CLOSED-FORM MOMENT DISTRIBUTION SOLUTION FOR A
SIMPLE FRAME

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To Albert Schrimsher, your legacy will continue to push me to be a better engineer and a better person, I miss you.
ABSTRACT OF THE THESIS

Closed-Form Moment Distribution Solution for a Simple Frame
by
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Moment Distribution is a structural analysis method published by Hardy Cross in the 1930’s and is based on the principle that member-end-moments in a joint in static equilibrium sum to zero. The method is typically taught in undergraduate structural analysis courses and is recognizable to most civil engineers for its use of distribution factors derived from the relative rotational stiffness of the members framing into a joint, and the distribution of moments from joint-to-joint. Before the advent of computer-aided structural analysis, moment distribution was the dominant method for solving complex, indeterminate beam and frame systems due to its use of simple calculations, which can be solved without the use of simultaneous equations.

In this thesis, a closed-form solution for finding the final member-end-moments of a simple frame is derived using equations specific to each joint, without distribution of moments or solving simultaneous equations. This thesis is intended as the first step to closed-form analysis of two and three dimensional frame systems commonly found in civil structures.
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CHAPTER 1

INTRODUCTION

Both the Direct Stiffness Method and Moment Distribution present effective methods of structural analysis of frame structures. However, their adoption into structural engineering followed very different arcs. Due to its ease of calculations, which rely primarily on basic multiplication and addition allowing easy hand calculations, moment distribution was the favored method of structural analysis for 50 years from its advent in the early 1930’s through the proliferation of mainframe computing in the early ‘80’s. Meanwhile, the simultaneous equations required by matrix calculations made the direct stiffness method unwieldy for hand calculations and, therefore, less preferred for analysis of civil structures, even while it was favored for aerospace analysis. With advancements in the power of computer calculations, large matrix computation ceased to be a significant hurdle to the adoption of the direct stiffness method. Meanwhile difficulty programming distribution cycles and added complexities introduced by sway, axial deformations and shear deformations reduced the appeal of moment distribution for computer applications. The limited use of moment distribution in computer applications has relegated it, primarily, to a simple method for hand-checking computer analysis results or for performing analysis when a computer is not available.

Since its publication there have been many attempts to develop a closed-form, or one-step, solution for moment distribution. Differentiated by its ability to find final member-end-moments at any joint in a continuous beam or bridge frame is the method developed by Robert Dowell (2009), which is capable of producing exact member-end-moments at any joint for a structure with any number of spans. Dr. Dowell’s method has been further developed to capture more advanced behavior such as shear deformations (Dowell and Johnson 2011) and, in its incremental form, non-linear seismic bridge analysis (Dowell 2009). Although these methods are often unwieldy for hand calculations, they significantly reduce the time necessary for computer calculations, allowing non-linear time history
earthquake analysis to be completed in a fraction of a second rather than hours using the direct stiffness method yielding results that are also more stable.

In spite of its obvious advantages, the current closed-form method yields a specific limitation, its confinement to continuous beams and bridge-type frames that have only one level. This thesis presents a method to expand the closed-form method to a simple two-dimensional, sway-inhibited, four-sided frame. This frame can represent a two-story building with one bay. Derived directly from moment distribution and the existing closed-form equations it enables calculation of exact member-end-moments at any of the joints.

### 1.1 Moment Distribution Review

In moment distribution, members that come together at joints are given distribution factors based on their relative rotational stiffness which is found from member length and end fixity, resulting in $\frac{4EI}{L}$ when the opposite member end is fixed and $\frac{3EI}{L}$ for members with pin or roller opposite end conditions. Distribution factors are always less than one and sum to one at each joint. All internal joints are treated as fixed from rotation during the calculation of distribution factors and fixed-end-moments. Applied loads on vertical or horizontal members cause fixed-end-moments to develop for the loaded member and the sum of these fixed-end-moments at a joint where the members connect typically will not be zero. The resultant is called an out-of-balance moment. Out-of-balance moments are applied to one or more joints, originating either from applied loads to members or from moments applied directly to joints. The out-of-balance moment is multiplied by -1 to reverse its sign, and then multiplied by each distribution factor in the joint. This yields a total value at the joint equal to the out-of-balance moment and opposite in sign. When the two moments are summed, the total moment at the joint equals zero, thereby satisfying joint equilibrium. One-half of the developed moments at the joint are then distributed to the opposite end of the member. These carry-over moments then contribute to the out-of-balance moments at those joints. The process is repeated with the values of the out-of-balance moments and incremental member moments reducing in each cycle, until a desired level of accuracy is achieved. Final member-end-moments are then found by summing all incremental moments by superposition.
1.2 Scope

This thesis introduces the first step towards the goal of multistory frame analysis by the addition of a second story and the production of the closed-form equations for the exact solution of member-end moments caused by flexural deformations in a simple frame structure. The frame is composed of a lower beam and an upper beam joined by two columns and does not include sway, shear deformation or axial deformation in the analysis.

1.3 Organization

The thesis is organized into seven chapters and two appendices. Chapter One covers the background and goals of the thesis. A literature review, recapping the available material which informed the direction and provided the foundation of the thesis, is found in Chapter Two. Chapter Three covers the physical properties of the frame system used in the thesis as well as the naming conventions used throughout the thesis. The derivation of the closed-form equations for the solution of the member-end-moments in the frame is the subject of Chapter Four. Chapter Five provides practical applications for the thesis as it is currently composed and provides potential areas for developing the thesis in the future. Three examples are listed in Chapter Six which demonstrate the accuracy of the closed-form method developed in Chapter Four. Chapter Seven is a conclusion which recaps the findings of the thesis as well as providing a brief “road map” for the continuation of the derivations contained in the thesis.

Following the seven primary chapters are two shorter appendices. Appendix A provides a derivation for a generalized version of the outgoing moment factor. Finally Appendix B demonstrates the use of symmetry to obtain the closed-form equations for moments applied to different joints. A list of the equations not provided in the previous chapters may be found at the end of Appendix B.
CHAPTER 2

LITERATURE REVIEW

Moment Distribution was first laid out in a paper by Hardy Cross in the Transactions of the ASCE 1932 (Cross 1932) explaining the basic moment distribution method for calculating flexural deformations and its basis in the static equilibrium of joints.

The initial closed-form method by Robert K. Dowell was first published in the journal Engineering Structures in 2009 (Dowell 2009). The publication lays out the closed-form equations necessary to find final member-end-moments produced by flexural deformations without distributing moments or solving simultaneous equations.

The closed-form method was further modified in 2011 by Robert K. Dowell and Timothy P. Johnson to incorporate shear and bending flexibility. In 2012 the capability to determine shear flow in cell-based cross-sections was also added (Dowell and Johnson 2012).

In 2012, nonlinear, incremental earthquake analysis based on the closed-form moment distribution method was introduced (Dowell 2012). This method is both faster and more stable than available direct stiffness method techniques.

Moment distribution allowed analysis of larger and more redundant structures more readily than any other method available at its time. The advances made to the closed-form moment distribution method to date have made it a very efficient system for quickly analyzing single-story bridge structures in all conditions of loading. Expansion to multistory frame systems, as proposed in this thesis, would make the closed-form moment distribution method applicable to buildings as well as bridges.

Just as the current closed-form moment distribution method has increased its capability by incrementally solving analysis problems of increasing complexity, such as analyzing shear deformations, torsional deformations, and seismic loading, the application of closed-form moment distribution to multistory frame structures must be approached in an incremental manner. This thesis introduces the first step towards the goal of multistory frame analysis by the addition of a second story and the production of the closed-form equations for
the exact solution of member-end moments in a simple frame structure composed of a lower beam and an upper beam joined by two columns.
CHAPTER 3

DEVELOPMENT OF THE FRAME MODEL

3.1 BUILDING FRAME

Traditional moment distribution begins with the calculation of distribution factors, which are based on the relative rotational stiffness of members connected at a joint. Since the current theory is intended to be applicable to all frame systems, regardless of member dimensions, the distribution factors are replaced with generic coefficients that are used during the derivation and replaced with the distribution factor values applicable to a specific frame for analysis. It is also necessary to label each joint in the frame to differentiate the coefficients to account for possible variations in the dimensions of specific frames to be analyzed. For instance, differing floor heights between levels or differing widths of bays would change the distribution factors. The naming convention as applied to an example frame may be seen in Figure 3.1.

![Figure 3.1. Joint and coefficient labeling system for a sample frame.](image)

To maintain continuity with the previous closed-form moment distribution method (Dowell 2009), the left and right coefficients for the distribution factors, “r” and “t”, were retained. However, it was necessary to add coefficients to account for the previously non-existent columns. The coefficients chosen were “c” for “column”, applied to the upper
vertical coefficient of each joint, and “f” for “foundation”, applied to the lower vertical coefficient of each joint. (Figure 3.2)

![Diagram of a joint with distribution variables](image)

Figure 3.2. Example joint with distribution variables.

To differentiate between joints, it was necessary to adopt a numbering format. Due to its familiarity to mathematicians and therefore engineers, the matrix numbering system was chosen. Matrix numbering is a two-digit system consisting of a row (or in this case a story) number followed by a column (in this case a bay) number, beginning with the lower left joint on the first floor and ending with the upper-most joint on the right. For instance, Joint 12 would refer to a joint located on the first floor and the second bay of a building (Figure 3.1). A further benefit of this numbering system is the ease with which it can be extended to a third dimension, i.e., the long term goal of the closed-form moment distribution method outlined in this thesis.

With the joint numbering system and the distribution factor coefficient labeling system in place, it is now possible to apply an “address” to each distribution factor in a frame. By applying the joint number in subscript to a distribution factor coefficient a specific distribution factor can now be referenced. For instance, the column framing into the bottom of Joint 12 would have the coefficient “$f_{12}$” (Figure 3.1). Note: See Section 6.3 for the application of the coefficient system to a frame with defined dimensions.

To make the derivation manageable, the frame needed to be reduced to its most basic constituent element, a single-bay, single-story element (Figure 3.3). Since this thesis deals only with flexural deformations, the element is braced against both vertical and horizontal displacement by a pin and against side-sway by a roller. This configuration allows the frame
to be approached with the basic form of moment distribution outlined by Cross in his original publication (Cross 1932).

### 3.2 The Simple Frame

Using the extracted element (Figure 3.3) and following the joint numbering system outlined in 3.1, a frame model was developed for use in the derivation. (Figure 3.4) The frame is fixed from translation by the pin at Joint 11 and fixed from sway by the roller at Joint 22. Each member has infinite axial stiffness and therefore resists any axial deformation meaning the roller at Joint 22 also restrains lateral translation of Joint 21. Additionally, all joints are fixed so the members are restricted from independent rotation. In this manner flexural deformation of the members and, by extension, the basic moment distribution method outlined by Cross (1932) can be used for the derivation of the member-end moments.
CHAPTER 4

DERIVATION OF CLOSED-FORM EQUATIONS

Moment distribution begins with all joints fixed from rotation, graphically represented by boxes around each joint and frequently referred to as a “clamp”. In Step 1a a moment is applied at one or more joints, as in Figure 4.1, either from a member load or from a moment applied directly to the joint (Figure 4.1). A unit moment is applied at Joint 21, for the purpose of this derivation.

4.1 Step 1

![Figure 4.1. Step 1a, moment applied.](image)

The next step in traditional moment distribution is to find the distribution factors at each joint. This step is unnecessary since the distribution factors are replaced with coefficients for this derivation, as outlined in 3.1.

At Step 1b, a joint with an out-of-balance moment is selected and its rotational fixity, or clamp, is released, allowing the joint to rotate freely and the moment to be distributed to the members that frame into the joint. The moment is distributed to each member depending on their distribution factors, a function of their relative rotational stiffness (Figure 4.2). A unit moment is applied to Joint 21, for this derivation.
A simplified graphic representation of this process depicts each joint with its distribution factors arranged next to their corresponding member (Figure 4.3). Representing the frame in this manner allows the moments to be distributed and their sums to be tabulated clearly without interference from the joint conditions or other structural figures.

The clamp is momentarily released and the moment distributes to the members connected to the joint. For computation, this is represented by reversing the sign of the moment and multiplying the moment by the distribution factors at the joint: one half of the moment distributes to the opposite end of each member. For more legible presentation, lines are placed beneath the balanced moments at Joint 21 and arrows to the distributed moments at Joints 11 and 22, respectively. Additionally, vertical lines, which have no physical significance, are added between the distribution factors at Joints 11 and 12 in order to separate the results at each location. At this point, the clamp on the first joint is replaced and
each adjacent joint, still fixed from rotation, holds an out-of-balance moment. In other words, the sum of the moments on all members framing into the joint do not equal zero (Figure 4.4).

![Diagram](image)

**Figure 4.4. First clamp is released and the moment is distributed.**

At Step 1c, two, non-adjacent joints are holding out-of-balance moments the clamps at both joints, Joint 22 and Joint 11 can be released simultaneously (Figure 4.5) and the out-of-balance moments, reversed in sign, can be distributed to the two members (Figure 4.6). As with the previous step, half of the moments distribute to the other end of their respective members and arrive at the adjacent joints, in this case, Joint 12 and Joint 21. The goal of this process in moment distribution is for the out-of-balance moments to approach zero. Since the distribution factors are always less than one, this can be achieved assuming care is exercised in the order of the release of the clamps (Section 4.1). The goal of this process is to produce geometric series at each side of each joint using the distribution factor coefficients defined in Section 3.1.

Following step 1c (Figure 4.5 and 4.6), out-of-balance moments collect at Joint 21 and Joint 12. This is the point at which the derivation diverges from the traditional moment distribution method. Instead of releasing the clamps from both Joint 21 and Joint 12, only the clamp at Joint 21 is released (Figure 4.7), repeating step 1a, and the out-of-balance moment is allowed to remain at Joint 12 (Figure 4.8), this keeps the geometric series at each joint manageable. This method is not efficient for reaching a solution in traditional moment distribution. However, in the closed-form derivation, the method allows the second story’s
Figure 4.5. Step 1c, clamps are released from joints holding out-of-balance moments.

Figure 4.6. Step 1c, clamps are released from Joints 22 and 11.

Figure 4.7. Step 1d, clamp is released from Joint 21.
beam and the first bay’s column to interact in a manner necessary to distribute moments between perpendicular members, a necessity in a multi-story structure.

As in step 1c, clamps are released at Joint 11 and Joint 22 and the out-of-balance moments distributed (Figure 4.9). By repeating the pattern outlined in steps 1a through 1e infinite series can be recognized at each joint. After repeating steps 1a through 1c one further time, equation 4.1, which represents the sum of all moments at position \( t_{12} \) (collected as an equation) can be written. Since the equation represents the moment at \( t_{12} \) from an applied moment at Joint 21, the equation is given the name 2112r:

\[
2112r = -\frac{f_{21}r_{11}}{4} - \frac{f_{21}c_{11}r_{11}}{16} - \frac{r_{21}t_{22}f_{21}r_{11}}{64} - \frac{f_{21}c_{11}^2r_{11}}{64} - \frac{r_{21}t_{22}f_{21}c_{11}r_{11}}{64} - \frac{r_{21}t_{22}^2f_{21}c_{11}r_{11}}{64} - \ldots
\]  

(4.1)

It is likely that seeing this series, with its readily identifiable pattern, represents the proof that a closed-form equation can be found. This revelation surely would have intrigued Hardy Cross. By combining like terms equation 4.2 is formed:

\[
2112r = -\frac{f_{21}r_{11}}{4} - \frac{f_{21}c_{11}r_{11}}{16} - \frac{r_{21}t_{22}f_{21}r_{11}}{64} - \frac{2r_{21}t_{22}f_{21}c_{11}r_{11}}{64} - \frac{r_{21}t_{22}^2f_{21}c_{11}r_{11}}{64} - \frac{r_{21}t_{22}^2f_{21}r_{11}}{64} - \ldots
\]  

(4.2)

Factoring out the first terms yields equation 4.3:

\[
2112r = -\frac{f_{21}r_{11}}{4} \left[ 1 + \frac{f_{21}c_{11}}{4} + \frac{r_{21}t_{22}}{16} + \frac{f_{21}c_{11}^2}{16} + \frac{2r_{21}t_{22}c_{11}}{16} + \frac{r_{21}t_{22}^2}{16} + \ldots \right]
\]  

(4.3)
After combining terms in the infinite series equation 4.4 is produced:

\[
2112r = -\frac{f_{21}r_{11}}{4} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} \right] \quad (4.4)
\]

Using the solution form for a geometric series from calculus (Larson, Hostetler, and Edwards 2007):

\[
\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1 \quad (4.5)
\]

The equation for the infinite series of equation 2112t can be written as:

\[
2112r = -\frac{f_{21}r_{11}}{4} \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = -\frac{f_{21}r_{11}}{4} \left( 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right) \quad (4.6)
\]

By collecting the terms of \( c_{12} \) in the same manner, the equation for a moment at \( c_{12} \) from an applied moment at 21, referred to as 2112c, is derived, beginning with the summation of terms:

\[
2112c = -\frac{r_{21}t_{22}}{4} - \frac{f_{21}c_{11}r_{21}t_{22}}{16} - \frac{r_{21}^2t_{22}f_{21}}{16} - \frac{f_{21}^2c_{11}^2r_{21}t_{22}f_{21}}{64} - \frac{r_{21}^2t_{22}^2f_{21}c_{11}f_{22}}{64} - \frac{r_{21}^2t_{22}^2f_{21}c_{11}f_{22}}{64} - \ldots \quad (4.7)
\]

By a similar method to equations 4.2, 4.3, and 4.4, the following equation is derived:
Using equation 4.5, the solution to equation 4.8 is found as:

\[ 2112c = -\frac{r_{21} f_{22}}{4} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}t_{22}^2}{16} + \ldots \right] \]  

(4.8)

Following the method outlined in steps 1a through 1e it can be observed that, as one half of the moments from \( f_{22} \) distribute to \( c_{12} \), the equation for the moments collected at \( f_{22} \) is simply two times the moments collected at \( c_{12} \) yielding equation 2122\( f \):

\[ 2122f = -\frac{r_{21} f_{22}}{2} \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = -\frac{r_{21} f_{22}}{2} \left( \frac{1}{1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4}} \right) \]  

(4.10)

In the same manner, the equation for the collected moments at \( r_{11} \) due to an applied moment at 21 can be derived:

\[ 2111r = -\frac{f_{21}r_{11}}{2} \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = -\frac{f_{21}r_{11}}{2} \left( \frac{1}{1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4}} \right) \]  

(4.11)

The summed moments at \( c_{11}, f_{21}, r_{21}, \) and \( r_{22} \) all take the form of alternating series, a geometric series in which the terms alternate between positive and negative values. Beginning with Joint 11 and moving to Joint 22 the derivation of the member-end moments adjacent to the applied moment at Joint 21 is as follows:

\[ 2112c = \frac{f_{21}c_{11}}{2} - \frac{f_{21}^2 c_{11}^2}{8} + \frac{r_{21}^2 t_{22} f_{21}}{8} - \frac{r_{21}^3 t_{22}^2 f_{21} c_{11}}{8} + \frac{f_{21}^3 c_{11}^2}{32} + \frac{2r_{21} t_{22}^2 f_{21}^2 c_{11}}{32} + \frac{r_{21}^2 t_{22}^3 f_{21} c_{11}}{32} + \frac{f_{21}^3 c_{11}^3}{32} + \frac{2r_{21}^2 t_{22}^2 f_{21}^2 c_{11}^2}{32} + \frac{r_{21}^3 t_{22}^3 f_{21}^2 c_{11}}{32} + \ldots \]  

(4.12)

Initially, the positive and negative terms must be separated. Due to the associative property, the sum of the infinite series of positive terms and the infinite series of negative terms will be the same as the series of positive and negative terms grouped together. Therefore the equation can be presented as:
\[
211lc = \left[ \frac{f_{21} + \frac{f_{21}^2}{2} + \frac{r_{21}t_{22}f_{21}}{8} + \frac{f_{21}^3c_{11}}{32} + \frac{2r_{21}t_{22}f_{21}^2c_{11}}{32} + \frac{r_{21}^2t_{22}^2f_{21}}{32}}{2} \right] + \\
\left[ \frac{f_{21}c_{11} - \frac{f_{21}^2c_{11}}{2} - \frac{r_{21}t_{22}f_{21}c_{11}}{8} - \frac{f_{21}^3c_{11}}{32} + \frac{2r_{21}t_{22}f_{21}^2c_{11}}{32} - \frac{r_{21}^2t_{22}^2f_{21}c_{11}}{32} \right] \\
\tag{4.13}
\]

By factoring and combining like terms, the equation takes the form:
\[
211lc = \frac{f_{21}}{2} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] - \\
\frac{f_{21}c_{11}}{2} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] \\
\tag{4.14}
\]

Collecting the infinite series and applying equation 4.5, equation 4.14 can be presented as:
\[
211lc = \left( \frac{f_{21}}{2} - \frac{f_{21}c_{11}}{2} \right) \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = \frac{f_{21} - f_{21}c_{11}}{2 \left( 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right)} \\
\tag{4.15}
\]

At position \( t_{22} \) we can perform the same process as was just demonstrated for position \( c_{11} \), to derive equation 2122t, beginning with the alternating series at \( t_{22} \):
\[
2122r = \frac{r_{21} - r_{21}t_{22}}{2} + \frac{f_{21}c_{11}r_{21}}{8} + \frac{r_{21}^2t_{22}}{8} - \frac{f_{21}c_{11}r_{21}t_{22}}{8} - \frac{r_{21}^2t_{22}^2}{8} + \\
\frac{f_{21}^2c_{11}r_{21}}{32} + \frac{2r_{21}t_{22}f_{21}c_{11}}{32} + \frac{r_{21}^2t_{22}^2f_{21}c_{11}}{32} - \frac{f_{21}^3c_{11}^2r_{21}}{32} - \frac{2r_{21}t_{22}f_{21}^2c_{11} + r_{21}^2t_{22}^2}{32} + \ldots \\
\tag{4.16}
\]

As in equations 4.13 and 4.14, the positive and negative terms in equation 4.16 are separated and the like terms factored out:
\[
2122r = \frac{r_{21}}{2} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] - \\
\frac{r_{21}t_{22}}{2} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] \\
\tag{4.17}
\]

In a manner similar to equation 4.15, equation 4.5 may be applied to produce:
\[
2122r = \left( \frac{r_{21}}{2} - \frac{r_{21}t_{22}}{2} \right) \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = \frac{r_{21} - r_{21}t_{22}}{2 \left( 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right)} \\
\tag{4.18}
\]

The two remaining equations that can be derived from steps 1a through 1e are 2121r and 2121f, i.e., the moments at Joint 21 from an applied moment at Joint 21. Just as equations
4.12 and 4.16, the sum of the moments collected at \( r_{21} \) and \( f_{21} \) take the form of alternating geometric series and are, therefore, solved in a similar manner. Beginning with the series at \( r_{21} \), which is the equation 2121r:

\[
2121r = r_{21} - \frac{r_{21}t_{22}}{4} + \frac{f_{21}c_{11}r_{21}}{4} - \frac{r_{21}^2t_{22}}{4} - \frac{f_{21}c_{11}r_{21}t_{22}}{16} - \frac{r_{21}^2t_{22}^2}{16} + \ldots (4.19)
\]

As in equation 4.17, the positive and negative terms in equation 4.1.19 are separated and the like terms factored out:

\[
2121r = r_{21} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] - \\
\frac{r_{21}t_{22}}{4} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] (4.20)
\]

Equation 4.5 can be applied to equation 4.20 as follows to produce 2121r:

\[
2121r = \left( r_{21} - \frac{r_{21}t_{22}}{4} \right) \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = \frac{4r_{21} - r_{21}t_{22}}{4 \left( 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right)} (4.21)
\]

Finally, in a similar manner to 2121r, 2121f can be derived from the series formed by the summation of the moments at \( f_{21} \):

\[
2121f = f_{21} - \frac{f_{21}c_{11}}{4} + \frac{f_{21}^2c_{11}^2}{4} + \frac{r_{21}t_{22}f_{21}}{4} - \frac{f_{21}^2c_{11}^2}{4} + \frac{r_{21}t_{22}^2f_{21}c_{11}}{16} - \frac{r_{21}^2t_{22}^2f_{21}c_{11}}{16} + \ldots (4.22)
\]

As in equation 4.20, the positive and negative terms are separated and the like values factored out:

\[
2121f = f_{21} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] - \\
\frac{f_{21}c_{11}}{4} \left[ 1 + \frac{f_{21}c_{11} + r_{21}t_{22}}{4} + \frac{f_{21}^2c_{11}^2 + 2r_{21}t_{22}f_{21}c_{11} + r_{21}^2t_{22}^2}{16} + \ldots \right] (4.22)
\]

Finally, by applying equation 4.5 to equation 4.22:
\[
2121f = \left( f_{21} - \frac{f_{21}c_{11}}{4} \right) \times \sum_{n=0}^{\infty} \left[ \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right]^n = \frac{4f_{21} - f_{21}c_{11}}{4\left( 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \right)} \tag{4.23}
\]

Opportunities for simplification can be found upon completion of the initial equations 2121f (equation 4.23), 2121r (equation 4.21), 2122t (equation 4.18), 2122f (equation 4.10), 2111c (equation 4.15), 2111r (equation 4.11), 2112t (equation 4.6), and 2111c (equation 4.9). All the equations share a common term that relates to the joint at which the moment was applied. Therefore this term is referred to as an “outgoing moment factor”:

\[
R_{21} = 1 - \frac{f_{21}c_{11} + r_{21}t_{22}}{4} \tag{4.24}
\]

This factor is similar in form to the cycle factors outlined in Dowell’s first closed-form moment distribution paper (Dowell 2009):

\[
R_a = 1 - \frac{r_{a-1}t_a}{4R_{a-1}} \tag{4.25}
\]

Due to this similarity, the naming convention was maintained despite the function providing a somewhat different purpose in the equations.

After simplification, the equations for a moment applied at Joint 21 with a fixed Joint 12 are as follows:

\[
2112r = \frac{f_{21}r_{11}}{4R_{21}} \tag{4.26}
\]

\[
2112c = -\frac{r_{21}f_{22}}{4R_{21}} \tag{4.27}
\]

\[
2122f = -\frac{r_{21}f_{22}}{2R_{21}} \tag{4.28}
\]

\[
2122r = \frac{r_{21} - r_{21}t_{22}}{2R_{21}} \tag{4.29}
\]

\[
2111r = \frac{f_{21}r_{11}}{2R_{21}} \tag{4.30}
\]

\[
2111c = \frac{f_{21} - f_{21}c_{11}}{2R_{21}} \tag{4.31}
\]
The outgoing moment factor can be presented in coefficient form as:

\[ R_{ab} = 1 - \frac{c_{ab} f_{(a+1)b} + r_{ab} f_{(a+1)b} + f_{ab} c_{(a-1)b} + r_{ab} r_{a(b-1)}}{4} \]  

(Appendix A) presents the derivation of the coefficient form of the outgoing moment factor equation and rules for its use.

With the coefficient form of the outgoing moment factor equation A.8 it is possible (through inspection) to formulate equations for a moment applied at Joint 12 with a fixed Joint 21, or a moment applied to Joint 22 with a fixed Joint 11, or a moment applied at Joint 11 with a fixed Joint 22, depending on the symmetry of the system. Moments applied at any one of the four joints impose moments on the three other joints. These moments are predictable based on coefficients (Appendix B for a brief derivation and list of equations). These equations are important for the next step of the derivation since they allow a direct calculation of the moments at each joint without repeating the derivation outlined in 4.1.

**4.2 Step 2**

The goal of Steps 2 through 5 is to develop an infinite series of the equations derived in Step 1. The approach chosen uses the member end moment equations developed in Section 4.1 and Appendix B, which assume a fixed joint opposite a joint with an applied unit moment. In the case of Section 4.1 these are Joint 12 and Joint 21 respectively. By (1) working from corner to corner across the frame (Figure 4.10), (2) scaling the moment equations by the out-of-balance moment collected at the fixed joint, and (3) superimposing the new values over the previous values, a geometric series composed of the infinite series derived in Section 4.1 and Appendix B is developed. Once a pattern is found (often in as little as 3 cycles) a closed form equation that can accurately represent the behavior of an infinite amount of moment distribution cycles is derived.

After the derivations in Section 4.1 the moments at Joint 12 can be calculated exactly as long as Joint 12 remains a fixed joint. However, the simple frame model described in
Section 3.2 (Figure 3.4) stipulates that Joint 12 is a roller joint and is therefore free to rotate. The free rotation of Joint 12 allows the moments to interact between the beam and column joined at the joint and return, or "bounce back", towards Joint 21. Once these "bounce back" moments are considered, the equations above (Figure 4.11) are no longer reliable and new equations must be derived to account for this behavior.

As in Step 1d, the derivation once again diverges from traditional moment distribution. At this point, there is a collected, out-of-balance moment at Joint 12, which, due to the derivation in Section 4.1, is the product of an infinite amount of releases of the clamps at Joints 21, 11, and 22. Therefore, the out-of-balance moment can be reversed in sign and applied at Joint 12.

Following the derivations in Appendix A and Appendix B, we can multiply the out-of-balance moment at Joint 12 by each of the closed-form equations in Section 4.1 for a unit moment applied at Joint 12 (Figure 4.12). The collected moments have been reversed in sign but the equations are somewhat unwieldy (Figure 4.13 and 4.14). At this point the equation for the collected moments is given an arbitrary name (i.e., a moment collected at Joint 12) \( M_{12} \):
Figure 4.11. Moment equations for applied moment at Joint 21, beginning of Step 2.

Figure 4.12 Moment equations for applied moment at Joint 12, beginning of Step 2.
Now there is an out-of-balance moment collected at Joint 21, as in Step 1 (Figure 4.15). This moment is the "bounce back" moment from an infinite amount of clamp releases on Joints 12, 22, and 11, respectively and therefore represents all of the moment distributions between those joints. As in Step 2, the out-of-balance moment is (1) collected at the fixed joint (in this case Joint 21), (2) reversed in sign and (3) distributed. To save space, an arbitrary name is substituted for the out-of-balance moment, in this case $M_{21}$:
Figure 4.15. Equations for moment applied at Joint 12 scaled by $M_{12}$.

$$M_{21} = \frac{1}{2} \left[ c_{11}t_{12} + c_{12}t_{22} \right]$$ (4.35)

This moment is multiplied by the equations for an applied moment at Joint 21 (Figure 4.16).

Figure 4.16. Result of an applied moment of $M_{21}$ at Joint 21
4.4 STEP 4

At this point the pattern mentioned at the beginning of Step 2 starts to appear as the out-of-balance moment to be applied at Joint 12 is a multiple of $M_{21}$, $M_{12}$ and a collected moment which is equal to $M_{12}$ (Figure 4.17).

\[ M_{21}M_{12} \times \frac{1}{2} \left[ \frac{f_{21}r_{11} + r_{21}f_{22}}{2R_{21}} \right] = M_{21}M_{12}^2 \]  
(4.36)

Once more, the equations for a moment applied at Joint 12 can be scaled by the out-of-balance moment and distributed (Figure 4.18).

Figure 4.17. End of Step 4 with out-of-balance moment collected at Joint 21.
4.5 STEP 5

At this stage of the derivation, a definite and predictable pattern is forming. The out
of balance moment is a multiple of \( M_{21} \), \( M_{12} \), and the constituent equation of \( M_{21} \).

\[
M_{21}M_{12}^2 \times \frac{1}{2} \left[ \frac{c_{11}f_{12} + c_{12}f_{22}}{2R_{12}} \right] = M_{21}^2 M_{12}^2 \quad (4.37)
\]

The geometric series becomes clear by scaling the equations for a moment applied at
Joint 21, from Section 4.1, and superimposing them over the previous equations (Figure
4.18).

4.6 STEP 6

The equations can now be collected, written as geometric series, and solved. An
examination of Figure 4.18 reveals that all the member-end-moment summations are
alternating geometric series. Since the derivations of the equations all follow the same
format, two sample derivations are listed below and the rest of the equations are listed after
the derivations.

As the equations presented are final member-end-moments, including bounce back
moments, the naming convention is altered. The two-digit joint number at which the moment
is applied precedes the capitalized letter and two-digit joint number of the member-end
position at which the moment is being calculated. For instance, the first equation presented is
the final member-end-moment at position \( c_{11} \) from a moment applied at Joint 21, so the
equation name reads \( 21C11 \). \( 21C11 \), written in series form with arbitrary equation
designations \( M_{21} \), equation 4.35, and \( M_{12} \), equation 4.34, is:

\[
21C11 = \frac{f_{21} - f_{21}c_{11}}{2R_{21}} - M_{12} \left[ \frac{c_{11}f_{12}}{2R_{12}} \right] + M_{21}M_{12} \left[ \frac{f_{21} - f_{21}c_{11}}{2R_{21}} \right] - M_{21}M_{12}^2 \left[ \frac{c_{11}f_{12}}{2R_{12}} \right] \ldots \quad (4.38)
\]

Using the same method applied to previous alternating series, the positives and negatives are
separated and like terms are factored out:

\[
21C11 = \frac{f_{21} - f_{21}c_{11}}{2R_{21}} \left[ 1 + M_{21}M_{12} + M_{21}^2M_{12}^2 + \ldots \right] M_{12} \left[ \frac{c_{11}f_{12}}{2R_{12}} \right] + M_{21}M_{12} + \ldots \quad (4.39)
\]

Equation 4.39 can therefore be expressed as a scalar multiple and a series.
By substituting the terms \( M_{21} \) and \( M_{12} \) and applying equation 4.1.5, the final member-end-moment equation 21C11 is:

\[
21C11 = \left[ \frac{f_{21} - f_{21}c_{11}}{2R_{21}} - M_{12} \left[ \frac{c_{11}t_{12}}{2R_{12}} \right] \right] \sum_{n=0}^{\infty} \left[ M_{21}M_{12}^{n} \right]
\]

(4.40)

At the \( r_{12} \) position, the geometric series is expressed as:

\[
21R11 = - \frac{f_{21}r_{11}}{2R_{21}} + M_{12} \left[ \frac{t_{12} - t_{12}r_{11}}{2R_{12}} \right] - M_{21}M_{12} \left[ \frac{f_{21}r_{11}}{2R_{21}} \right] + M_{21}M_{12}^{2} \left[ \frac{t_{12} - t_{12}r_{11}}{2R_{12}} \right] - \frac{M_{21}^{2}M_{12}^{2}}{2R_{21}} + ... \]

(4.42)

After separating the positive and negative terms:

\[
21R11 = \left[ M_{12} \left[ \frac{t_{12} - t_{12}r_{11}}{2R_{12}} \right] - \frac{f_{21}r_{11}}{2R_{21}} \right] + M_{21}M_{12} + M_{21}M_{12}^{2} + ... \]

(4.43)

21R11 can be re-written as a geometric series:

\[
21R11 = \left[ M_{12} \left[ \frac{t_{12} - t_{12}r_{11}}{2R_{12}} \right] - \frac{f_{21}r_{11}}{2R_{21}} \right] \sum_{n=0}^{\infty} M_{21}M_{12}^{n}
\]

(4.44)

By substituting the terms \( M_{21} \) and \( M_{12} \) and applying equation 4.1.5, equation 4.6.7 reduces to:

\[
21R11 = \left[ \frac{(f_{21}r_{11} + r_{21}f_{22})(t_{12} - t_{12}r_{11})}{8R_{21}R_{12}} \right] - \frac{f_{21}r_{11}}{2R_{21}} \frac{1}{1 - \frac{(f_{21}r_{11} + r_{21}f_{22})(c_{11}t_{12} + c_{12}t_{22})}{16R_{21}R_{12}}}
\]

(4.45)

The other six final member-end-moment equations all share very similar derivations and are therefore listed below in their reduced form.
\[ 2T'12 = \frac{\left( f_{21} \bar{r}_{11} + r_{21} \bar{f}_{22} \right) \left( 4t_{12} - t_{12} \bar{r}_{11} \right)}{16R_{21}R_{12}} - \frac{f_{21} \bar{r}_{11}}{4R_{21}} \]  
\[ 2F'21 = \frac{4f_{21} \bar{f}_{22} \bar{c}_{11}}{4R_{21}} - \frac{\left( f_{21} \bar{r}_{11} + r_{21} \bar{f}_{22} \right) \left( c_{11}t_{12} + c_{12}t_{22} \right)}{16R_{21}R_{12}} \]  
\[ 2R'21 = \frac{4r_{21} \bar{r}_{21}t_{22}}{4R_{21}} - \frac{\left( f_{21} \bar{r}_{11} + r_{21} \bar{f}_{22} \right) \left( c_{11}t_{12} + c_{12}t_{22} \right)}{16R_{21}R_{12}} \]  
\[ 2F'22 = \frac{\left( f_{21} \bar{r}_{11} + r_{21} \bar{f}_{22} \right) \left( c_{12} - c_{12} \bar{f}_{22} \right)}{8R_{21}R_{12}} - \frac{r_{21} \bar{f}_{22}}{2R_{21}} \]  
\[ 2T'22 = \frac{r_{21} \bar{r}_{21}t_{22}}{2R_{21}} - \frac{\left( f_{21} \bar{r}_{11} + r_{21} \bar{f}_{22} \right) \left( c_{12}t_{22} \right)}{8R_{21}R_{12}} \]

4.7 **SIMPLIFYING THE FINAL MEMBER-END-MOMENT EQUATIONS**

The eight final member-end-moment equations all prove accurate to 10 significant figures (See Chapter 6) but, in the form presented above, are not easily expressed. However, all eight equations share similar functions, and the process of identifying and naming those functions makes the equations manageable.

The first function that is shared by all of the equations identifies the path from the joint at which the moment is applied, or the origin of the moment, to the joint farthest from it, i.e., the joint considered fixed at the beginning of the derivation. For this reason, it is referred to as the origin factor. It is represented by a capital Q, to avoid confusion between capital and lower-case 'o' and zero, and followed by the two digit joint number of its origin. For instance,
the origin factor from a moment applied at Joint 21 is $Q_{21}$. To understand the function of the origin factor, it is best to envision each moment as a discreet entity traveling through the members during the distribution process; the origin factor acts as a "road map" from the joint at which the moment is applied to the farthest joint from it. For instance, the origin cycle factor for Joint 21 is:

$$Q_{21} = \frac{(f_{21}r_{11} + r_{21}f_{22})}{4R_{21}}$$

and implies that one moment travels out from member-end $f_{21}$ through $r_{11}$ where as the other travels from $r_{21}$ through $f_{22}$ to arrive at the opposite joint. The origin cycle factor for Joint 12 is also present in the equations in Section 4.6:

$$Q_{12} = \frac{(c_{11}f_{12} + c_{12}f_{22})}{4R_{12}}$$

When combined, these two origin cycle factors account for "bounce back" moments that travel from corner to corner along linear paths and also the moments that travel clockwise and counter-clockwise around the frame. Since the infinite series of the two origin factors multiplied by one another captures the movement of moments around the square frame they will be referred to as "box factors". A capital 'B' followed by a subscript with two-digit joint numbers that describe the opposite corners of the frame is used as notation. In the case above, $B_{2112}$ captures the movement of moments around the box bounded by Joints 21 and 12 and is equivalent to:

$$B_{2112} = 1 - Q_{21}Q_{12}$$

Due to the commutative property of multiplication, $B_{2112}$ works for moments applied at Joint 21 and for moments applied at Joint 12.

Applying these two cycle factors simplifies equations 4.41, 4.45, and 4.46-4.51 drastically. By applying the naming conventions developed in Section 4.1 and Appendix B the equations become much more manageable. Note: positive and negative signs are included in the definition of moment equations, so addition is used in all equations:

$$21C1 = \frac{211c + Q_{21}211c_{B_{2112}}}{211c}$$

$$21R1 = \frac{Q_{21}211r + 211r_{B_{2112}}}{211c}$$
Equations 4.55 through 4.62 represent the eight final member-end-moment equations in their simplest form. They produce exact unit member-end-moments for each member-end in the simple frame under an applied moment at Joint 21.
CHAPTER 5
APPLICATIONS AND EXTENSION

5.1 APPLICATION

Although not a very realistic structural feature on its own, the square frame used for this derivation is the basic building block of any frame structural system. For instance, with a simple modification the equations can be adapted for a two story moment frame structure (Figure 5.1). Future modifications may allow added bays and floors to the frame.

![Figure 5.1. 2-story moment frame.](image)

The other immediate application of the box frame is for superposition onto the existing method developed by Dowell (2009) to create portions of 2-story bridge or truss structures.

5.2 EXTENSION

The R-factors are capable of being immediately extended to a joint with four member connections and the equation for this usage is presented previously (Equation A.8). By
following the polynomial pattern of the R-factor equation presented, the R-factors for joints
with 6 member connections (3-dimensional), 10 member connections (3-dimensional with
bracing) or more may be possible. With some effort, the generalized R-factor should be
sufficient to extend the single box to a 1x2 system and then to a 2x2 system, using the
principle of superposition. Theoretically, an expansion to an $\infty \times \infty$ system of box frames is
possible, although the equations, in their current format, are not manageable for that
approach. Expanding to a 1x2 or a 2x2 system may indicate some simplifications to the
equations in order to facilitate presentation in an a/b format, as the R-factor equation is
currently presented in (in fact, extending to a 2x2 system enabled the presentation of the R-
factor equation as an a/b equation).
CHAPTER 6

EXAMPLE

The following is an example box frame (Figure 6.1) made of members with the same cross-sections and material properties. The frame experiences no axial or shear deformations and no side-sway, and has an applied moment at joint 21. The frame is solved using three methods, (1) hand-calculated moment distribution, (2) commercially available structural analysis software, and (3) the new method proposed in this thesis.

![Figure 6.1. Simple frame for solution.](image)

6.1 HARDY CROSS MOMENT DISTRIBUTION

The first method will be the solution of the frame through hand-calculated moment distribution (Table 6.1).
Table 6.1. Step 1: Distribution Factors

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>k</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>21-22</td>
<td>4EI/6</td>
<td>.4</td>
</tr>
<tr>
<td></td>
<td>21-11</td>
<td>4EI/4</td>
<td>.6</td>
</tr>
<tr>
<td></td>
<td>Σ</td>
<td>5/3</td>
<td>1.000</td>
</tr>
<tr>
<td>22</td>
<td>22-21</td>
<td>4EI/6</td>
<td>.4</td>
</tr>
<tr>
<td></td>
<td>22-21</td>
<td>4EI/4</td>
<td>.6</td>
</tr>
<tr>
<td></td>
<td>Σ</td>
<td>5/3</td>
<td>1.000</td>
</tr>
<tr>
<td>11</td>
<td>11-12</td>
<td>4EI/6</td>
<td>.4</td>
</tr>
<tr>
<td></td>
<td>11-21</td>
<td>4EI/4</td>
<td>.6</td>
</tr>
<tr>
<td></td>
<td>Σ</td>
<td>5/3</td>
<td>1.000</td>
</tr>
<tr>
<td>12</td>
<td>12-11</td>
<td>4EI/6</td>
<td>.4</td>
</tr>
<tr>
<td></td>
<td>12-22</td>
<td>4EI/4</td>
<td>.6</td>
</tr>
<tr>
<td></td>
<td>Σ</td>
<td>5/3</td>
<td>1.000</td>
</tr>
</tbody>
</table>

- **Step 2: Moment distribution.** The joint release order starts with Joint 21, followed by the simultaneous release of Joints 22 and 11 (Figure 6.2) (arrows indicate the path of moment distribution for the first two distribution cycles on each member). After the initial two joint releases, simultaneous releases of Joints 21 and 12 followed by simultaneous releases of Joints 11 and 22 were repeated until a desired level of accuracy was achieved. Although all digits on the calculator were carried, the calculations are no more accurate than the largest digit of smallest incremental moment at each joint.
6.2 SAP 2000 Analysis

The analysis was performed on the commercially available finite element analysis software SAP2000. The model is formatted to be as close to the frame used for Sections 6.1 and 6.3. It is four foot tall by six foot wide, all members are composed of the same material and the same cross section. The frame is pinned at all four corners to prevent sway and axial deformations (Figure 6.3). Only analysis in the xz-plane was performed and a large modifier is set for the cross-sectional area when performing shear calculations, effectively eliminating shear deformations (Figure 6.4). A moment of 1000 kip-in. is applied to the top left joint and the analysis is run with the applied moment as the only considered load. All SAP2000 results in Table 6.2 have been listed to 10 digits of accuracy.

6.3 New Closed-Form Method

The solution gained using the proposed closed-form method outlined in the thesis is shown in Table 6.3.
Figure 6.3. SAP2000 model.

Figure 6.4. Deformed shape SAP2000.

Table 6.2. Relevant Results from SAP2000

<table>
<thead>
<tr>
<th>Variable</th>
<th>Joint</th>
<th>CaseType</th>
<th>M2</th>
<th>FrameElem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{12}$</td>
<td>3</td>
<td>LinStatic</td>
<td>12.1212121212</td>
<td>1</td>
</tr>
<tr>
<td>$f_{22}$</td>
<td>4</td>
<td>LinStatic</td>
<td>-121.2121212121</td>
<td>1</td>
</tr>
<tr>
<td>$r_{21}$</td>
<td>5</td>
<td>LinStatic</td>
<td>412.1212121212</td>
<td>2</td>
</tr>
<tr>
<td>$t_{22}$</td>
<td>4</td>
<td>LinStatic</td>
<td>121.2121212121</td>
<td>2</td>
</tr>
<tr>
<td>$r_{11}$</td>
<td>6</td>
<td>LinStatic</td>
<td>-121.2121212121</td>
<td>3</td>
</tr>
<tr>
<td>$t_{12}$</td>
<td>3</td>
<td>LinStatic</td>
<td>-12.1212121212</td>
<td>3</td>
</tr>
<tr>
<td>$f_{21}$</td>
<td>5</td>
<td>LinStatic</td>
<td>587.8787878788</td>
<td>4</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>6</td>
<td>LinStatic</td>
<td>121.2121212121</td>
<td>4</td>
</tr>
</tbody>
</table>
Table 6.3. Distribution Factors for Closed-Form Solution (same as Hardy Cross Solution)

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
<th>( r )</th>
<th>( t )</th>
<th>( c )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
<td>0</td>
<td>0.6</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0.4</td>
<td>0.6</td>
<td>0</td>
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<td>1</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0.4</td>
<td>0</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 6.3 lists the distribution factors for each joint. Using the previously outlined equations unit moment results are produced:

\[
21F'21 = \frac{2121f + Q_{21} 1221f}{B_{2112}} = 0.587878788
\]

\[
21R21 = \frac{2121r + Q_{21} 1221r}{B_{2112}} = 0.412121212
\]

\[
21T22 = \frac{2122r + Q_{21} 1222r}{B_{2112}} = 0.121212121
\]

\[
21F22 = \frac{Q_{21} 1222f + 2122f}{B_{2112}} = -0.121212121
\]

\[
21C12 = \frac{Q_{21} 1212c + 2112c}{B_{2112}} = 0.012121212
\]

\[
21T12 = \frac{Q_{21} 1212r + 2112r}{B_{2112}} = -0.012121212
\]

\[
21R11 = \frac{Q_{21} 1211r + 2111r}{B_{2112}} = -0.121212121
\]

\[
21C11 = \frac{2111c + Q_{21} 1211c}{B_{2112}} = 0.121212121
\]

Multiplying these results attained using a unit moment by the value of the applied moment (1000 kip-in.) produces the exact member-end-moments, which match the results from SAP2000 to the ten significant figures seen in the program (Table 6.4).
Table 6.4. Final Member End-Moments

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
<th>( r )</th>
<th>( t )</th>
<th>( c )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-121.2121212121</td>
<td>0</td>
<td>121.2121212121</td>
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<td>2</td>
<td>0</td>
<td>-12.1212121212</td>
<td>12.1212121212</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>412.1212121212</td>
<td>0</td>
<td>0</td>
<td>587.8787878788</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>121.2121212121</td>
<td>0</td>
<td>-121.2121212121</td>
</tr>
</tbody>
</table>
CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

7.1 CONCLUSION

Based on the moment distribution method developed by Cross (1932) and the closed-form moment solution method developed by Dowell (2009), this thesis presents the closed-form equations for the exact solution of all member-end-moments for a simple frame. The intent is to extend the current closed-form moment distribution solutions from single-story, continuous beam or bridge structures to multi-story frame structures starting by adding a second story.

Using incremental moment distribution methods and superposition, infinite geometric series were developed at each end of the frame members. Since the value of the distribution factors are always less than one, the geometric series converge, allowing basic calculus concepts to be employed to find the exact, closed-form equations, which produce results that require an infinite amount of moment distribution cycles in the traditional method.

The equations are derived considering only flexural deformations. The 4-sided frame used is braced from side-sway and experiences no axial deformation. Joint labels and corresponding distribution factor coefficients have been defined in order to use existing moment distribution methods to develop closed-form equations that will be applicable to any four-sided frame. Naming conventions are likewise introduced for outgoing moment factors, origin factors, and a box factor, which allow simplified presentation of the member-end-moment equations. In addition to the derivation of equations for the one-story, one bay frame, an additional derivation is presented which allows the outgoing moment factor to be extended to a two-story two bay frame.

Using this method, solutions for any member-end-moments on the four-sided frame can be found without the use of simultaneous equations or distributions. As derived, the equations require only scalar multiplication, addition, and subtraction to solve, making them very computationally efficient. In order to determine their accuracy, the derived equations
were compared to hand-calculations and commercially available structural analysis software and the results proved exact to 10 significant figures, ensuring the reliability of the method.

7.2 Future Work

This thesis represents the first step towards a long-term goal of developing a closed-form solution to frames of any size. Expansion to two bay or two story frame structures is the most practical, immediate development for this thesis. In the short term, the introduction of the effects of sway is eased by the $c$ and $f$ distribution factors and adds another level of accuracy to the results. The addition of shear and axial deformations represent incremental improvements based on the framework of this thesis. An expansion to three dimensions, using the existing torsional deformation work from Dowell & Johnson (2012) would represent the final stage of the expansion of the thesis.
REFERENCES


APPENDIX A

DERIVATION OF THE GENERAL FORM OF THE OUTGOING MOMENT FACTOR
The derivation process for the general form of the outgoing moment factor is presented below. The derivation utilizes a 2x2 frame constructed of four component frames of the same type described in Section 3.2 (Figure 3.4) arranged as below (Figure A.1):

![Figure A.1. 2x2 frame system.](image)

With fixed connections at the corners, Joints 11, 13, 31, and 33, and rollers at Joints 12 and 23 to prevent sway, meaning the analysis can be performed in the same manner as the derivations in Chapter 4, using only flexural deformations (Figure A.2).

![Figure A.2. Applied moment at Joint 22](image)

Since the only concern in this derivation is the outgoing moment factor for a joint with all four distribution factors, a unit moment will be applied at Joint 22 only and distributed to the adjacent joints. (Figure A.3.)
The derivation progresses in a manner very similar to the derivation in 4.1 except the moment radiates out from the initial joint, Joint 22, in four directions rather than two. (Figure A.4)

The clamps at Joints 12, 21, 23, and 32 are released and the out-of-balance moments at those joints redistribute leaving out-of-balance moments at the four corner joints and at the center joint, Joint 22. (Figure A.4)
Once more, the clamp at Joint 22 is released and the moment is allowed to distribute. The rapid increase in terms that occurs with each cycle of clamp releases can be seen in the figure above (Figure A.5) and, after this cycle, the equations become too difficult to represent legibly. For this reason, a representative position, in this case \( c_{22} \), is chosen and it's series is presented below as 2222c, following the naming protocol introduced in Section 4.1:

\[
2222c = c_{22} - \frac{c_{22}f_{32}}{4} + \frac{c_{22}^2f_{32} + c_{22}r_{22}t_{23} + c_{22}f_{22}c_{12} + c_{22}r_{22}r_{21}}{4}
\]

\[
\frac{c_{22}^2f_{32} + c_{22}f_{32}r_{22}t_{23} + c_{22}f_{32}c_{12} + c_{22}f_{32}r_{22}r_{21}}{16} + ...
\]

In the same manner as equation 4.1.20, the positive and negative terms can be separated and the like terms factored out yielding:

\[
2222c = c_{22} \left[ 1 + \frac{c_{22}f_{32} + r_{22}t_{23} + f_{22}c_{12} + t_{22}r_{21}}{4} + ... \right] - \frac{c_{22}f_{32}}{4} \left[ 1 + \frac{c_{22}f_{32} + r_{22}t_{23} + f_{22}c_{12} + t_{22}r_{21}}{4} + ... \right]
\]

(A.2)

Which presents matching series with a familiar form, writing the next step in the series reveals the pattern:
\begin{align*}
1 &+ \frac{c_{22} f_{32} + r_{22} t_{23} + f_{22} c_{12} + r_{22} t_{21}}{4} + \frac{c_{32}^2 f_{32}^2}{16} + \frac{2c_{22} f_{32} r_{22} t_{23}}{16} + \frac{2c_{22} f_{32} r_{22} c_{12}}{16} + \\
&\frac{2c_{22} f_{32} t_{22} r_{21}}{16} + \frac{r_{22} f_{23}}{16} + \frac{2r_{22} f_{23} c_{12}}{16} + \frac{2r_{22} t_{23} r_{22} r_{21}}{16} + \frac{f_{22} c_{12}^2}{16} + \frac{2f_{22} c_{12} r_{22} r_{21}}{16} + \frac{t_{22}^2 r_{21}^2}{16} + \ldots \\
\text{(A.3)}
\end{align*}

Applying equation 4.1.5 to equation A1.2 with the series expansion notated in A1.3:

\begin{align*}
2222e &\left[ c_{22} - \frac{c_{22} f_{32}}{4} \sum_{n=0}^{\infty} \left[ c_{22} f_{32} + r_{22} t_{23} + f_{22} c_{12} + t_{22} r_{21} \right]^n \right] \\
2222e &\left[ \frac{4c_{22} - c_{22} f_{32}}{4 \left( 1 - c_{22} f_{32} + r_{22} t_{23} + f_{22} c_{12} + r_{22} t_{21} \right)} \right] \\
\text{(A.4)}
\end{align*}

Since all of the equations originating from an applied moment at Joint 22 share the common multiple in the denominator:

\begin{align*}
1 - \frac{c_{22} f_{32} + r_{22} t_{23} + f_{22} c_{12} + t_{22} r_{21}}{4} \\
\text{(A.6)}
\end{align*}

It is replaced with the outgoing moment factor named $R_{22}$:

\begin{align*}
R_{22} & = 1 - \frac{c_{22} f_{32} + r_{22} t_{23} + f_{22} c_{12} + t_{22} r_{21}}{4} \\
\text{(A.7)}
\end{align*}

Considering equation 4.1.24:

\begin{align*}
R_{21} & = 1 - \frac{f_{21} c_{11} + r_{21} t_{22}}{4} \\
\text{(4.24)}
\end{align*}

A pattern can be established and the outgoing moment factor can be written with coefficients which will allow it to be adapted easily to any generic Joint $ab$.

\begin{align*}
R_{ab} & = 1 - \frac{c_{ab} f_{(a+1)b} + r_{ab} f_{a(b+1)} + f_{ab} c_{(a-1)b} + t_{ab} r_{a(b-1)}}{4} \\
\text{(A.8)}
\end{align*}

Equation 4.24 demonstrates the application of the coefficient outgoing moment equation. In the derivation in 4.1 $R_{21}$ has only $f$ and $r$ distribution factors, therefore the values of $c_{ab}$, $c_{21}$ in this case, and $t_{ab}$, or $t_{21}$, would be zero, eliminating the coefficients and, by multiplication, the non-existent $f_{(a+1)b}$ and $r_{a(b-1)}$. 

APPENDIX B

MOMENT EQUATIONS BASED ON JOINT NUMBER
Figure B.1 and Figure B.2 are a brief derivation of the moment equations based on a moment applied at Joint 22 and will demonstrate the adaptability of the moment equations based on the symmetry of the system.

Given the experience with the derivation in Section 4.1 it will take significantly less cycles to recognize the series as it will merely be a re-ordering of the coefficients based on the origination of the moment and the fixed joint.

After four cycles the pattern is evident enough to start writing the equations. The $2211c$ equation can be identified from the moment distribution cycles in Figure B.3.
Figure B.3. Third and fourth moment distribution cycles.

$$221 l_c = -\frac{t_{22} f_{21}}{4} \sum_{n=0}^{\infty} \left[ \frac{t_{22} r_{21} + f_{22} c_{12}}{4} \right]^n = -\frac{t_{22} f_{21}}{4 R_{22}} \quad (B.1)$$

The other equations can be inferred by comparison to the derivation in Section 4.1 and the symmetry of the system:

$$221 l_r = -\frac{f_{22} t_{12}}{4 R_{22}} \quad (B.2)$$

$$2221 f = -\frac{t_{22} f_{21}}{2 R_{22}} \quad (B.3)$$

$$222 l_r = t_{22} - t_{22} r_{21} \quad (B.4)$$

$$2212 r = -\frac{f_{22} t_{12}}{2 R_{22}} \quad (B.5)$$

$$2212 c = -\frac{f_{22} - f_{22} c_{12}}{2 R_{22}} \quad (B.6)$$

$$2222 f = 4 f_{22} - f_{22} c_{12} \quad (B.7)$$

$$2222 r = 4 t_{22} - t_{22} r_{21} \quad (B.8)$$

For moments applied at Joint 11:

$$1122 f = -\frac{r_{11} c_{12}}{4 R_{11}} \quad (B.9)$$
\[ 112r = -\frac{c_{11} r_{21}}{4R_{11}} \]  
(B.10)

\[ 112lr = -\frac{c_{11} r_{21}}{2R_{11}} \]  
(B.11)

\[ 1121f = \frac{c_{11} - c_{11} f_{21}}{2R_{11}} \]  
(B.12)

\[ 1112c = -\frac{r_{11} c_{12}}{2R_{11}} \]  
(B.13)

\[ 1112t = \frac{r_{11} - r_{11} t_{12}}{2R_{11}} \]  
(B.14)

\[ 111lc = \frac{4c_{11} - c_{11} f_{21}}{4R_{11}} \]  
(B.15)

\[ 111lr = \frac{4r_{11} - r_{11} t_{12}}{4R_{11}} \]  
(B.16)

For moments applied at Joint 12:

\[ 122lr = \frac{c_{12} f_{22}}{4R_{12}} \]  
(B.17)

\[ 1221f = -\frac{c_{11} f_{12}}{4R_{12}} \]  
(B.18)

\[ 121lc = -\frac{c_{11} f_{12}}{2R_{12}} \]  
(B.19)

\[ 1222f = \frac{c_{12} - c_{12} f_{22}}{2R_{12}} \]  
(B.20)

\[ 1222t = -\frac{c_{12} t_{22}}{2R_{12}} \]  
(B.21)

\[ 121lr = \frac{t_{12} - t_{12} r_{12}}{2R_{12}} \]  
(B.22)

\[ 1212c = \frac{4c_{12} - c_{12} f_{22}}{4R_{12}} \]  
(B.23)
\[1212r = \frac{4I_{12} - I_{12}r_{11}}{4R_{12}}\] (B.24)

Origin cycle factors for Joints 11 and 22:

\[Q_{11} = \frac{(r_{11}c_{12} + c_{11}r_{21})}{4R_{11}}\] (B.25)

\[Q_{22} = \frac{(f_{22}t_{12} + t_{22}f_{21})}{4R_{22}}\] (B.26)

Final member end moment equations for a moment applied at Joint 12:

\[12C_{11} = \frac{Q_{12}2111c + 1211c}{B_{2112}}\] (B.27)

\[12R_{11} = \frac{1211r + Q_{12}2111r}{B_{2112}}\] (B.28)

\[12C_{12} = \frac{1212c + Q_{12}2112c}{B_{2112}}\] (B.29)

\[12T_{12} = \frac{1212r + Q_{12}2112r}{B_{2112}}\] (B.30)

\[12F_{21} = \frac{Q_{12}2121f + 1221f}{B_{2112}}\] (B.31)

\[12R_{21} = \frac{Q_{12}2121r + 1221r}{B_{2112}}\] (B.32)

\[12F_{22} = \frac{1222f + Q_{12}2122f}{B_{2112}}\] (B.33)

\[12T_{22} = \frac{Q_{12}2122r + 1222r}{B_{2112}}\] (B.34)

Final member end moment equations for a moment applied at Joint 11:

\[11C_{11} = \frac{1111c + Q_{11}2211c}{B_{1122}}\] (B.35)

\[11R_{11} = \frac{1111r + Q_{11}2211r}{B_{1122}}\] (B.36)
$$11C12 = \frac{Q_{12}^{1} 2212c + 1112c}{B_{1122}}$$

(B.37)

$$11T12 = \frac{1112r + Q_{11}^{1} 2212r}{B_{1122}}$$

(B.38)

$$11F21 = \frac{1121f + Q_{11}^{1} 2221f}{B_{1122}}$$

(B.39)

$$11R21 = \frac{Q_{11}^{1} 2221r + 1121r}{B_{1122}}$$

(B.40)

$$11F22 = \frac{Q_{11}^{1} 2222f + 1122f}{B_{1122}}$$

(B.41)

$$11T22 = \frac{Q_{11}^{1} 2222r + 1122r}{B_{1122}}$$

(B.42)

Final member end moment equations for a moment applied at Joint 22:

$$22C11 = \frac{Q_{22}^{1} 1111c + 2211c}{B_{1122}}$$

(B.43)

$$22R11 = \frac{Q_{22}^{1} 1111r + 2211r}{B_{1122}}$$

(B.44)

$$22C12 = \frac{2212c + Q_{22}^{1} 1112c}{B_{1122}}$$

(B.45)

$$22T12 = \frac{Q_{22}^{1} 1112r + 2212r}{B_{1122}}$$

(B.46)

$$22F21 = \frac{Q_{22}^{1} 1121f + 2221f}{B_{1122}}$$

(B.47)

$$22R21 = \frac{2221r + Q_{22}^{1} 1121r}{B_{1122}}$$

(B.48)

$$22F22 = \frac{2222f + Q_{22}^{1} 1122f}{B_{1122}}$$

(B.49)

$$22T22 = \frac{2222r + Q_{22}^{1} 1122r}{B_{1122}}$$

(B.50)