi_{||}) - (\omega + \bar{\omega}) \cos(\theta_{||})}{1 + |\omega|^2} \\
\frac{(\omega + \bar{\omega}) \sin(\theta_{||}) \sin(\phi_{||}) + i(\omega - \bar{\omega}) \sin(\theta_{||}) \cos(\phi_{||})}{1 + |\omega|^2}
\end{array} \right].
\]

(3.13)

Using the substitution of trigonometric identities \( \cos(\phi_{||}) = \frac{1}{2} \left[ e^{i\phi_{||}} + e^{-i\phi_{||}} \right] \) and

\( \sin(\phi_{||}) = \frac{1}{2i} \left[ e^{i\phi_{||}} - e^{-i\phi_{||}} \right] \) results in the product:

\[
H_{anis} = i\gamma k m_{||} \left[ \cos(\theta_{||}) \omega - \frac{1}{2} \sin(\theta_{||}) \left[ e^{i\phi_{||}} - e^{-i\phi_{||}} \right] \omega^2 \right].
\]

(3.14)
for the anisotropy term. Applying Eq. (3.4) to Eq. (3.8), we get the demagnetization term:

\[-\gamma \hat{m} \times \vec{H}_{\text{demag}} \cdot \begin{bmatrix}
\frac{1 + |\omega|^2}{2} \\
\frac{i(1 + |\omega|^2)}{2} \\
\frac{-\omega(1 + |\omega|^2)}{2}
\end{bmatrix} = \frac{1}{1 + |\omega|^2} \left[ -2i\gamma \pi S_o(2\omega - 2\omega^2\bar{\omega})N_3 - 2i\gamma \pi S_o(-\omega N_2 + \omega^2 N_2 - \omega N_1 - \bar{\omega} N_1 + \omega^2 \bar{\omega} N_1 - \omega^3 N_2 + \omega^3 N_1) \right]. \tag{3.15}\]

Applying Eq. (3.4) to Eq. (3.9) produces the following applied field term:

\[-\gamma \hat{m} \times \vec{H}_{\text{applied}} \cdot \begin{bmatrix}
\frac{1 + |\omega|^2}{2} \\
\frac{i(1 + |\omega|^2)}{2} \\
\frac{-\omega(1 + |\omega|^2)}{2}
\end{bmatrix} = i\gamma h_{a3}\omega + \frac{\gamma h_{a2}}{2} + \frac{\gamma h_{a2}\omega^2}{2}. \tag{3.16}\]

For the damping term, we apply Eq. (3.4) to Eq. (3.10):

\[\lambda \hat{m} \times \frac{d\hat{m}}{dt} \cdot \begin{bmatrix}
\frac{1 + |\omega|^2}{2} \\
\frac{i(1 + |\omega|^2)}{2} \\
\frac{-\omega(1 + |\omega|^2)}{2}
\end{bmatrix} = i\lambda \dot{\omega}. \tag{3.17}\]

Applying Eq. (3.4) to Eq. (3.12) produces the spin torque term

\[-\gamma a \hat{m} \times (\hat{m} \times \vec{S}_P) \cdot \begin{bmatrix}
\frac{1 + |\omega|^2}{2} \\
\frac{i(1 + |\omega|^2)}{2} \\
\frac{-\omega(1 + |\omega|^2)}{2}
\end{bmatrix} = -\gamma a \omega. \tag{3.18}\]

Combining terms Eq. (3.14), Eq. (3.15), Eq. (3.16), Eq. (3.17), and Eq. (3.18) results in this expression of \(\dot{\omega}\) [33]:

\[\dot{\omega} = i\lambda \dot{\omega} - \gamma a \omega + i\gamma h_{a3}\omega + \frac{\gamma h_{a2}}{2} + \frac{\gamma h_{a2}\omega^2}{2} + i\gamma \kappa m_{\parallel} \left( \cos(\theta_{\parallel}) \omega - \frac{1}{2} \sin(\theta_{\parallel}) \left[ e^{i\phi_{\parallel}} - e^{-i\phi_{\parallel}} \omega^2 \right] \right) + \frac{-2i\gamma \pi S_o}{1 + |\omega|^2} \left[ 2\omega N_3 - 2\omega^2 \bar{\omega} N_3 - \omega N_2 + \omega^2 \bar{\omega} N_1 - \omega^3 N_2 + \omega^3 N_1 \right]. \tag{3.19}\]
Collecting $\dot{\omega}$’s on the left hand side and simplifying produces an expression of the Landau-Lifshitz equation in complex stereographic coordinates [20, 38, 54]:

\[
(1 - i\lambda)\dot{\omega} = -\gamma (a - ih_{a3})\omega + \frac{\gamma h_{a2}}{2}(1 + \omega^2) + im||\kappa\gamma \left[ \cos \theta ||\omega \\
- \frac{1}{2} \sin \theta || (e^{i\phi} - \omega^2 e^{-i\phi}) \right] - \frac{i\gamma 4\pi S_o}{(1 + |\omega|^2)} \left[ N_3 (1 - |\omega|^2)\omega \\
- \frac{N_1}{2} (1 - \omega^2 - |\omega|^2)\omega - \frac{N_2}{2} (1 + \omega^2 - |\omega|^2)\omega - \frac{(N_1 - N_2)}{2} \omega \right].
\]

The result in Eq. (3.20) is consistent with the finding in “Landau-Lifshitz Equation of Ferromagnetism: Exact Treatment of the Gilbert Damping”, which states that the difference of adding the Gilbert damping term to the Landau-Lifshitz-Gilbert equation is simply a time scale of $(1 - i\lambda)$ [33]. Figure 3.2 shows the difference between plots using the LL equation in a three dimensional coordinate system labeled $M_x$, $M_y$, and $M_z$ and the Complex Stereographic Projection using complex coordinates $\omega_1$ and $\omega_2$.

![Figure 3.2](image_url)

**Figure 3.2.** Trace of free magnetic field direction of a (Left) Spherical Projection as a (Right) Complex Stereographic Projection. Source: J.A. Turtle, *Numerical exploration of the dynamics of coupled spin torque nano oscillators*, (2012).

### 3.2 Center Manifold Reduction

A system’s linear part is made up of multiple invariant subspaces known as stable ($E^S$), unstable ($E^U$), and center ($E^C$) subspaces when eigenvalues are present. As the eigenvectors are expanded in a system, the eigenvalues can either have a positive real part, a negative real part, or a zero real part. Orbits possessing eigenvalues with a negative real part...
decay to zero as time goes to infinity; therefore, stability exists within the orbit. Orbits of eigenvalues with a positive real part become unbounded as time goes to infinity and therefore, the orbit is unstable. The subspace of the linearized system containing eigenvalues equal to zero neither decay nor grow exponentially as time goes to infinity. Therefore, $E^C$, the subspace corresponding to eigenvalue’s real-parts equalling zero are analyzed to find stability for the asymptotic behavior [57]. The subspace, which both passes through the fixed point to where the system is restricted and is also invariant to $E^C$, is the center manifold.

To reduce the number of dimensions in the STNO system, the center manifold reduction is used. The system must be linearized about an equilibrium point where an expansion of power series is taken. As the system is restricted to the manifold, the behavior can be analyzed. The equations with eigenvalues with negative real parts are integrated into the rest of the system of equations reducing the total number of equations [43]. To use this method, first a Jacobian matrix in Jordan canonical form is calculated using the real parts of the LLG equation’s complex stereographic form [48]. The Jacobian of this equation about the equilibrium, where symmetry exist in the STNO system, gives the matrix of the linearized function as

$$ J = \begin{bmatrix} -\lambda h_{a3} - \mu I_{DC} & -h_{a3} + \mu I_{DC} \lambda \\ h_{a3} - \mu I_{DC} \lambda & -\lambda h_{a3} - \mu I_{DC} \end{bmatrix} $$

and the coupling term as

$$ \frac{\delta H(x_j, x_j)}{\delta x_j} = \frac{-\mu J_{DC} A_{GMR}}{N} \begin{bmatrix} 1 & -\lambda \\ \lambda & 1 \end{bmatrix} \sum_{j=1}^{N} \left( \frac{1 - r_j^2}{1 + r_j^2} \right) + \frac{4\mu J_{DC} A_{GMR}}{N(1 + r_j^2)^2} \begin{bmatrix} (x_j - \lambda y_j) x_j & (x_j - \lambda y_j) y_j \\ (y_j + \lambda x_j) x_j & (y_j + \lambda x_j) y_j \end{bmatrix} . $$

Multiplying by an invertible transition matrix $P$ gives the Jordan Canonical Form as

$$ L = P^{-1} J P = \begin{bmatrix} J + A & O_2 & O_2 & O_2 \\ O_2 & J + A & O_2 & O_2 \\ O_2 & O_2 & J + A & O_2 \\ O_2 & O_2 & O_2 & J + A + 4B \end{bmatrix} = \begin{bmatrix} J_1 \\ \vdots \\ J_2 \end{bmatrix} . $$

Two blocks are formed in this matrix, one pertaining to eigenvalues with negative real parts. The other block matrix contains eigenvalues with zero real parts, which are the eigenvalues of interest when studying dynamics for an extended time period. The system of motion can then be written as

$$ \frac{dX_j}{dt} = L X_j + f(X) $$

where $L$ is the Jacobian in Jordan Canonical Form $X_j = (x_j, y_j)$ so that $L X_j$ represents the linear part of the dynamics, and $f(x)$ the nonlinear component. The system of motion can be separated into two
equations, one corresponding to eigenvalues with negative real parts and the other corresponding to
eigenvalues with zero real parts. These equations take the form:

$$\frac{dx}{dt} = J_1 x + g_x(x, y),$$ (3.21)

$$\frac{dy}{dt} = J_2 y + g_y(x, y)$$ (3.22)

where $J_1$ represents the block of $\tilde{L}$ containing eigenvalues with no real parts and $J_2$ represents the
block of $\tilde{L}$ containing eigenvalues with a negative real part. The center manifold can be written as

$$W_c = \{(x, y) | y = h(x)\}$$ (3.23)

for small values of $x$. Since the center manifold is invariant, $y$ satisfies

$$\frac{dy}{dt} = Dh \frac{dx}{dt}.$$ (3.24)

Therefore, taking the power series of Eq. (3.23) giving

$$h(x) = a + bx^2 + cx^3 + O(x^4)$$ (3.25)

and solving for the derivative of Eq. (3.25) obtains

$$Dh = 2bx + 3cx^2 + O(x^3).$$ (3.26)

Substituting the values of Eq. (3.21), Eq. (3.22), Eq. (3.25), and Eq. (3.26) into Eq. (3.24) obtains

$$B(a + bx^2 + cx^3 + O(x^4) + g_y(x, h(x)) = (2bx + 3cx^2 + O(x^3)) [Bx + g_x(x, h(x))].$$

This equation is solved for the unknown coefficients, which form the solution of $h(x)$. These values
are then substituted back into Eq. (3.21) reducing the system. However, using the center manifold
reduction for Eq. (3.20) for any number of oscillators is too complex to solve numerically or
computationally. In order to reduce the number of dimensions in the system by restricting the system to
the center manifold, the symmetry of the system must be observed near the Hopf bifurcation points.
The general theory of Hopf bifurcation with symmetry was developed by Golubitsky, Stewart, and
Schaeffer. A system can be reduced on the center manifold based on the symmetry of the system about
a Hopf bifurcation [23, 24]. This simplifies the system about $X = 0$, where the STNO system has a
one-to-one correspondence with the reduced normal form equation containing $S_N$ symmetry for
$N \geq 4$ oscillators [16]. The system about other Hopf Bifurcations are still too complex for this
technique. To find this form, Eq. (3.20) is rewritten as

$$\omega_j = f(\omega_j) - a(\omega_1, ..., \omega_N)\omega_j \equiv g(\omega_j).$$ (3.27)
The internal dynamic function is represented as
\[
f(\omega_j) = i\gamma h_{o3}\omega_j + \frac{\gamma h_{o2}}{2}(1 + \omega_j^2) + im_j|\kappa\gamma\left[\cos\theta_j|\omega_j - \frac{1}{2}\sin\theta_j\left(e^{i\theta_j} - \omega_j^2 e^{-i\phi_j}\right)\right]
\]
\[- i\gamma 4\pi S_o \left(\frac{1}{1 + |\omega_j|^2}\right) N_3 (1 - |\omega_j|^2)\omega_j - \frac{N_1}{2} (1 - \omega_j^2 - |\omega_j|^2)\omega_j - \frac{N_2}{2} (1 + \omega_j^2 - |\omega_j|^2)\omega_j - \frac{(N_1 - N_2)}{2}\omega_j\],
where
\[
\gamma = \frac{\gamma}{1 - \lambda_i} = \gamma \left(1 + \frac{\lambda_i}{1 + \lambda^2}\right).
\]
The spin torque magnitude term in Eq. (3.11) becomes
\[
a(\omega_1, ..., \omega_N) = \left(\frac{\hbar}{2S_oVe}\right) I_{dc} \left(1 + \frac{A_{GMR}}{N} \sum_{j=1}^{N} \frac{1 - |\omega_j|^2}{1 + |\omega_j|^2}\right)
\]
after assuming the resistance, \(R_0\) and magnetoresistance, \(\Delta R\), are the same for each oscillator due to all-to-all coupling. The value of \(A_{GMR}\) is equal to \(\frac{\Delta R}{(R + R_{\text{eff}})/N}\) [52]. The spin torque term is the coupling parameter for the system, since it contains both the input DC current value, \(I_{dc}\) and the \(A_{GMR}\) term.

The initial conditions used are \(e_|| = (0, 0, 1)^T, \theta_|| = 0, \phi_|| = 0, m_|| = m\cdot e_|| = \frac{1-|\omega_j|^2}{1+|\omega_j|^2}\).
\(N_1 = N_2\), and \(h_{o2} = 0\). After substituting and simplifying Eq. (3.27) becomes:
\[
\dot{\omega}_j = \gamma h_{o3}i\omega_j + \gamma i\left[\kappa + 4\pi S_o (N_3 - N_1)\right] \left(\frac{1 - |\omega_j|^2}{1 + |\omega_j|^2}\right)\omega_j
\]
\[- \gamma \mu I_{dc} \left[1 + \frac{A_{GMR}}{N} \sum_{j=1}^{N} \frac{1 - |\omega_j|^2}{1 + |\omega_j|^2}\right]\omega_j\] \hspace{1cm} (3.28)
where
\[
\mu = \frac{\hbar}{2S_oVe}.
\]
The system has all-to-all coupling since the system of STNOs has \(S_N \times S_1\) symmetry. Therefore, the system has C-axial isotropy subgroups, \(S_N \times S_1\), which in this case are isotropy subgroups with a two-dimensional fixed-point subspace [16]. In order to derive the normal form of this equation, using the theorem in Dias et. al. [16], we have the following theorem:

**Theorem 3.1.** The action of \(S_N \times S^1\) on \(C^{N,0}\) and the function \(H_i : C^{N,0} \rightarrow C^{N,0}\), for \(i = 1, ..., N:\n\]
\[
H_i(\omega) = (h_i(\omega), h_i((12)\omega), ..., h_i((1N)\omega))
\]
where \(\omega = (\omega_1, ..., \omega_N) \in C^{N,0}\) and
\[
h_1(\omega) = |\omega_1|^2\omega_1 - \frac{1}{N}\sum_{j=1}^{N} |\omega_j|^2\omega_j, \quad h_2(\omega) = \bar{\omega}_1 \sum_{j=1}^{N} \omega_j^2, \quad h_3(\omega) = \omega_1 \sum_{j=1}^{N} |\omega_j|^2.
\] \hspace{1cm} (3.29)

Then, if \(N \geq 4\), the functions \(H_i\) for \(i = 1, 2, 3\) constitute a basis of the complex vector space of the \(S_N \times S^1\)-equivariant functions with homogeneous polynomial components of degree 3.
Since the system has $S_N$-symmetry, the equations for each oscillator are the same due to the equations being all equally dependent upon each other. To show Eq. (3.28) is $S_N \times S^1$-equivariant under the $S_N \times S^1$ action, using the exponential function as stated by Dias et al [16]:

$$(\phi, \theta)(\omega_1, ..., \omega_N) = e^{i\theta} (\omega_{\phi^{-1}(1)}, ..., \omega_{\phi^{-1}(N)}),$$

where $(\phi, \theta) \in S_N \times S^1$ and $(\omega_1, ..., \omega_N) \in C^{N,0}$. The exponential function acts as a rotation on the system. Since the system is equal to the rotated system, it contains $S_1$ symmetry thereby, it will be equivariant. This is because $S^1$ symmetry, also referred to as the circle group, has the property that the magnitude remains the same when any rotation is applied. Therefore, the system is $S_N \times S^1$-equivariant. A Taylor series expansion on Eq. (3.28) is used to find the terms represented in Eq. (3.29). The Taylor Series Expansion gives

$$\dot{\omega}_j = \tilde{\gamma}_a \phi_i \omega_j + \tilde{\gamma} i \left[ \kappa + 4\pi S_0 (N_3 - N_1) \right] \left[ 1 - 2 |\omega_j|^2 + O(4) \right] \omega_j$$

$$- \tilde{\gamma} \mu I_{dc} \left[ 1 + \frac{A_{GMR}}{N} \sum_{j=1}^{N} \left( 1 - 2 |\omega_j|^2 + O(4) \right) \right] \omega_j.$$ (3.30)

Collecting coefficients of like powers of the value $\omega_j$ for (3.30) gives:

$$\dot{\omega}_j = \left( \tilde{\gamma}_a \phi_i + \tilde{\gamma} i \left[ \kappa + 4\pi S_0 (N_3 - N_1) \right] - \tilde{\gamma} \mu I_{dc}(1 + A_{GMR}) \right) \omega_j$$

$$- \left( 2\tilde{\gamma} i \left[ \kappa + 4\pi S_0 (N_3 - N_1) \right] |\omega_j|^2 + 2\tilde{\gamma} \mu I_{dc} \frac{A_{GMR}}{N} \sum_{j=1}^{N} |\omega_j|^2 \right) \omega_j \equiv \tilde{g}(\omega_j).$$ (3.31)

The irreducible representations of $S_N \times S_1$ are $[1, ..., 1]^T$, which is the trivial case, and

$$\{(\omega_1, ..., \omega_N) \in C^N : (\omega_1 + ... + \omega_N) = 0 \}. \text{ This nontrivial irreducible representation will be the restriction on the space } C^{N,0}, \text{ the center manifold, which will be used to reduce the total number of dimensions in the system by one. Therefore, the normal equation restricting the system onto the center manifold has the form:}$$

$$\dot{\omega}_j = \tilde{g}|_{C^{N,0}} - \frac{1}{N} \sum_{k=1}^{N} \tilde{g}_k|_{C^{N,0}} \left[ \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right]_{N-1}.$$ (3.32)

Substituting Eq. (3.31) in Eq. (3.32) gives

$$\dot{\omega}_j = A \omega_j + B |\omega_j|^2 \omega_j + C (|\omega_j|^2 + ... + |\omega_j|^2) \omega_j - \frac{1}{N} \left[ A(\omega_1 + ... + \omega_N) \\ B(|\omega_1|^2 \omega_1 + ... + |\omega_N|^2 \omega_N) + C(|\omega_1|^2 + ... + |\omega_N|^2)(\omega_1 + ... + \omega_N) \right],$$ (3.33)

where constant values of each term are grouped together in the forms:

$$A = \mu(z) = \tilde{\gamma} h_a i + \tilde{\gamma} i \left[ \kappa + 4\pi S_0 (N_3 - N_1) \right] - \tilde{\gamma} \mu I_{dc}(1 + A_{GMR}),$$

$$B = 2\tilde{\gamma} i \left[ \kappa + 4\pi S_0 (N_3 - N_1) \right],$$

$$C = 2\tilde{\gamma} \mu I_{dc} \frac{A_{GMR}}{N}.$$ (3.34)
Therefore, using the restriction \[ \sum_{j=1}^{N} z_j = 0 \] to simplify Eq. (3.33) results in

\[
\dot{\omega}_j = A\omega_j + B\left[|\omega_j|^2\omega_j - \frac{1}{N}(|\omega_1|^2\omega_1 + ... + |\omega_N|^2\omega_N)\right] + C\left[|\omega_1|^2 + ... + |\omega_N|^2\right]\omega_j, \tag{3.35}
\]

where \( j \) represents the \( j \)th oscillator from 1 to \( N - 1 \). This equation can be compared directly with the normal form from Eqs. (3.34) found in Dias et al. [16]. The combination of parameters in \( A, B, \) and \( C \) are directly related to similar parameters of the normal forms. They determine the conditions for bifurcation and stability properties of the bifurcating solutions. Therefore, this general form of the STNO system can be used to form the normal equations of a system with at least four oscillators for the bifurcation point at \( I_{dc} \approx -30 \).

To simplify the complex stereographic projection equation using the center manifold reduction method, the magnetic shape parameters for the anisotropy field are set to \( N_1 = .5, N_2 = .5, \) and \( N_3 = 0 \). When \( N_1 \) and \( N_2 \) are set equal, the magnetic shape of the anisotropy field lies only in the \( z \) direction and therefore oscillations do not exist in the system. When the values are slightly skewed, having \( N_1 = .4 \) and \( N_2 = .6 \), as in Figure 3.3 with 3 oscillators, small ranges of oscillations emerge starting from \( I_{DC} = 0 \). These oscillations take similar shape to the cases of \( N_1 = 1, N_2 = 0, \) and \( N_3 = 0 \). It can be seen that for the value of \( \theta_h = 0 \), oscillations occur between 0 and -500, while for the case of \( N_1 = 1, N_2 = 0, \) and \( N_3 = 0 \), oscillations exist between the ranges of 0 to -1700. This demonstrates that the center manifold reduction can be used for parameter values of \( N_1 = N_2 \) containing small perturbations.

\[ 0 \]

\[ 0.5 \]

\[ 1 \]

\[ 1.5 \]

\[ 2 \]

\[ 2.5 \]

\[ 3 \]

\[ -2500 \]

\[ -2000 \]

\[ -1500 \]

\[ -1000 \]

\[ -500 \]

\[ 0 \]

\[ 500 \]

\[ 1000 \]

\[ 1500 \]

\[ 2000 \]

\[ 2500 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

\[ 0 \]

\[ 0.5 \]

\[ 1 \]

\[ 1.5 \]

\[ 2 \]

\[ 2.5 \]

\[ 3 \]

\[ -2500 \]

\[ -2000 \]

\[ -1500 \]

\[ -1000 \]

\[ -500 \]

\[ 0 \]

\[ 500 \]

\[ 1000 \]

\[ 1500 \]

\[ 2000 \]

\[ 2500 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

\textbf{Figure 3.3. Coherence for values of } N_1 = .4, N_2 = .6, \text{ and } N_3 = 0 \text{ for a 3 oscillator system}
3.3 **Series Circuit**

In order to generate a large power output, an \( N \) number of STNO devices are either placed in series or parallel coupled to an injected DC current and to a resistive load. For a series circuit, the STNOs are connected to one another in series, while connected in parallel to the resistive load and the injected DC current. This is seen in Figure 3.4, where \( R_1, ..., R_N \) represents each oscillator, \( I \) represents the injected input DC current, and \( R_C \) represents the resistive load. The direction of the two ferromagnetic layers are pertinent in calculating the resistance in the circuit. To calculate the

![Series circuit diagram](image)

**Figure 3.4. Series circuits for STNO configurations.**

resistance, the two extreme cases of the correlation of the magnetic directions, either a parallel or anti-parallel configuration, are used to find an average resistance of the system giving the equation:

\[
R_{0i} = \frac{(R_{APi} + R_{Pi})}{2}. \tag{3.36}
\]

Resistance in anti-parallel, \( R_{APi} \), is the amount of resistance in an STNO, when the direction of the fixed ferromagnetic layer is opposed to that of the free ferromagnetic layer. When the fixed and free ferromagnetic layers are parallel to one another, the resistance in parallel, \( R_{Pi} \), causes a minimum amount of resistance. These resistances are due to the GMR effect. The unpolarized input DC current consists of randomized spin directions, which can be assumed that half of the spin directions are spin up and half are spin down. Therefore, half of the electrons change spin directions through the first magnetic layer irrelevant to the magnetic direction. When the input DC current goes through the second layer, the free magnetic layer, the parallelized case will cause a minimum resistance, since the spin of the electrons are aligned with the magnetic direction. The antiparallel case has a maximum resistance, since the electron spin opposes the magnetic direction causing electron scattering. Figure 3.5 shows the amount of resistance as compared to the angle of difference between the two magnetic directions.
Figure 3.5. Plot of the different resistance for change in direction of magnetization between the two ferromagnetic layers.

The difference in the resistance from an extreme resistance to the average resistance is

\[ \Delta R_i = \frac{(R_{API} - R_{Pi})}{2}. \]

For a series circuit, the Complex Stereographic Projection is further derived where the value of the resistance of the \( i \)th oscillator is determined by the difference between the average resistance, \( R_{0i} \), and the difference in resistance, \( \Delta R_i \), based on the angle between the easy and fixed axis, \( \theta_i(t) \), as

\[ R_i(t) = R_{0i} - \Delta R_i \cos \theta_i(t). \] (3.37)

The input DC current divider is made up of the resistive load, \( R_C \), which is assumed to be 50Ω, the difference in resistance, and the average resistance for each oscillator resulting in

\[ \beta \Delta R_i = \frac{\Delta R_i}{(R_C + \sum_{i=1}^{N} R_{0i})}. \]

Kirchhoff’s Current Law and Ohm’s Law are then applied to calculate the input DC current in the system. Kirchhoff’s Current Law states there must be conservation of electric charge in a circuit; hence, the total input DC current into a node must equal the amount of input DC current flowing out. Ohm’s Law then states that the input DC current in a circuit is directly proportional to the potential difference between two points. Thereby, the input DC current is equal to the ratio of voltage over resistance. Using these laws produce [28]:

\[ I_j = \frac{R_c}{\sum_{i=1}^{N} R_i + R_c} I_0. \] (3.38)
Substituting Eq. (3.37) into Eq. (3.38) produces the value of $I(t)$:

$$I(t) = \frac{R_c}{R_c + \sum_{i=1}^{N} R_0} \left( \frac{\sum_{i=1}^{N} \Delta R_i \cos \theta_i(t)}{R_c + \sum_{i=1}^{N} R_0} \right).$$

This can be expanded into a Taylor series and simplified to produce:

$$I(t) = I_{DC} \left( 1 + \sum_{i=1}^{N} \beta \Delta R_i \cos \theta_i(t) \right).$$

After manipulating the equation for the value for the input DC current in the system, this equation can be substituted into the LLG equation, Eq. (2.1), resulting in [52]:

$$\frac{d\hat{m}_j}{dt} = -\gamma_0 \hat{m}_j \times \vec{H}_{eff} + \alpha \hat{m}_j \times \frac{d\hat{m}_j}{dt} + \gamma_0 I_{DC} \left[ 1 + \sum_{i=1}^{N} \beta \Delta R_i \cos \theta_i(t) \right] \hat{m}_j \times (\hat{m}_j \times \hat{S}_p).$$

For a series circuit, the input DC current varies for each oscillator because the resistance of each oscillator is dependent on the proceeding oscillators. This causes the system to be all-to-all coupling. Therefore, the series circuit is used for the exploration of the dynamics of the system to find stability and synchronization.

### 3.4 Parallel Circuit

The structure of a parallel circuit has each STNO in the circuit connected in parallel to both the resistive load and injected DC current, assumed to be 50Ω. This can be seen in Figure 3.6, where $R_1, \ldots, R_N$ represents each STNO, $I_j$ represents injected DC current in to the circuit, and $R_C$ is the resistive load. To calculate the Landau-Lifshitz equation using a parallel circuit, the resistance and input DC current is calculated. Ohm’s Law is used to calculate the input DC current giving the equation

$$I_j = \frac{R_{eqj}}{R_{eqj} + R_j} I_0,$$

where $R_{eqj}$, the equivalent resistance, is the value of the total resistance in the system. The value of $R_{eqj}$ is therefore

$$R_{eqj} = \frac{1}{\sum_{i=1, j \neq i}^{N} \frac{1}{R_i} + \frac{1}{R_e}}. \quad (3.40)$$

This value is then substituted into the coupling term, $a_j$, in Eq. (3.11). Giving the equation

$$a = \frac{hA I}{2S_0 Ve} = \frac{hA}{2S_0 Ve} \cdot \frac{1}{1 + R_j \left( \sum_{i=1, j \neq i}^{N} \frac{1}{R_{eqj}} + \frac{1}{R_e} \right)}.$$
This new coupling term is then inserted into the LLG equation giving the equation to use for the STNOs coupled in a parallel array

$$\frac{d\hat{m}}{dt} = -\gamma \hat{m} \times \vec{H}_{\text{eff}} + \lambda \hat{m} \times \frac{d\hat{m}}{dt} - \gamma a \hat{g} \left( P, \hat{m} \cdot \hat{S}_p \right) \hat{m} \times \left( \hat{m} \times \hat{S}_p \right). \quad (3.41)$$

### 3.5 Power

In order to calculate the power output in the system, Eq. (3.39) is solved for $\theta_i(t)$, the time dependent angle between the easy axis and $\hat{m}$. The Fast Fourier Transform is used to find a maximum peak of the power output for a defined frequency. As seen in Figure 3.7, where for a circuit in parallel, the average frequency of the uncoupled oscillators, shown in blue, is similar to that of the frequency of the coupled oscillators, shown in green. This is not the case for the oscillators in series, which can be seen in Figure 3.8, where the average frequency of the uncoupled oscillators is significantly less than that of the coupled oscillators. The values in power output of the coupled oscillators results in one peak, since the oscillators are synchronized causing both phase and frequency locking. This peak has a much larger power output having a difference from the uncoupled power by a value of $N^2$, where $N$ is the number of oscillators in the system. This difference ratio can be seen in both the parallel and series cases [9].
For any number of oscillators, the power ratio between the coupled and uncoupled oscillators stays consistent with the power law, increasing by $N^2$, illustrated in Figure 3.9. In this figure, the plotted values represent the power ratio measured against the number of oscillators in the system. The curve shown in green represents the calculated values. This is compared to the value of the number of oscillators, $N$, squared shown in blue. It can be seen from this plot that the value of the power output ratio is similar to the value of $N^2$ for any number of oscillators validating the use of the power law in the system. Therefore, in order to generate the largest power output, regions of synchronization for a high number of oscillators must be used.

Synchronization can either occur through phase locking or coupling of the oscillators. Phase locking is a natural occurrence in nature as seen in Josephson Junctions, rhythmic flashing of the lights of fireflies, and in oscillations of a system of two pendulum clocks coupled together. In order for a
complex dynamical system to have synchronization, the elements must exhibit a nonlinear response to the agent causing the action. The properties of phase-locking are narrowing of the signal line width and an increase of power. In STNOs, the magnetic precession, which is the source of the microwave signal, is nonlinear, thereby, natural phase-locking exists in the system [31]. However, in a complex system, phase locking has little effect. Another way for synchronization to occur in the system is through coupling between magnetic oscillations by either the use of ferromagnetic layers or by oscillators connected electrically through the input DC current [28]. This is the main cause of synchronization in the STNO system. However, as the amount of oscillators are increased, the range of synchronization decreases. Therefore, a way to determine the synchronization regions in the system for a specified number of oscillators must be found.

![Figure 3.9. Comparison of the number of oscillators squared to the Power Ratio of Coupled to Uncoupled oscillators (Left) for a parallel circuit ranging between 3 and 20 oscillators and (Right) series circuit ranging between 3 and 9 oscillators.](image)

### 3.6 Bifurcations

A bifurcation diagram occurs when a system causes a qualitative or topological change in behavior due to a change made to a parameter value. As the parameters cross through critical thresholds, changes in stability properties of equilibria, periodic orbits, and other invariant sets can be analyzed. Therefore, bifurcation diagrams help develop understanting of a system as a whole and distinguish isolated regions of synchronization [8]. A three oscillator configuration in series with applied field $\theta_h = 0$ is considered. The input current, $I_{DC}$, is varied, while all other parameters are treated as constants. As the input DC current is increased, various limit cycles appear as shown in Figure 3.10. For values of $\theta_h$ between 0 and $\pi/2$, oscillations exist in regions where the value of the input DC current is negative. The reason for this is due to the gyromagnetic ratio, in the LLG equation containing a negative value as introduced by Lakshmanan [33]. For the large and small values of input DC current, the only stable solutions found are the equilibrium points. The input DC current values
spanning between these two states consist of mainly unsynchronized regions, while synchronized regions exist near the equilibrium points [53].

Figure 3.10. Different limit cycles in series for a range of input DC current for 3 oscillators, Source: J. A. Turtle, Numerical exploration of the dynamics of coupled spin torque nano oscillators, (2012).

To view the stability and synchronization of oscillators of a $N = 2$ system for one parameter, a bifurcation diagram is created using the software package AUTO [17]. AUTO is a software package designed to numerically find equilibria, bifurcations, and limit cycles. A bifurcation diagram shows the beginning, evolution, and end of attracting or repealing sets of fixed or periodic steady state in a system [8]. The magnitude of input current, $I_{DC}$, affects the magnetic precession creating different orbital cycles, which is illustrated in Figure 3.11. These bifurcation diagrams each illustrate bifurcation diagrams in one dimension of the system, where the left diagram represents the dynamics with the initial condition of the x coordinate and the right diagram represents the dynamics with initial condition of the y coordinate. The bifurcation diagrams for $X_1$ is the same as for $X_2$ and the bifurcation diagram for $Y_1$ is the same as for $Y_2$.

The parameter change, of $I_{DC}$, can result in extreme behavior changes including high periodicities, quasiperiodicity, and synchronization in both bifurcation diagrams. For high and low values of input DC current, the magnetization direction is relaxed to an equilibrium state indicated by red lines. For low levels of input DC current, $I_{DC} \approx -1700$, two saddle node bifurcations occur. Two fixed points branch off at each of these bifurcations, a nontrivial equilibrium point indicated as a red line and an unstable equilibrium point indicated by a dashed line. As each of the nontrivial equilibrium points become unstable, back to back Hopf bifurcations form depicted as $HB_2, HB_3$ and $HB_4, HB_5$. The stable oscillations are shown as filled circles, while unstable limit cycles are represented as hollow circles. At the bifurcations $HB_2$ and $HB_4$, an out-of-phase orbit emerges indicated by blue dots. These bifurcations occur before the bifurcations $HB_1$ and $HB_3$ where in-phase orbits branch off indicated by green dots. Out-of-phase oscillations have similar waveforms and amplitudes, however, the period of the oscillators are offset by a half period. The in-phase oscillations are amplitude and
Figure 3.11. Bifurcation diagrams for $N = 2$ oscillators with parameters $\gamma = 0.0176 \text{Oe ns}$, $\gamma = 0.008$, $\theta|| = 0 \text{rad}$, $\theta_h = 0 \text{rad}$, $h_a = 300 \text{Oe}$, $\kappa = 45 \text{Oe}$, $S_o = 8400/4\pi \text{Oe}$, $N = [1, 0, 0]^T$, $R_{oi} = 0.1 \omega$, $R_c = 50 \Omega$, and $\Delta R_i = 0.03 \Omega$, Source: J.A. Turtle

phase synchronized. Symmetry breaking bifurcations are also evident, where the in-phase state preserves $S_2$ symmetry, while the out-of-phase oscillations break symmetry. When $\theta_h = 0$, the applied field is directed along the z axis causing reflectional $Z_2$ symmetry as seen in Figure 3.11 [52]. As the value of $I_{DC}$ increases, the amplitude of the limit cycles increase until they merge into a gluing bifurcation near $I_{DC} = -100$ as theorized by Gambaudo [21]. The final Hopf bifurcation occurs along the trivial equilibrium, $I_{DC} \approx -30$, and is labeled $HB_1$ [52].

In bifurcation diagrams, two unstable sets of oscillations cannot coexist next to each other. If a system started between these two sets of oscillations, it would be repelled away from one set of oscillations and therefore would have to be attracted to the other. However, this is not the case for the bifurcation diagrams represented in this thesis. Firstly, the eigenvalues are studied to observe the dynamics in each region between the orbits. The eigenvalues of the Jacobian are a way to determine stability in the local region of a system [8]. The eigenvalues for the region $I_{DC} < -1700$ are all negative, meaning stability in the oscillations. The eigenvalues for the region $I_{DC} > -1675$ are positive, meaning the oscillators are unstable. The region between these two values of $I_{DC}$ are have positive and half near zero. This means the oscillators are unstable in two dimensions and are neither stable or unstable and therefore are periodic in two dimensions. This regions has bistability which is the ability of a system to rest in two states that can be both stable or unstable.

The second reason the unstability branches can exist near each other is the system with $N = 2$ oscillators is a four dimensional system. The bifurcation diagrams is two dimensional accounting for one dimension in the system and an added dimension, the inputted DC current. Therefore, the
dynamics in the bifurcation diagrams can not be fully interpreted by one bifurcation diagram or completely analyzed on a plane. The system starting in the bifurcation diagram in the region between the unstable in-phase oscillations and the unstable equilibrium points may be attracted to the stable out-of-phase oscillations in a different dimension of the manifold.

Another way to view the dynamics in the system is by a bifurcation diagram of the norm of the system, labeled as $||Z||$, where $||Z|| = \sqrt{X_1^2 + X_2^2 + Y_1^2 + Y_2^2}$, in Figure 3.12. The same dynamics in the regions of $-1700 < I_{DC} < -1675$ between the unstable in-phase oscillation and the unstable equilibrium are the same as in the bifurcation diagrams with $X_1$ and $Y_1$. For the regions about $I_{DC} = -40$ a similar dynamic can be seen. The eigenvalues for $I_{DC} < -38$ are negative, causing stable oscillations attracting the oscillators. The eigenvalues for $I_{DC} > -38$ are all positive meaning unstable oscillations repelling oscillators. The set of oscillations occurring at this branch are mutually stable periodic. Meaning the oscillations are periodic however in multiple dimensions, these periodic orbits are stable attracting the oscillator. Similar dynamics are in the bifurcation diagrams of $\theta_h = \pi/4$ and $\theta_h = 3\pi/4$.

![Bifurcation Diagram](image)

**Figure 3.12. Bifurcation diagram for $N = 2$ oscillators with parameters as Fig. 3.11 for the Norm of the system against input current, Source: J.A. Turtle**

To further study the dynamics, an angle is added to the applied magnetic field breaking $Z_2$ symmetry, as seen in Figure 3.13, a bifurcation diagram where the value of $\theta_h$ is equal to $\pi/4$. However, even though the symmetry along $X_1 = 0$ is broken, there still exists back-to-back Hopf bifurcations where in-phase and out-of-phase oscillations branch off similar to the diagram of $\theta_h = 0$. 
The coordinates of the oscillation states approach the North Pole on the 3 dimensional sphere model until the cycles form together into one cycle, which creates a bifurcation. However, the gluing bifurcation is much more difficult to obtain without $Z_2$ symmetry and, therefore, it may be necessary to simultaneously vary two parameters.

Figure 3.13. Bifurcation diagram for N=2 with same parameters as Fig.3.11 and $\theta_h = \pi/4$, Source: J.A. Turtle

For the bifurcation diagram with the value of $\theta_h = 3\pi/4$, the value of input DC current, $I_{DC}$, must be positive for oscillations to occur as seen in Figure 3.14. For this case, it can be seen that the order of the in-phase and out-of-phase branches are swapped. This is critical, since the Hopf bifurcation of the in-phase branch occurs at higher magnitudes of $I_{DC}$. Therefore, a region of globally-stable synchronization exists as compared to the system where $\theta_h = 0$ and $\theta_h = \pi/4$. Since the direction of the applied field is not along the $z$-axis, as in the case of $\theta_h = \pi/4$, the $Z_2$ symmetry is broken causing a shift in dynamics in the system. However, a bifurcation still exists near the values of $I_{DC} \approx 100$ followed by another Hopf bifurcation at $I_{DC} \approx 30$. For $I_{DC} \lesssim 30$ and $I_{DC} \gtrsim 2275$, oscillations do not exist and only the nontrivial equilibrium points are present [52].

Creating bifurcation diagrams with AUTO is tedious and can become numerically unstable. To study the stability of synchronized limit cycles with a more robust numerical algorithm, we consider Transverse Lyaponuv Exponents (TLE) [14, 34]. The real equations of Eq. 3.20, the Complex
Stereographic Equation, after the separation of the real and imaginary parts gives [48]

$$
\begin{align*}
\dot{x}_j &= f(x_j, y_j) - ax_j + a\lambda y_j \\
\dot{y}_j &= g(x_j, y_j) - a\lambda x_j - ay_j.
\end{align*}
$$

The terms with the variable $a$ contain all of the coupling for the system. The internal dynamics of the $x_j$ and $y_j$ equations are

$$
\begin{align*}
f(x_j, y_j) &= -\gamma h a_3 x_j - h a_3 y_j - \kappa \left( \frac{1 - r_j^2}{1 + r_j^2} \right) (\lambda x_j + y_j) \\
&\quad + \frac{h a_2}{2} (1 + x_j^2 - 2\lambda x_j y_j - y_j^2) + \frac{4\pi S_0}{1 + r_j^2} (\lambda x_j^3 + 2x_j^2y_j - \lambda x_j y_j^2 - \lambda x_j) \\
g(x_j, y_j) &= h a_3 x_j - \lambda h a_3 y_j + \kappa \left( \frac{1 - r_j^2}{1 + r_j^2} \right) (x_j - \lambda y_j) \\
&\quad + \frac{h a_2}{2} (\lambda + \lambda x_j^2 + 2\lambda x_j y_j - \lambda y_j^2) + \frac{4\pi S_0}{1 + r_j^2} (-x_j^3 + 2\lambda x_j y_j + x_j y_j^2 + x_j)
\end{align*}
$$

where $r_j^2 = x_j^2 + y_j^2$. Looking at small perturbations in the synchronization manifold where $u_1 = u_2$, the transverse coordinates $x_\perp = u_1 - u_2$ is used, where $u_1$ and $u_2$ represent the real equations for each oscillator in a system of 2 oscillators. This begins the calculation for the TLEs in the basin of attraction.
for the synchronized solution. This substitution gives

\[ \dot{x}_\perp = f(x_1, y_1) - f(x_2, y_2) - ax_\perp + a\lambda x_\perp, \]

\[ \dot{x}_\perp = g(x_1, y_1) - g(x_2, y_2) - ax_\perp - ax_\perp. \]

Neglecting higher order terms, the process of expanding the first function about the second coordinate is used for determining the values of \( f, g, \) and \( a. \) This causes the linearization of Eq. (3.42) to be transverse to the synchronized manifold in matrix form

\[ \dot{x}_\perp = (J + K)x_\perp, \]

where \( J \) is the Jacobian matrix of \( F = (f(x, y), g(x, y)) \) evaluated at the synchronization manifold and \( K \) is the matrix resulting from the linearization of the coupling terms. Therefore, the synchronization state is said to be stable if \( x_\perp \to 0 \) as \( t \to \infty \) or if the TLEs are negative.

As seen in Figure 3.15, where the top figure, represents the parameter \( \theta_h = 0, \) the system is unstable for input DC current between -1700 and -1660. This region is the area between Hopf bifurcations, where only the out-of-phase branches exist illustrated in the bifurcation diagram in Figure 3.11. As the input DC current increases, the value of the sum of the TLEs decrease increasing in stability until reaching the gluing bifurcation near \( I_{DC} = -100 \) where a dip occurs. This is caused by the coordinates in the system nearing the South Pole for the spherical 3 dimensional model. The coordinates approaching the South Pole are projected off to infinity onto the Complex Stereographic Projection. The system then becomes less stable until the Hopf bifurcation near \( I_{DC} \approx -30. \) Above this input DC current value are nontrivial equilibrium states. Figure 3.15, the bottom figure, illustrates the stability of the system when the applied magnetic field has an angle of \( 3\pi/4. \) Due to this shift, oscillations in the system only occur at positive input DC current values as in the bifurcation diagram 3.14. Since the order of the stable in-phase and unstable out-of-phase oscillations are opposite of the order of the branches in the bifurcation diagrams of \( \theta_h = 0 \) and \( \theta_h = \pi/4, \) stability exists throughout the system. This is due to the stable in-phase oscillations occurring at a higher magnitude of DC current than the out-of-phase oscillations. Near the input DC current value \( I_{DC} = 150, \) a bifurcation appears causing a dip in stability. This is similar to the case in the system \( \theta_h = 0 \) [48].
Figure 3.15. TLE plots for stability for $N = 2$ for (top) $\theta_h = 0$ and (bottom) $\theta_h = 3\pi/4$
CHAPTER 4
SYNCHRONIZATION

The idea of calculating the synchronization of a complex system of oscillators was first thought of by Norbert Wiener in 1958. However, it was Art Winfree who first created a model for the calculation of synchronization of interacting limit-cycle oscillators [50]. For his model to work, the oscillators had to be practically identical and have weak coupling simplifying the problem. The coherence was therefore determined by the phase which was able to change due to the weak coupling and changes in frequency based on the interaction of other oscillators [19]. Winfree’s work inspired Yoshiki Kuramoto, who started on the problem in 1975 and went another step farther to create the Kuramoto Model. In this model, the phase is calculated by coupling the oscillators and adding an input frequency. This allow one to see the change in the phase throughout time for each oscillator. This is similar to the coupled Equations (3.42) where the phase is calculated by determining the angle between a reference point and the complex coordinate. In order to calculate the coherence, a fourth order Runge Kutta method is implemented to solve the LLG equation for the value of the magnetic direction for each oscillator in the system. These values are then used to calculate the phase of each oscillator and insert these values into the order parameter equation to determine synchronization. The order parameter is given as [25]

\[ r(t) e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}, \]

where \( \psi(t) \) is the average phase, \( \theta_j(t) \) is the angle between the magnetic direction, \( \hat{m} \), of each oscillator compared to the easy axis throughout time, and \( r(t) \) is the measure of the phase coherence.

One can visualize a dot representing each oscillator moving about a unit circle. When all of the dots are moving at the same speed at the same position, then they are synchronized and as the dots are equally spaced, they are asynchronized. If it is assumed that each value of \( \theta_j \) is a point along a circle, then the values of \( \theta_j \) are the same and the value of \( \psi \) will be the same as the total average of \( \theta \). This would mean \( r = 1 \) and the oscillators are synchronized. If the values of \( \theta_j \) are equally spaced apart along the circle, the value on the right hand side, the average value of the complex term \( e^{i\theta_j} \), would equal zero, where dividing by the complex term, \( e^{i\psi} \), would cause \( r = 0 \), meaning that the oscillators are out-of-phase [50].

4.1 Coherence Through Time

There are a variety of ways oscillators can be synchronized. This includes complete synchronization, where all oscillators are identical, lag synchronization, where oscillators are identical except for a lapse in time, and phase synchronization, where oscillators are entrained in phase but not in amplitude [45]. STNOs contain phase synchronization, since the oscillators need to have phases that
are the same and therefore generate frequency locking to generate a significant increase in power output [41]. For calculating phase coherence, the oscillators are connected in series, since there is all-to-all coupling in the system. In this case, each oscillator is dependent on the others, which increases the probability of synchronization after a period of time [41]. Due to phase transitions, the coherence in oscillators will vary throughout time before reaching a steady state [19]. Figure 4.1 is the measure of coherence with respect to time for a system of two oscillators for the angle of the applied magnetic field direction equal to $3\pi/4$. For the system of STNO oscillators, the synchronization of the oscillators through time is independent of the initial period of the oscillators. However, for a majority of input DC current values, the system of oscillators is not synchronized. Since the coherence value is time dependent, the coherence theoretically is determined as time goes to infinity to ensure a consistent coherence value. However, for computational purposes, the coherence parameter is determined when a value is consistent within $\pm 1e^{-5}$ for a time length of 200.

![Figure 4.1](image)

**Figure 4.1.** Coherence of $N = 2$, $I_{DC} = -2300$, $\theta_h = 0$, and $0 \leq t \leq 800$.

### 4.2 Coherence Through DC Current

The phase synchronization is based on both initial conditions and the coupling parameter [45]. The coherence is dependent on the value of the coupling parameter in the LLG equation. As the coupling increases, the regions of synchronization increases. However, as the value of the coupling parameter is greater than the critical level, the oscillators are forced into a state of synchronization becoming unstable. The injected DC current, $I_{DC}$, which is one of the coupling parameters, is varied in the system to find a coupling parameter at which synchronization occurs [50].

In order to evaluate the coherence for a set of values of input DC current for a specific angle of the applied magnetic direction, $\theta_h$, the program is set to contain a loop to calculate the value of the coherence parameter for each value of input DC current and storing the result. For each input DC
current value, the coherence parameter is calculated at each time step, which is increased by 10 until the variance of the coherence parameter is less than $\pm 10^{-5}$ for a length of 200. The coherence plot when no angle is implemented to the applied magnetic field, as seen in the left of Figure 4.2, contains the values of the coherence with varying values of input DC current. As mentioned earlier, the value of $r = 1$ means the oscillations are in-phase and for $r = 0$ means the oscillations are out-of-phase. For values beyond the equilibrium points, near $I_{DC} = -20$ and $I_{DC} = -1700$, oscillations do not exist. These values are equal to 1 because all the oscillators have a phase of 0; and therefore, all the oscillators are equal. The only values of input DC current, where oscillations are formed for the case of $\theta_h = 0$, are negative. As the input DC current is varied, the value of the coherence parameter alters repeatedly between these two phases. This shows that the oscillators are easily pushed in and out of synchronization based on a small change in initial conditions. As seen in the bifurcation diagrams, the basins of attraction for in-phase oscillators and out-of-phase oscillators are close together causing the oscillators to easily change from one basin of attraction to the other. A small change in the applied field can alter the region of synchronization in the system, as seen on the right of Figure 4.2, a coherence plot where the parameter value $\theta_h = \pi/4$. The oscillators have less fluctuation from the synchronized state causing the basin of attraction of the in-phase state to be greater.

![Figure 4.2](image)

**Figure 4.2.** Coherence parameter of a system where (Left) $\theta_h = 0$ for input DC current from -1800 to 0 and (Right) $\theta_h = \pi/4$ for input DC current from -1500 to 0.

The coherence parameter case of $\theta_h = 3\pi/4$ using the two oscillators can be seen in Figure 4.3. The values of the input DC current range from 0 to 2300. The oscillations for this parameter exist only for positive input DC current values. The region of synchronization is greater for the parameter of $\theta_h = 3\pi/4$, because the oscillations of the out-of-phase and in-phase oscillations are switched. For the input DC current values between 0 and 200, the coherence parameter fluctuates because of the changes in stability surrounding the gluing bifurcation and the Hopf bifurcation. For the values between 1800 and 2275, the in-phase oscillations are stable, while the out-of-phase oscillations are unstable for positive initial conditions. However, for negative initial conditions, the stability is swapped. Therefore, since the basin of attraction for the in-phase and out-of-phase exist at these points, the coherence parameter fluctuates between synchronization and asynchronization.
4.3 PARALLEL PROGRAMMING

We now investigate the changes in synchronization when two parameters, input DC current and applied magnetic field, change. For a two parameter system, the region examined ranges for the input DC current from -2400 to 2400 and for the applied magnet angle from 0 to $\pi$. Figure 4.4 demonstrates the coherence of this case of varying input DC current and applied magnetic angle with $N = 2$. The values of the applied magnetic angle from $\pi$ to $2\pi$ are symmetric to these values since these angles represent the same degrees from the easy axis except with a reversed current. Figure 4.4 demonstrates the coherence of a system by varying the input DC current and the applied magnetic angle with initial conditions of $N = 2$ and the initial conditions used for the bifurcation diagrams. The areas indicated in red are synchronized, the regions indicated in green are asynchronized, and the regions indicated in blue are regions where no oscillations exist. The coherence values that have not stabilized to completely out-of-phase or synchronized states based on a large enough time scale are the regions indicated in yellow. The areas of synchronization are between the Hopf bifurcations of the in-phase and out-of-phase oscillations near $I_{DC} = -1700$ for $\theta_h = 0$, $I_{DC} = -2250$ for $\theta_h = \pi/4$, and $\theta_h = 2275$ for $\theta_h = 3\pi/4$ similar to the synchronized regions on the bifurcation diagram. The highest regions of synchronization are in the areas about the values of $\theta_h = 3\pi/4$ and along the maximum current, $I_{DC} = -2300$ and 2300, neighboring the equilibrium points. For negative values of input DC current, oscillations exists in the regions of $\theta_h < \pi/2$, while for $\theta_h > \pi/2$, oscillations exists for positive values of input DC current.

As the number of oscillators increase, the region of synchronization decreases. As Figure 4.5 shows, the synchronization plots have the same initial conditions as Figure 4.4, except $N$ is increased to three oscillators for the figure on the left and $N$ is increased to four oscillators for the figure on the right. The synchronization regions for $N = 5$ is smaller than those of $N = 3$. These plots also show...
Figure 4.4. Figure of the coherence parameter for values of input DC current from -2400 to 2400 and values of $\theta_h$ from 0 to $\pi$ for N=2 oscillators. The value of 1 represents synchronized regions, value of 0 represents out-of-phase oscillations, and blue regions are where no oscillations exists.

that the oscillations change repeatedly between synchronization and asynchronization. This is related back to the bifurcation diagram Figure 3.11, proving an easy transition between the two basins of attraction between the synchronized and asynchronized oscillations. However, the areas of synchronization are those close to the equilibrium and close to $\theta_h=\pi/2$, similar to the synchronized regions of the system $N = 2$.

To have a better time efficiency to calculate the coherence for a large data set a parallelized code is implemented. In order to parallelize the code, the varying values of the DC current were split into subgroups to be run on different processors. Each subgroup contained loops of both values of the DC current and the angle of the applied field direction from the easy axis. The calculated values for the coherence on each processor were combined together and then outputted to a data file. The Dulcinea cluster from the Computational Science Department was used for this process. Each node consisted of 8 physical cores, 10 of which were Dual Intel E5520 with a CPU of 2.27 GHz and 24 GB of RAM and 2 of which were Dual Intel E5620 with a CPU of 2.40 GHz with 48 GB of RAM. In order to find the fastest runs, the serial code for $N = 2$ was run on both types of processors. The difference in computing time can be seen in Table 4.1. The code ran on these processors contained a data set of 29,760 variables. The nodes with the most memory and highest CPU rate had a faster computing time. However, since 10 of the nodes have 24 GB and only 2 have 48 GB, the following runs of the parallel coherence program were run on the 24 GB nodes to have consistent timings.
Figure 4.5. Figure of the coherence parameter for values of input DC current from -2400 to 2400 and values of $\theta_h$ from 0 to $\pi$ for (top) three oscillators and (bottom) five oscillators. The value of 1 represents synchronized regions, value of 0 represents out-of-phase oscillations, and blue regions are where no oscillations exists.

To analyze the performance of a parallel program, it is compared to the computation time of the serial program. The most widely used methods of performance analysis are speedup and efficiency. Speedup, $S(n,p)$, where $n$ is the size of the data set and $p$ is the number of processors, is the ratio of the run time of the serial program to the run time of the parallel program. There is much ambiguity in this ratio for which serial code is being compared, either the most efficient serial program or the parallel program run on one processor. For this case, the parallel program for calculating coherence was run on one processor. The value of the speedup ranges between $0 \leq S(n,p) \leq p$. When the value
Table 4.1. A table listing the time measurement of serial runs for the coherence calculation

<table>
<thead>
<tr>
<th>Ram</th>
<th>CPU</th>
<th>Time (sec)</th>
<th>Ratio ((T_{48GB}/T_{24GB}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 GB</td>
<td>2.27 GHz</td>
<td>119244.055929</td>
<td>-</td>
</tr>
<tr>
<td>48 GB</td>
<td>2.40 GHz</td>
<td>113750.195457</td>
<td>.953928</td>
</tr>
</tbody>
</table>

of the speedup is equal to the number of processors, the program is optimized and is known as a linear speedup. In Table 4.2, the value of each execution time and speedup is listed along with the number of processors used to parallelize the code. The actual speedup values deteriorate as compared to the linear speedup values as the number of oscillators increase. This is caused by the cost of using the call MPI_Gather and the cost of transferring information between nodes. The value of the efficiency is the ratio of the speedup to the number of processors

\[ E(n,p) = \frac{S(n,p)}{p}. \]

This calculates the utilization in a parallel program in reference to the serial program. This value should be between zero and one, since the maximum value of the speedup is \( p \). When \( E(n,p) < 1/p \), then the program exhibits slow down, where the run time of the parallel program is slower than the serial program [40]. However, this is not the case, as shown in Table 4.2. In this table the parallel program was run on processors with 24 GB of RAM and a CPU clock rate of 2.27 GHz. The data size reflects the amount of data calculated on each processor. The values of efficiency for the different number of processors are all fairly close to 1.

Table 4.2. A table demonstrating the performance of the parallel program for a variety of different processors used.

<table>
<thead>
<tr>
<th>Num. of Processors</th>
<th>Data size</th>
<th>Time (sec)</th>
<th>Speedup</th>
<th>Linear Speedup</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31 × 960</td>
<td>128779.364420</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>31 × 480</td>
<td>71364.929546</td>
<td>1.8045</td>
<td>2</td>
<td>.9022</td>
</tr>
<tr>
<td>4</td>
<td>31 × 240</td>
<td>36529.270391</td>
<td>3.5254</td>
<td>4</td>
<td>.8813</td>
</tr>
<tr>
<td>8</td>
<td>31 × 120</td>
<td>19207.198023</td>
<td>6.7047</td>
<td>8</td>
<td>.8381</td>
</tr>
<tr>
<td>16</td>
<td>31 × 60</td>
<td>9373.621007</td>
<td>13.7385</td>
<td>16</td>
<td>.8587</td>
</tr>
<tr>
<td>32</td>
<td>31 × 30</td>
<td>4703.392363</td>
<td>27.3801</td>
<td>32</td>
<td>.8556</td>
</tr>
</tbody>
</table>

According to the Amdahl law, it is very difficult to obtain a speed up value equal to the number of processors, and having an efficiency equal to 1, because each program, in terms of run time, has a fraction that can not be parallelized and must be executed. Another important indicator of calculating efficiency is the execution cost representing the total processor time used to solve the problem. The elements representing this type of cost are load imbalance and communication between processors. The load imbalance is generated by the unbalance of tasks that are assigned to different processors. In the case of the coherence calculations, the processors calculating coherence of fixed point regions, will
finish faster than other processors and will wait in an idle state until the other tasks are complete. The last task for a parallel program is for the processors to communicate with each other to obtain a final result on one processor. Therefore, a large amount of data on each processor may increase the execution cost by gathering the data on one processor [7].
CHAPTER 5

CONCLUSION

Microwave signals have a wide range of applications including cellular communications, GPS, audio and video broadcasting, satellite communications, radar, and electronic warfare. A microwave signal is produced using a STNO device consisting of layers of ferromagnetic and nonmagnetic material, which exploits the GMR effect and STT. The GMR is a fluctuation in resistance caused by electron switching due to the varying magnetic direction of distinct ferromagnetic layers. The STT uses an applied magnetic field to alter the polarized current with a fixed spin direction, which aligns with the direction of the free ferromagnetic material. The applied force causes precession in the electron, due to conservation of angular momentum, and therefore causing a precession in the free magnetic layer.

The power output of one STNO is about 1nW, which is not enough power for use in applications. For practical purposes, multiple oscillators must be connected together either in series or parallel within a device. When these oscillators are coupled together, the power output of the coupled oscillators is equal to the number of oscillators squared times greater than the uncoupled oscillators. Therefore, regions of synchronization are pertinent for discerning the maximum amount of power output for the system. Since the series circuit contains all-to-all coupling, the series case is used for the evaluation of the dynamics of the system.

The Landau-Lifshitz-Gilbert-Slonzewski-Berger equation is used in order to model the magnetic precession of the free ferromagnetic layer creating microwave signals where the Gilbert term adds damping to the system and the Slonczewski-Berger term takes into consideration the spin torque. The equation is simplified from a three dimensional spherical equation to a two dimensional complex stereographic equation, where the precession is projected onto a complex plane. The $2 \times N$ dimensional system can be reduced further by projecting the system onto the center manifold. However, the equations are too complex to apply the center manifold reduction method computationally, therefore a technique using the irreducible representation, where the summation of all the oscillators equal zero, is used. Once the equation has been simplified by reducing the amount of dimensions, the dynamics of the system can be explored to find synchronization regions.

One approach to determine synchronization in the oscillators is through a bifurcation diagram, which is produced using the program AUTO. The bifurcation diagrams show the stability and synchronization in the oscillations through different values of the injected DC current and initial conditions. For the angle of the applied magnetic direction referenced to the easy axis equal to 0 and $\pi/4$, stable synchronized and unstable asynchronized branches exist in the negative realm of DC current. While for the applied angle equal to $3\pi/4$, these branches, existing in the positive realm of DC current, are switched causing more synchronization in the system. Another way to look at the stability is through Transverse Lyaponuv Exponents.
To find the regions of synchronization in the system the order parameter developed by Kuramoto is used. The order parameter compares the phases of each oscillator in the system to determine if they are the same. When the oscillators are synchronized the coherence parameter is one and when asynchronous is zero. This technique has shown that the largest regions of synchronization occurs at the angle of the applied field direction equal to $3\pi/4$, supporting the results from the bifurcation diagrams. The values of input DC current, where the highest probability of synchronization occurs is for values near the nontrivial stable points. This is where the coupling parameter contained in the LLG equation approaches the critical point. However, as the number of oscillators increase in the system, these regions of synchronization shrink.

Further work would include adding a time delay, addition of white noise to the system, and the exploration of the symmetry in the oscillators. In simulations created by Persson et al. [41] a time delay at a peak of 10$\text{ps}$, which is a phase shift of about 90°, causes an increase in synchronization in a system of two STNOs. A pair of electrically coupled STNOs have larger regions of synchronization when allowed to lock to their self-generated ac with a phase difference of 90°. In order to create more realistic simulations, white noise can be added to the system. White noise corresponds to a random signal within any frequency band of a fixed width, which can account for a variety of different disturbances in the system [37]. In order to simplify the system and find new ways to explore synchronization regions, we can explore the symmetry of the coupled system. This would include finding a way to simplify the LLG equation in order to apply the center manifold reduction on equilibrium points not focused at the origin. This would help gain a better understanding of the system and be able to apply other computational techniques. For computational purposes, a quicker way to calculate the Transfer Lyaponuv Exponents and Order Parameter is parallelizing on a GPU.
BIBLIOGRAPHY


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[40] P. S. Pacheco, Parallel programming with MPI, Morgan Kaufmann, 1997.


