AN ANALYTIC AND GEOMETRIC APPROACH
FOR EXAMINING THE STABILITY OF
LINEAR DIFFERENTIAL EQUATIONS WITH TWO DELAYS

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Rie Tsujiwaki Wilsterman
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The Undersigned Faculty Committee Approves the
Thesis of Rie Tsujiwaki Wilsterman:

An Analytic and Geometric Approach
For Examining The Stability of
Linear Differential Equations With Two Delays

Joseph Mahaffy, Chair
Department of Mathematics and Statistics

Robert Grone
Department of Mathematics and Statistics

Roger Whitney
Department of Computer Science

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Rie Tsuiwaki Wilsterman
DEDICATION

This thesis is dedicated to my husband Jim Wilsterman, an extraordinarily successful Aztec alumni and Professor, whose unwavering support of my studies allowed a childhood dream of studying advanced mathematics become a reality.
ABSTRACT OF THE THESIS

An Analytic and Geometric Approach
For Examining The Stability of
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by
Rie Tsujiwaki Wilsterman
Master of Arts in Mathematics
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This thesis focused on the stability analysis for the linear differential equation with two delays:

\[ \dot{y}(\tau) + A y(\tau) + B y(\tau - 1) + C y(\tau - R) = 0. \] (1)

This equation has four parameters, and its analysis is surprisingly complex. We build upon previous work of Joseph Mahaffy and Timothy Busken in locating the stability region. Their earlier work provided the theoretical framework and important numerical results supporting the idea that when the delay R is rational, the region of stability is enlarged. Previously developed computer programs demonstrated how the region of stability in the ABC-parameter space evolves with changing R. This thesis provides some analytic proofs to support the enlarged region for \( R = \frac{1}{3} \). Our analysis shows how sensitive the region of stability is to the delay, R. This work may assist mathematical modelers with the sensitivity analysis and complex solutions observed for some models with multiple delays.
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CHAPTER 1

INTRODUCTION

Delay differential equations (DDEs) have been used for many years in control theory, and recently they have been applied to biological models. Investigation into the stability of these delay differential equations is a complex and important problem. Studies suggest that the region of stability in differential equations with two delays becomes larger when the delays are rationally related. This thesis examines the stability region of the following linear two delay differential equation:

$$\dot{y}(t) + ay(t) + by(t - r_1) + cy(t - r_2) = 0,$$

as the coefficient parameters $a, b, c \in \mathbb{R}$ vary, and the ratio of the delays, $\frac{r_2}{r_1}$, varies over $(0, 1)$.

The study of Eqn. (1.1) and its stability properties appear to be simple. Equilibrium solutions of (1.1) and its characteristic equation are very easy to find. However, the analysis of the characteristic equation is very complicated. It is a transcendental equation with infinitely many solutions. Examining the roots of this exponential polynomial and knowing when the real part of the eigenvalues become positive or negative are difficult tasks. Busken [8] states that it is practically impossible to describe or illustrate complex bifurcation surfaces when they intersect to form a five-dimensional stability region. He breaks down the stability space of Eqn. (1.1) into a geometric study of subspaces in two, three, and four-dimensions. The complexities of the bifurcation analysis point to the potential of rich dynamics from models with two delays.

Zaron [43] has proved that when the ratio of the delays is a constant, Eqn. (1.1) has a minimum region of stability. However, the region of stability varies with different delays, and the size of the stability region with rationally related delays is larger than one with delays nearby that are irrationally related. Mahaffy and Busken [27] and Busken [8] provide strong evidence supporting the idea of enlarged regions of stability for rational delays.

This thesis continues work from the Master’s thesis of Busken [8]. Busken developed detailed numerical routines to study Eqn. (1.1). Its primary contribution was a systematic approach to finding the stability region for Eqn. (1.1) and developing numerical algorithms, establishing a larger region of stability for special rationally dependent delays, $\frac{r_2}{r_1}$. This thesis builds on the numerical intuition of Busken [8] and provides a more rigorous, analytical approach that justifies some of his observations.
As an introduction to this thesis, we highlight the main results of Busken [8]. Both his thesis and this one use the definitions and results developed in Mahaffy et al. [29] as a basis for the study of the stability region for Eqn. (1.1). These analyses center on the study of the characteristic equation. Our studies concentrate on changes in the stability region as the parameters in Eqn. (1.1) vary. Since changes in stability occur when eigenvalues cross the imaginary axis, it is important to determine the image of the imaginary axis for the characteristic equation of Eqn. (1.1) in the parameter space. Busken [8] developed sophisticated MatLab programs, which follow this image as the parameters \(a, b, c\) and \(\frac{r_2}{r_1}\) vary. For fixed \(R = \frac{r_2}{r_1}\), he developed 3D pictures, which allowed visualization of the stability boundary in the \(abc\)-parameter space, extending the results of Mahaffy et al. [29].

One interesting numerical study showed how spur like projections jut from the stability surface at certain values of \(a\). These unusual projections have a significant effect in extending the region of stability outside a pyramidal -shaped stable region centered on the positive \(a\)-axis. They also produce disconnected regions of stability in the \(bc\)-cross sections for certain \(a\) values.

Busken [8] performed a detailed numerical study of how the stability surface changes as \(R\) varies. His results showed an orderly progression of changes in the stability region. However, despite general continuity of the stability region as the delays vary, there are singularities at rational values of the delays. The image of the imaginary axis produces a countable collection of “bifurcation” surfaces in the \(abc\)-parameter space for fixed \(R\). At rational \(R\) these “bifurcation” surfaces become organized and often only a small number lie on the stability boundary. This results in a larger stability region. Busken [8] concentrated his numerical studies near \(R = \frac{1}{4}\), demonstrating the enlarged region of stability. This thesis provides analytical proofs for some of the surface organization. The proofs detail how the stability region becomes enlarged.

Chapter 2 discusses some background information on two delay differential equations pointing out interesting and relevant examples of how two-delay differential equations can be applied in various fields of study. Chapter 3 provides the basic definitions and theorems, which are used to describe the evolution of the stability surfaces, and applies them to prove the geometric ordering of stability surfaces. Chapter 4 contains the main results of this thesis. In this chapter we establish analytic results that prove the sequential ordering of “bifurcation” surfaces, which suggest that rationally dependent delays have larger regions of stability. In Chapter 5, we summarize our results and suggest possible future research.
CHAPTER 2
BACKGROUND

Differential equations including time delays have been incorporated into various scientific fields, such as biological population models, epidemiology, dynamic models of economics, and mechanical systems. Delay differential equations (DDEs) have been used for more complicated scientific phenomenon. Biological systems often have intrinsic delays. When biological systems are modeled, delay differential equations are frequently employed. For example, in population models there are delays in maturation or gestation. Large sequences of biochemical reactions may be modeled with a delay for the long sequence of intermediate reactions.

Hutchinson [20] was the first person to investigate the logistic growth model with a delay. Later, Cooke and Yorke [9], Nussbaum [35], Silkowski [40], and Stech [41] advocated delving much deeper into logistic population models requiring a second or multiple time delays. MacDonald [25, 26] and Murdoch et al. [34] applied two delay differential equations to explain interspecies competition and parasitoid-host models, which offset at different maturation times. Cooke and Yorke [9] and McDonald [24] constructed a two delay model to examine the transmission of a gonorrhea epidemic, explaining differing incubation times between males and females.

Multiple delay differential equations have been applied to tumor treatment. Villasana and Radunskay [31] allowed for cycle-phase specific drugs to be monitored for effects and interactions between tumors and immune cells. Delay differential equations established clear differentiation between phases, allowing for treatment with a specific cycle-phase drug regimen. Likewise, mathematical physiologists have incorporated two delays into their models for disease development. Bélair, Mackey, and Mahaffy [4] used a model with the two delays to take account for the regulation of mammalian erythropoiesis. MacDonald [24, 25] applied two delays to a model in the production of granulocytes, a category of white blood cells. Mammalian platelet production with a two-delay differential equation were modeled by Bélair [4].

Glass, Beuter, and Larocque [11], Gopalsamy and Leug [12], Guzelis, Cuneyt and Chua [13], and Mohamad and Gopalsamy [33] analyzed motor-controlled differential equations using non-linear variable delays. Mechanical and robotic control problems [17] have been modeled using multiple delay differential equations. In examining their optic
problems, Mizuno and Ikeda [32] and Marriot et al. [30] combined multiple-time delay compartments within control loops.

Often dynamic economic models require the application of DDEs. The work of Keller [21] explores this, and he states: “Dynamic economic models generally consist of difference or differential behavioral equations. Several arguments are in favor of continuous time systems: the multiplicity of decisions overlapping in time, a more adequate formulation of market adjustments and distributed lag processes, the properties of estimators, etc.” [21]. Integrodifferential equations with two delays were used by Belair and Mackey [3] to analyze the pricing dynamics of singular commodity markets. Time delays were applied to their analysis of supply and demand price functions in order to explain the relationship between consumers’ memories of past prices and market equilibrium. Key to understanding this relationship were time delays caused by production lags and storage policies. Howroyd and Russell [19] examined competing firms in which adjustments of output are subject to delays. The stability of the oligopoly problem is considered under the Cournot strategy where firms calculate their own optimal output based on knowledge of their production at that time and of their competitors’ outputs at a previous time. Under such conditions, stability is not affected by the information delays. Howroyd and Russell found that if the information available to all firms was subject to delays, then stability will be affected, and the probability of stability increases with decreasing delays.

Many researchers [1, 10, 15, 16, 23, 29, 35] have examined the asymptotical behavior of the boundary of a stability region of (1.1), using analytical and geometric methods. Many scientists [7, 14, 22, 23, 35, 36, 37, 38, 39, 42] have applied bifurcation theorem to DDEs with two delays where \( a = 0 \). Hale and Huang [15] conducted a global stability analysis of (1.1) by a geometrical approach where they restricted the parameters. Braddock and van den Driessche [7] proposed two delay logistic equation where its coefficients and delay terms are positive constants. They developed linear stability regions for a two delay logistic equation in order to examine stable limit cycles when the ratio of the delays is sufficiently large.

The construction of the above models may result in complicated dynamics; however, linear stability analysis is often used to find bifurcations in behavior. The change in stability of (1.1) is associated with the eigenvalues of the characteristic equation. Several special cases have been proved, which establish necessary and sufficient conditions on the asymptotic stability of Eqn. (1.1). For example, Hayes [18] solved the stability of a one delay problem \((c = 0)\). Nussbaum [35] and Ruiz-Claeyssen [38] resolved it for \( a = 0 \) and \( r_1 = 1 \) with other specific restrictions. Many researchers analyzed Eqn. (1.1) for special cases. For example, Bélaire and Campbell [2], Braddock and van de Driessche [6], and Hale [14] solved the more
complicated case, \( a = 0 \). This case is very important, but \( a \neq 0 \) is more difficult to characterize.

Zaron [43] presented a clear stability analysis for Eqn. (1.1) in \( abc \)-parameter space. Hale and Huang [15] followed with a geometric approach to describe the asymptotical behavior of the boundary of a stability region for Eqn. (1.1) in the \( r_1r_2 \)-parameter space. The majority of these analyses study the two-dimensional stability structure of Eqn. (1.1). Since Eqn. (1.1) has five parameters, these studies often scale one parameter to unity, set another parameter equal to zero, then fix a third parameter. The stability region is then determined for the remaining two parameters. Two-dimensional analyses frequently observe disconnected stability regions for (1.1). Elsken [10] established that the stability region of Eqn. (1.1) is connected in the \( abc \)-parameter space for fixed \( r_1 \) and \( r_2 \). Mahaffy, Zak, and Joiner [28, 29] scaled out one of the delays and studied Eqn. (1.1) for an interval of delay ratio values, examining two-dimensional stability regions and developing complete three-dimensional bifurcation surfaces in the coefficient-parameter space.

This thesis geometrically examines a scaled version of the scalar linear first order differential equation with two delays, (1.1). It focuses on proving several results from an analytical approach that Busken [8] researched and showed numerically. His work provides strong evidence supporting the idea of enlarged regions of stability for rational delays. The following work provides some detailed proofs to justify his claim. The next chapter offers key definitions for describing the evolution of stability surfaces. Subsequently, detailed proofs on the geometric ordering of stability surfaces are provided showing simplified regions of stability, which are larger in size than an asymptotic minimum region of stability.
CHAPTER 3
TOOLS AND TECHNIQUES FOR ANALYSIS
OF THE TWO-DELAY PROBLEM

This chapter sets the basic definitions and theorems in order to explain how the region of the stability changes. We examine the image of the characteristic equation for our DDE and define bifurcation surfaces. We define the limited ways that the bifurcation surfaces change positions and effect the boundary of the stability surface.

In Chapter 1, we introduced a scalar linear first order differential equation with two delays given by Eqn. (1.1). In order to reduce the five parameters in Eqn. (1.1) to four parameters, we rescale as follows: \( R = \frac{r_1}{r_2}, A = r_1a, B = r_1b \) and \( C = r_1c \), then we have:

\[
\dot{y}(\tau) + Ay(\tau) + By(\tau - 1) + Cy(\tau - R) = 0.
\]  

(3.1)

The primary topic of this thesis is studying the stability region of the delay differential equation (3.1). The stability surface is very complex, and the work of Mahaffy et al. [29] provides a framework for the analysis of such a surface. The focus of this chapter is to provide key definitions and theorems needed to describe the stability region, so that this thesis is mostly self-contained.

Equilibria for (3.1) occur when \( \dot{y} = 0 \). Whenever \( A + B + C \neq 0 \), it follows that \( y(t) \equiv 0 \) is the unique equilibrium for (3.1). In the particular case, where \( A + B + C = 0 \), then any constant solution \( y(t) \equiv k \) is an equilibrium for (3.1). This thesis provides details for analyzing the region of stability for an equilibrium of (3.1) in the \( A, B, C, \) and \( R \) parameter space, \( \mathbb{R}^3 \times (0, 1) \). When the delay parameter, \( R \), is fixed, many bifurcation surfaces intersect to create a connected, unbounded stability surface found in the three-dimensional coefficient-parameter space of Eqn (3.1).

Stability properties of the linear DDE (3.1) can be analyzed by studying the eigenvalues of the characteristic equation. If we assume a solution of (3.1) in the form \( y(\tau) = \xi e^{\lambda \tau} \), then we obtain the following characteristic equation:

\[
\lambda + A e^{-\lambda} + B e^{-\lambda R} = 0.
\]  

(3.2)

The characteristic equation is an exponential polynomial with an infinite number of roots, \( \lambda \). Thus, there are infinity many linearly independent characteristic solutions of the DDE, as we would expect. Stability is lost when the real part of the eigenvalue, \( \lambda \), becomes positive.
When $\lambda = 0$, the characteristic equation is satisfied with any point of the plane $A + B + C = 0$. It follows that traversing this plane results in an eigenvalue changing sign, so this plane creates a bifurcation surface. Below we examine how (3.1) changes stability for a given $R$ in the $ABC$-parameter space.

One important region for our study is an absolute region of stability, which is independent of the delay. This provides a basic structure to compare against as $R$ varies.

**Theorem 3.1. Minimum Region of Stability (MRS)** For $A > |B| + |C|$, all solutions $\lambda$ to Eqn. (3.2) have $\Re(\lambda) < 0$, which implies that Eqn. (3.1) is asymptotically stable inside the pyramidal-shaped region centered about the positive $A$-axis, independent of $R$.

**Proof.** The proof of the theorem can be found in both Zaron [43] and Boese [5].

Stability is lost when the eigenvalues cross the imaginary axis. Since one face of the MRS is located on the $A + B + C = 0$ plane, one part of the stability boundary always runs along a portion of the zero root-crossing surface. Since the boundary of the stability region is an image of the imaginary axis in the $ABC$-parameter space, the next step of the analysis can be determined by finding the image of the imaginary axis.

Let $\lambda = \omega i$ ($\omega \in \mathbb{R}$). Eqn. (3.2) can be written as

$$[A + B \cos(\omega) + C \cos(\omega R)] + i [\omega - B \sin(\omega) - C \sin(\omega R)] = 0. \quad (3.3)$$

By separating (3.3) into its real and imaginary parts and solving for $B$ and $C$, we have the following parametric equations:

$$B(\omega) = \frac{A \sin(\omega R) \cos(\omega R)}{\sin(\omega(1 - R))}, \quad (3.4)$$

$$C(\omega) = -\frac{A \sin(\omega R) \cos(\omega R)}{\sin(\omega(1 - R))}, \quad (3.5)$$

which are defined for $\omega \neq \frac{j \pi}{1 - R}$. Since $B(-\omega) = B(\omega)$ and $C(-\omega) = C(\omega)$, both $B(\omega)$ and $C(\omega)$ are even functions in $\omega$, and one need only consider $j \in \mathbb{Z}^+$. As $A$ changes with $\omega \in \left(\frac{(j-1)\pi}{1-R}, \frac{j\pi}{1-R}\right)$, Eqns. (3.4) and (3.5) produce a surface in the $ABC$-parameter space. As $\omega$ increases, singularities of $B(\omega)$ and $C(\omega)$ occur at $\omega = \frac{j \pi}{1 - R}$, and these singularities induce the bifurcation surface change and lead to the following definition.

**Definition 3.1.** For $R \in (0, 1)$ Bifurcation Surface $j \Lambda_j$, is determined by Eqns. (3.4) and (3.5) and is defined parametrically for $\frac{(j-1)\pi}{1-R} < \omega < \frac{j\pi}{1-R}$ and $A \in \mathbb{R}$. This creates a separate parameterized surface representing pure imaginary, conjugate solutions of the characteristic
equation, (3.2), which can be sketched in the \(ABC\) coefficient-parameter space of Eqn. (3.1), for each positive integer, \(j\). For \(\lambda = 0 (\omega = 0)\), we have the special bifurcation surface, \(\Lambda_0\), which is the \(A + B + C = 0\) plane.

When the value of \(R \in (0, 1)\) is fixed, the real root-crossing plane, \(\Lambda_0\), and bifurcation surfaces, given by Eqns. (3.4) and (3.5), intersect within the \(ABC\) space and create a continuous three dimensional region of stability that comes from Eqn. (3.1). Because the MRS is centered about the positive \(A\)-axis, we often fix the \(A\) value and observe the stability region within the \(BC\)-plane cross section. The boundary of the stability region is easy to observe when the value of \(A\) is fixed. Thus, we have the related definition.

**Definition 3.2.** *(Bifurcation Curve \(j\), \(G_j\))* is determined by Eqns. (3.4) and (3.5) and is defined for \((j-1)\pi - R < \omega < j\pi - R\). With the values of \(R\) and \(A\) fixed, as \(\omega\) is allowed to sweep through its interval, a separate parameterized curve can be drawn on the \(BC\) plane for each \(j\).

With these preliminary definitions we begin describing the region of stability for DDE (3.1). Our approach is to show the evolution of the stability surface as \(A\) increases and develop more definitions and theorems to aid in our description.

The obvious place to begin is the smallest value of \(A\) where Eqn (3.1) has stability. Mahaffy *et al.* [29] showed that provided \(R > R_0 \approx 0.012\), the stability region comes to a point summarized by the following.

**Theorem 3.2.** *(Starting Point)* If \(R > R_0\), then the stability surface comes to a point at \((A_0, B_0, C_0) = \left(-\frac{R+1}{R}, \frac{R}{R-1}, \frac{1}{R(R-1)}\right)\), and Eqn. (3.1) is unstable for \(A < A_0\).

The proof of this theorem is given by Mahaffy *et al.* [29]. Following this starting point, there is a range of \(A\) values where the stability region is bounded exclusively by \(\Lambda_0\) and \(\Lambda_1\). Mahaffy *et al.* [29] showed that \(G_1\) intersects the real root-crossing plane, \(\Lambda_0\), at \(B = (AR + 1)/(1 - R)\) and \(C = -(A + 1)/(1 - R)\) as \(\omega \rightarrow 0^+\), which is the line

\[\frac{A + 1}{1 - R} = \frac{B - 1}{R} = -C\]  

(3.6) in \(ABC\) parameter space.

For a range of \(A\) values, \(A > A_0\), intersects \(\Lambda_0\) again with \(0 < \omega < \frac{\pi}{1 - R}\). (See Fig. 3.1b.) The next region of stability comes from a self intersecting bifurcation curve in the \(BC\)-plane. (See Fig. 3.1). Figure (3.1b) shows \(G_2\) self-intersecting and approaching \(G_1\) as \(A\) increases from \(A_0\). This disjoint region of stability in the \(BC\)-plane creates a stability spur, which joins the original region of stability at a particular value of \(A, A_1^*\).

**Definition 3.3.** *(Stability Spur)* If Bifurcation Surface \(j + 1\) self-intersects above the zero-root crossing plane as \(A\) increases, with the Cusp Point of Spur \(j\) denoted \(A_j^p\), then the
Figure 3.1: Transition, $A_1^\ast$. A sequence of illustrations (a-f) is presented of the stability boundary located in the $BC$ plane for $R = 0.25$ and $A$ values at the Starting Point (a), before (b,c,d), during (e) and after (f) the first spur-connecting transition, $A_1^\ast \approx -2.4184$. $\Gamma_1$ (blue), $\Gamma_2$ (green) and $\Gamma_0$ (purple) form the boundary of the stability region (Busken[8]).

quasi-cone-shaped stability spur has its cross-sectional area monotonically increase with $A$ until $A$ reaches a transitional value.

The first stability spur joins the main stability region bounded by $\Lambda_0$ and the $\Lambda_1$ plane at $A = A_1^\ast$, which we define to be the first transition (Figure 3.1e).

Definition 3.4. (Transition and Degeneracy Line) There are critical values of $A$ corresponding to where Eqns. (3.4) and (3.5) become indeterminate at $\omega = \frac{j\pi}{1-R}$. These transitional values of $A$ are denoted by $A_j^\ast$, where

$$A_j^\ast = -\frac{j\pi}{1-R} \cdot \cot\left(\frac{jR\pi}{1-R}\right), \quad j = 1, 2, \ldots.$$ (3.7)

At a transition, Curves $j$ and $(j + 1)$ coincide at the specific point $(B_j^\ast, C_j^\ast)$, where

$$B_j^\ast = (-1)^j \frac{(1-R) \cos\left(\frac{jR\pi}{1-R}\right) - jR^\pi \csc\left(\frac{jR\pi}{1-R}\right)}{(1-R)^2}$$
\[ C_j^* = -(-1)^j \frac{(1 - R) \cos \left( \frac{j\pi}{1 - R} \right) - j\pi \csc \left( \frac{j\pi}{1 - R} \right)}{(1 - R)^2}. \] (3.8)

All along the Degeneracy line, \( \Delta_j \),

\[ (B - B_j^*) + (-1)^j(C - C_j^*) = 0, \quad A = A_j^*, \] (3.9)

there are two roots of (3.2) on the imaginary axis with \( \lambda = \pm \frac{j\pi}{1 - R}i \). If \( \Lambda_j \) is on the boundary of the stability region for \( A \) slightly less than \( A_j^* \), then \( \Delta_j \) becomes part of the stability region’s boundary at Transition \( j \). After \( A_j^* \), \( \Lambda_{j+1} \) enters the boundary of the stability region.

Since Eqns. (3.7) and (3.8) with fixed \( R \) lead to a countably infinite number of transitions, some of them have a great influence over stability boundary change, but not all of them. There are only a few ways that the stability surface actually changes, and the two definitions for transferral and tangency provide descriptions for changes in the region of stability. Both tangencies and transferrals are situations where a new bifurcation surface enters the boundary of the stability region for larger \( A \), with most changes coming from tangencies.

**Definition 3.5. (Transferral and Reverse Transferral)** The transferral value of \( A = A_{i,j}^* \) is the value of \( A \) corresponding to the intersection of \( \Lambda_j \) (or \( \Gamma_j \)) with \( \Lambda_i \) (or \( \Gamma_i \)) at the \( \Lambda_0 \) plane with the \( \Lambda_j \) entering the boundary of the stability region for some range of \( A > A_{i,j}^* \). For some values of \( R \) the stability surface can undergo a reverse transferral, \( \tilde{A}_{j,i}^* \), which is a transferral characterized by \( \Lambda_j \) (or \( \Gamma_j \)) leaving the boundary, or a transferring back over to \( \Lambda_j \) (or \( \Gamma_j \)) the portion of the boundary originally taken by \( \Lambda_j \) (or \( \Gamma_j \)) at \( \Lambda_{i,j}^* \) (< \( \tilde{A}_{j,i}^* \))

Fig. 3.2 illustrates a transferral for \( R = 0.33 \) as \( A \) increases through \( A_{1,4}^* \approx 7.1 \). Here \( \Gamma_4 \) enters the boundary of the stability surface where \( \Gamma_1 \) intersects \( \Gamma_0 \) in the 4th quadrant.

**Definition 3.6. (Tangency and Reverse Tangency)** The value of \( A \) corresponding to the tangency of two surfaces \( i \) and \( j \) is denoted \( A_{i,j}^t \). \( \Lambda_j \) (or \( \Gamma_j \)) becomes tangent to \( \Lambda_i \) (or \( \Gamma_i \)), where \( \Lambda_i \) (or \( \Gamma_i \)) is a part of the stability boundary preliminary to \( A = A_{i,j}^t \). As \( A \) increases from \( A_{i,j}^t \), the new bifurcation surface, \( \Lambda_j \), is incorporated into the boundary of the stability region, separating segments of the bifurcation surface to which it was tangent. However, many times as \( A \) is increased \( \Lambda_j \) (or \( \Gamma_j \)), the same surface (curve), which entered the boundary through tangency \( A_{i,j}^t \), can be seen leaving the stability boundary via a reverse tangency, denoted \( \tilde{A}_{i,j}^t \).

Fig. 3.3 illustrates a tangency for \( R = 0.33 \) as \( A \) increases through \( A_{6,10}^t \approx 88 \). Here \( \Gamma_{10} \) (green) enters the boundary of stability by way of the tangency, \( A_{6,10}^t \), for the stability surface associated with \( R = 0.33 \). Prior to the tangency Fig. 3.3a, \( \Gamma_{10} \) is away from the boundary, and portions of \( \Gamma_1 \) (blue), \( \Gamma_2 \) (green), \( \Gamma_4 \) (red), and \( \Gamma_6 \) (green) can be found along
the stability boundary. As $A$ approaches $A_{6,10}$, $\Gamma_6$ (green) and $\Gamma_{10}$ (green) approach one another until becoming tangent Fig. 3.3b. Afterwards Fig. 3.3c, $\Gamma_{10}$ takes on a portion of stability boundary the between its two intersection with $\Gamma_6$.

The main focus of this thesis is the study of global asymptotic change in the stability surface for rational $R$. When the delay $R$ is rational, the curves defined by Eqn. (3.8) can be classified into families of bifurcation curves with similar characteristics. Since certain values of the delay have only a small number of families, the curves participating on the boundary of the stability become easy to determine. By the continuity of the characteristic equation, values of $R$ near a particular value, which has a small number of families, should exhibit similar regions of stability.

Because stability surfaces are given by the periodic nature of the sinusoidal functions, which include $R$, the bifurcation curves are classified in individual families of curves for each value of $R$.

**Definition 3.7. (Families of bifurcation curves)** For $A$ fixed, take rational $R = \frac{k}{n}$ and $j = n - k$. From Eqns. (3.4) and (3.5), one can see that the singularities occur at $\frac{n\pi}{j}, i = 0, 1, \ldots$. $\Lambda_i$ with $\frac{n(i-1)\pi}{j} < \omega < \frac{n\pi}{j}$ satisfies:

$B_i(\omega) = A \sin\left(\frac{kw}{n}\right) + \omega \cos\left(\frac{kw}{n}\right) / \sin\left(\frac{\omega}{n}\right)$, $C_i(\omega) = -A \sin(\omega) + \omega \cos(\omega) / \sin\left(\frac{\omega}{n}\right)$

Now consider $\Lambda_{i+2}$ with $\mu = \omega + 2n\pi$, then

$B_{i+2j}(\mu) = A \sin\left(\frac{kw}{n}\right) + \mu \cos\left(\frac{kw}{n}\right) / \sin\left(\frac{j\omega}{n}\right) = A \sin\left(\frac{kw}{n}\right) + (\omega + 2n\pi) \cos\left(\frac{kw}{n}\right) / \sin\left(\frac{j\omega}{n}\right)$

$C_{i+2j}(\mu) = -A \sin(\omega) + (\omega + 2n\pi) \cos(\omega) / \sin\left(\frac{j\omega}{n}\right)$

These equations show that $B_{i+2j}(\mu)$ follows the same trajectory as $B_i(\omega)$ with a shift of $2n\pi \cos\left(\frac{kw}{n}\right) / \sin\left(\frac{j\omega}{n}\right)$ for $\omega \in \left(\frac{(j-1)\pi}{1-R} : \frac{j\pi}{1-R}\right)$, while $C_{i+2j}(\mu)$ follows the same trajectory as $C_i(\omega)$ with a shift of $2n\pi \cos(\omega) / \sin\left(\frac{j\omega}{n}\right)$ over the same values of $\omega$. This related behavior of bifurcation surfaces separated by $\omega = 2n\pi$ creates $2j$ families of curves in the $BC$-plane for fixed $A$.

Thus, there is a quasi-periodicity among the bifurcation surfaces when $R$ is rational. For example, when $R = \frac{1}{2}$, there are two families of curves within the $BC$-plane. When $R = \frac{1}{3}$ and $R = \frac{1}{4}$, four and six families are established, respectively. Fig. (3.3) illustrates the four family structure near $R = \frac{1}{3}$. As $A$ becomes larger, it is especially important to classify bifurcation curves for each family, when we want to know the importance of the asymptotic structure of the stability region.
Figure 3.2: The stability boundary for $R = 0.33$ located in the $BC$ plane before, during and after the first transferral is given by the closed region enveloping the MRS (bold dashed line). The closed region is formed by the $\Gamma_0$ plane (purple) and $\Gamma_1$ (blue), $\Gamma_2$ (green), $\Gamma_3$ (black) and $\Gamma_4$ (red). This transferral shows $\Gamma_4$ entering the boundary of the stability region.

Figure 3.3: The stability boundary in the $BC$ plane for $R = 0.33$ is given during and after $A_{6,10}^r$. The color scheme for the curves is: $\Delta_0$ (purple). The first families, $\Gamma_1$, $\Gamma_5$ and $\Gamma_9$ (blue). The second family $\Gamma_2$, $\Gamma_6$ and $\Gamma_{10}$ (green). The third family, $\Gamma_3$, $\Gamma_7$ and $\Gamma_{11}$ (black). The fourth family $\Gamma_4$, $\Gamma_8$ and $\Gamma_{12}$ (red). This tangency shows $\Gamma_{10}$ entering the boundary of the stability region.
CHAPTER 4
ANALYTICAL RESULTS

The two delay differential equation (3.1) has a complex region of stability in the $ABC$-parameter space for specific values of $R$. Previous work has suggested that when $R$ is rational, then the region of stability is increased. The numerical studies of Busken [8] have shown this increased region of stability and given some insight as to why this is true. This thesis extends his results and provides some analytical proofs for the extension of the region of stability for $R$ in the rational form $\frac{1}{n}$.

The goal of our study is to analytically prove that the asymptotic stability region for $R = \frac{1}{n}$ reduces to only four boundary curves as $A \to +\infty$. The four boundary curves for $R \to \frac{1}{n}$ from below are $\Gamma_1, \Gamma_{n-1}, \Delta_{n-1}$ and the zero root crossing $\Gamma_0$. This limited set of curves results in an enlarged region of stability beyond the MRS, and its simple form allows numerical computation of how much larger the stability region becomes.

4.1 Example $R = \frac{1}{3}$

To illustrate our study we concentrate on the case $R = \frac{1}{3}$ and note that arguments generalize to other delays. For $R = \frac{1}{n}$, Definition 3.7 gives $2(n-1)$ distinct family of bifurcation curves, which implies $R = \frac{1}{3}$ has four distinct family members. The key bifurcation curves on the boundary of the stability region asymptotically, as $A^* \to +\infty$ with $R \to \frac{1}{3}$ from below, are $\Gamma_1$ and $\Gamma_2$ with $\Gamma_0$ and $\Delta_2$, creating the other two boundaries. $\Gamma_1$ and $\Gamma_2$ are clearly the first bifurcation curves of the first and second families, while $\Delta_2$ arises from the singular point $\omega = \frac{2\pi}{1-R}$ between $\Gamma_2$ and $\Gamma_3$.

Fig. 4.1 shows 16 bifurcation curves for $R = 0.33$ at $A^*_2 = 199.853$. By continuity of the characteristic equation (3.2), we expect similarities between $R = \frac{1}{3}$ and $R = 0.33$, particularly for $A < A^*_2$. Fig. 4.1 shows the four family structure we predict for $R = \frac{1}{3}$ (Note that $R = 0.33$ is predicted to have 134 families by Definition 3.7, which will ultimately result in a much closer approach of the stability region to the MRS as $A \to +\infty$). The figure shows the distinct ordering of family members within the four families and the characteristic pattern of each of the four families. The coloring pattern in Fig. 4.1 follows $\Gamma_0$ in purple and then the four successive families in blue, green, black, and red.

The Fig. 4.1a shows the organization of the first family with all members lying outside the boundary of the region of stability with each successive member further away. This Fig. 4.1a also shows the 4th family, which parallels the first family along the boundary in the
Figure 4.1: Four bifurcation curves for each of the four families ($\Gamma_1, \cdots, \Gamma_{16}$), $\Gamma_0$, and $\Delta_2$ for $R = 0.33$ at $A_{2m}^* = 199.853$ with close-ups at the corners of the stability region. The color scheme for the curves is: $\Gamma_1, \Gamma_5, \Gamma_9$, and $\Gamma_{13}$ are blue. $\Gamma_2, \Gamma_6, \Gamma_{10}$, and $\Gamma_{14}$ are green. $\Gamma_3, \Gamma_7, \Gamma_{11}$, and $\Gamma_{15}$ are black. $\Gamma_4, \Gamma_8, \Gamma_{12}$, and $\Gamma_{16}$ are red. $\Gamma_0$ and $\Delta_2$ are purple and dashed red, respectively.
In the 1st quadrant, the primary boundary is the degenerate line \((\Delta_2)\) formed at \(A^*_2\) at the conjunction of \(\Gamma_2\) and \(\Gamma_3\). Later in this chapter we prove that as \(R \to \frac{1}{3}\) this degenerate line approaches the line \(C - B = A^*_2\), which is visible in Fig. 4.1. Since \(R = 0.33 < \frac{1}{3}\), there is a small gap between the degenerate line and the boundary of the MRS, so we see a small segment of \(\Gamma_1\) on the boundary of the stability region intersecting \(\Delta_2\) which is visible in Fig. 4.1b. We see that outside the tiny segment of \(\Gamma_1\) (right of the MRS), the first and third families are outside the stability region running parallel to the lines \(C = \pm B\) through the 1st and 4th quadrants.

In the 1st quadrant, then diverges opposite the first family below \(\Gamma_0\). Thus, in the 4th quadrant the boundary of the stability region consists only of the curves from \(\Gamma_0\) and \(\Gamma_1\).

Numerically, our MatLab programs allow us to observe the stability region for any delay \(R\) slightly less than \(\frac{1}{3}\) at \(A^*_2\). The boundary of the stability region is almost identical to Fig. 4.1, except that \(A^*_2\) increases with \(R \to \frac{1}{3}\), expanding the scales of the \(B\) and \(C\) axes. To further understand the evolution of this stability surface as \(R \to \frac{1}{3}\) from below, we detail how Fig. 4.2 can be used to explain the process. As noted earlier, the key elements causing the bulges in the stability region are the limited number of families of bifurcation curves and the transitions, which distort the boundary. At \(R = \frac{1}{3}\), all the transitions \(A^*_{2n} \to +\infty, n = 1, 2, \ldots\), which is easily verified by Definition 3.4. Furthermore, it can be shown using perturbation analysis that the transitions have a distinct ordering over a limited range,

\[
A^*_2 > A^*_4 > \ldots > A^*_{2n} > A^*_{2(n+1)} > \ldots
\]

for delays \(R < \frac{1}{3}\). (Details of this proof are include later, but depend on \(n\) and \(R\), as would be expected.) This sequence allows one to readily determine changes to the boundary of the stability region.

The summary Table 4.1 shows the complete evolution of the stability surface for \(R = 0.33\) from its beginning at \(A_0 \approx -4.030\) until \(A^*_2 \approx 199.853\). We call attention to the sequence of reverse tangencies and reverse transferral for \(A < A^*_2 \approx 199.853\). These events all occur just prior to one of the transitions, \(A^*_{2n}\). The transition \(A^*_2\) creates the degeneracy line, \(B + C = B^*_2 + C^*_2\) at \(\omega = \frac{2\pi}{1-R}\), which becomes part of the boundary of the stability region.
Figure 4.2: Bifurcation curves: The four curves on the boundary of the stability region, for $R = 0.33$ at $A_2^* = 199.853$ are $\Gamma_0$ (violet), $\Gamma_1$ (blue), $\Gamma_2$ (green), and $\Delta_2$ (dashed).

<table>
<thead>
<tr>
<th>surface change</th>
<th>A</th>
<th>reverse tangency</th>
<th>$A_{10,6}^t \approx 170.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$-4.030$</td>
<td>reverse tangency</td>
<td>$A_{8,4}^t \approx 186$</td>
</tr>
<tr>
<td>$A_1^*$</td>
<td>$A_1^* \approx -0.10995$</td>
<td>reverse tangency</td>
<td>$A_{6,2}^t \approx 194$</td>
</tr>
<tr>
<td>transferral</td>
<td>$A_{1,4}^z \approx 7.1$</td>
<td>reverse tangency</td>
<td>$A_{4,1}^z \approx 198.8$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{1,6}^t \approx 30.7$</td>
<td>reverse transferral</td>
<td>$A_{6,10}^t \approx 88$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{4,8}^t \approx 57.5$</td>
<td>$A_2^*$</td>
<td>$A_5^* \approx 199.853$</td>
</tr>
</tbody>
</table>

Table 4.1: List of 2D boundary changes for $R = 0.33$ and $A \in [A_0, A_2^*]$.

Fig. 4.3 shows that the transition, $A_1^* = 199.41$, results in a degeneracy line, $\Delta_4$, passing through the point $(B_4^*, C_4^*) \approx (100.41, -300.43)$ with the line parallel and below $\Gamma_0$. At $A_4^*$, $\Gamma_1$ intersects $\Gamma_0$ at $(B, C) \approx (99.71, -299.12)$, which lies closer to the stability region than $(B_4^*, C_4^*)$. Thus, the transition pulls $\Gamma_4$ and $\Gamma_5$ outside the stability region, swapping the direction in which the curves go to infinity. This gives $\Gamma_5$ a flow paralleling $\Gamma_1$ and $\Gamma_0$ and maintaining its position outside the stability region. This distortion from the transition $A_4^*$ results in the reverse transferral $\tilde{A}_{i,1}^z \approx 198.8$, just prior to the transition.

In a similar fashion, $A_6^* \approx 198.68$ creates the degeneracy line, $\Delta_6$, which passes through $(B_6^*, C_6^*) \approx (-100.31, 300.92)$ and is parallel to $\Delta_2$. We note that $\Delta_2$ passes through $(B_2^*, C_2^*) \approx (-100.34, 300.11)$. Again, $(B_6^*, C_6^*)$ is outside the stability region pulling $\Gamma_6$ and
Figure 4.3: Bifurcation curves: Three and two bifurcation curves for the first (blue) and fourth (red) families, respectively, for $R = 0.33$ at $A^*_4 = 199.413$ are shown, including the degeneracy line, $\Delta_4$ (dashed red), between $\Gamma_4$ (red) and $\Gamma_5$ (blue) and $\Gamma_0$ (purple).
$\Gamma_7$ to their position paralleling but outside $\Gamma_2$ and $\Gamma_3$ in the second quadrant. Subsequently, $\Gamma_7$ falls in order with other members of the third family. The 3$^{rd}$ family (black) lined sequentially outside the region of stability for $\Gamma_{11}$, $\Gamma_{15}$, $\Gamma_{19}$, $\cdots$ with the 2$^{nd}$ family (green) paralleling $\Gamma_2$ along the upper left boundary of the stability region, then moving away, except for $\Gamma_2$, which is seen with the black curves of Fig. 4.4. The distortion from $A^\ast_6$ and reordering of the curves is what produces the reverse tangency, $\tilde{A}_{6,2}^t \approx 194$, simplifying the composition of the boundary of the stability region from 5 curves to only 4 curves. This reverse tangency causes $\Gamma_6$ to leave the region of stability. Thus, at $\tilde{A}_{6,2}^t \approx 194$, the boundary of stability region has its largest segment composed of $\Gamma_2$, which in the counter clockwise direction intersects $\Gamma_0$. The line $\Gamma_0$ then intersects a small segment of $\Gamma_4$, which the intersects a segment of $\Gamma_1$. Finally, the boundary is closed with $\Gamma_1$ intersecting $\Gamma_2$ in the 1$^{st}$ quadrant.

Figure 4.4: Bifurcation curves: Three and four bifurcation curves for the second (green) and third (black) families, respectively, for $R = 0.33$ at $A^\ast_6 = 198.679$ are shown, including the degeneracy line, $\Delta_6$ (red dashed), between $\Gamma_6$ and $\Gamma_7$. Note that $\Gamma_2$ creates the boundary of the stability region in this diagram, and constructing this portion of the boundary $\tilde{A}_{6,2}^t \approx 194$ has recently occurred, removing $\Gamma_6$ from the boundary of the stability region.
At $A_8^* \approx 197.65$, $\Delta_8$ passes through $(B_8^*, C_8^*) \approx (-100.55, -301.73)$, which is parallel to $\Gamma_0$. Now, $(B_8^*, C_8^*)$ is outside the stability region pulling $\Gamma_9$ to a position paralleling, but outside $\Gamma_5$. This pulls these curves outside the stability region, which earlier resulted in the reverse tangency $\tilde{A}_t^{10,4} \approx 186$ and $\Gamma_8$ leaving the boundary of the stability region. Figure 4.5 shows the boundary of the stability region at $A_8^* \approx 197.65$, and it consists of the four curves, $\Delta_0$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_4$.

![Figure 4.5: Bifurcation curves: Four and five bifurcation for the first (blue) and fourth (red) families, for $R = 0.33$ at $A_8^* = 197.65$ are shown, including the degeneracy line between $\Gamma_8$ and $\Gamma_9$(dashed red) and $\Delta_0$ (purple). Note that $\Gamma_1$ and $\Gamma_4$ remain close to the boundary of the stability region with $\Delta_0$, $\Gamma_1$ and $\Gamma_4$ constructing this portion of the boundary. $\tilde{A}_t^{10,4}$ recently occurred, removing $\Gamma_8$ from the boundary of the stability region.](image)

Fig. 4.6 shows the boundary of the stability region at $A_{10}^* \approx 196.32$, when $\Delta_{10}$ passes through $(B_{10}^*, C_{10}^*) \approx (-100.87, 302.71)$, which is parallel to $\Delta_6$. This pulls $\Gamma_{10}$ and $\Gamma_{11}$ outside the stability region and earlier resulted in the reverse tangency $\tilde{A}_t^{10,6} \approx 170.6$ with $\Gamma_{10}$ leaving the boundary of the stability region. The distortion from $A_{10}^*$ and reordering of the
Figure 4.6: Bifurcation curves: Four curves for the second and third families, respectively, for $R = 0.33$ at $A_{10}^* = 196.32$ are shown, including the degeneracy line between $\Gamma_1$ and $\Gamma_2$. Note that $\Gamma_2$ and $\Gamma_6$ remain close to the boundary of the stability region with $\Gamma_2$ constructing this portion of the boundary.
curves is what produces the reverse tangency $\tilde{A}_{10,6}$, simplifying the composition of the boundary of the stability region. Fig. 4.7 shows the boundary of stability region at the reverse tangency $\tilde{A}_{10,6}^t \approx 170.6$. This figure shows the upper corner in the 2$\textsuperscript{nd}$ quadrant of the stability region with only $\Gamma_2$ and $\Gamma_6$ on the boundary.

Figure 4.7: Bifurcation curves: Three curves for each the second (green) and third (black) families, respectively, for $R = 0.33$ at $\tilde{A}_{10,6}^t$ are shown. As $A$ is increased, the same curves, which entered the boundary through tangency $A_{6,10}^t$, can be seen leaving the stability boundary via a reverse tangency, $\tilde{A}_{10,6}^t$.

As seen in Table 4.1, there is an alternating pattern of reverse tangencies as we progress to higher values of $A$, and each reverse tangency simplifies the boundary of the stability region and organizes the families into the ones seen in Fig. 4.1 because of one of the $A_{2n}^*$ transitions. This same sequence of events occurs for each $R < \frac{1}{3}$ as $R \to \frac{1}{3}$ (sufficiently close) with more tangencies and reverse tangencies before $A_{2}^*$ as $R \to \frac{1}{3}$ from below and $A_{2}^*$ getting larger. Thus, the geometric orientation of the curves and the sequence of reverse
tangencies and transferrals are virtually identical to figures shown with \( R = 0.33 \), except the \( B \) and \( C \) scales increase as \( R \rightarrow \frac{1}{3} \). The continuity of the characteristic equation show that all delays \( R < \frac{1}{3} \), yet sufficiently close to \( R = \frac{1}{3} \), will generate a simplified stability region very similar to Fig. 4.2 at \( A_2^* \). As \( R \rightarrow \frac{1}{3} \) from below, the degeneracy line gets closer to the MRS, and the region of stability becomes more symmetric.

## 4.2 Analysis for Enlarged Stability Region

The previous section provided geometric arguments to illustrate how the region of stability for \( R = \frac{1}{3} \) is enlarged. In this section we provide analytic proofs that advance our arguments from the previous section toward the goal of proving the simple structure of the enlarged stability region for delays in the form of \( R = \frac{1}{n} \). We begin this section with a summary of results from work of Mahaffy and Busken [27].

Mahaffy and Busken [27] divided analysis into \( R = \frac{1}{2n} \) and \( R = \frac{1}{2n+1} \) cases, which have distinct geometric shapes. The previous section with \( R = \frac{1}{3} \) showed the evolution of the stability surface, which simplified to four curves. Fig. 4.2 is the generic shape for any delay of the form \( R = \frac{1}{2n+1} \). \( \Gamma_0 \) creates the lower boundary for region of stability. The other boundaries are successively in counter-clockwise direction \( \Gamma_1, \Delta_{2n}, \) and \( \Gamma_{2n} \). These boundaries have been shown numerically and illustrated in the previous section. Our goal is ultimately to prove these are the only curves asymptotically as \( R \rightarrow \frac{1}{2n+1} \) from below. Mahaffy and Busken [27] proved the following result about these curves.

**Lemma 4.1.** For \( R = \frac{1}{2n+1} \), one boundary of the region of stability is the limiting line

\[
B + C = A_{2n}^*,
\]

which lies on the MRS.

**Proof.** This proof is shown in Mahaffy and Busken [27].

As we mention before, for any delay, \( R, \Gamma_0 \) is always one part of the stability boundary, appearing in the 3rd quadrant of the \( BC \)-plane along the MRS with \( B + C = -A \). Fig. 4.2 shows that \( \Delta_{2n} \) approaches the MRS on the upper boundary of the stability region. This lemma shows that as \( R \rightarrow \frac{1}{2n+1} \), the limiting boundary from \( \Delta_{2n} \) matches the MRS symmetric and opposite \( \Gamma_0 \). Eqn. (3.6) describes the intersection of \( \Gamma_0 \) and \( \Gamma_1 \), which provides the lowest point on the region of stability for \( R \rightarrow \frac{1}{2n+1} \) from below at \( A_{2n}^* \). With \( R = \frac{1}{2n+1} \), Eqn. (3.6) gives this intersection point as
(B, C) = \left(\frac{A + 2n + 1}{2n}, -\frac{(2n + 1)(A + 1)}{2n}\right).

This shows the lower edge of the stability region for \( R = \frac{1}{2n+1} \) lengthening by a factor of \( \frac{1}{2n} \) times the length of a edge of the MRS. Thus, the region of stability is increased, provided this point creates a corner of the stability region. The lemma below provides the highest point of the region of stability.

**Lemma 4.2.** For \( R < \frac{1}{2n+1} \) and \( R \to \frac{1}{2n+1} \), the bifurcation curve \( \Gamma_{2n} \) comes to the point

\[
(B^{*}_{2n}, C^{*}_{2n}) = \left(-\frac{A^{*}_{2n} + 2n + 1}{2n}, \frac{(2n + 1)(A^{*}_{2n} + 1)}{2n}\right)
\]

with \( A^{*}_{2n} \to +\infty \).

**Proof.** This proof is shown in Mahaffy and Busken [27].

Lemma 4.2 shows that as \( R \to \frac{1}{2n+1} \) from below, and \( A^{*}_{2n} \to +\infty \), the point \((B^{*}_{2n}, C^{*}_{2n})\) tends toward a value symmetric with the origin to the intersection of \( \Gamma_1 \) and \( \Lambda_0 \).

The next step in our analysis is to show that bifurcation curves, \( \Gamma_1 \) and \( \Gamma_{2n} \), pass arbitrarily close to the point \((A^{*}_{2n}, 0)\) and \((-A^{*}_{2n}, 0)\) in the \( BC\)-plane as \( R \to \frac{1}{2n+1} \). The next lemma proves the asymptotic limit, giving more information about the symmetric shape of the region.

**Lemma 4.3.** For \( R < \frac{1}{2n+1} \) and \( R \to \frac{1}{2n+1} \), the bifurcation curves, \( \Gamma_1 \) and \( \Gamma_{2n} \), pass arbitrarily close to the points \((A^{*}_{2n}, 0)\) and \((-A^{*}_{2n}, 0)\), respectively, in the \( BC\)-plane with \( A^{*}_{2n} \to +\infty \).

**Proof.** This proof is shown in Mahaffy and Busken [27].

Lemma 4.3 shows that the stability region for \( R = \frac{1}{2n+1} \), asymptotically finds \( \Gamma_{2n} \) mirroring \( \Gamma_1 \) in the 2\(^{nd}\) quadrant. \( \Gamma_{2n} \) and \( \Gamma_1 \) cross the \( B \) axis at \(-A^{*}_{2n}\) and \( A^{*}_{2n}\), respectively, which creates two sides of the edge of the stability region when \( R \to \frac{1}{2n+1} \). Note that \((A^{*}_{2n}, 0)\) and \((-A^{*}_{2n}, 0)\) coincide with the point of the MRS at \( A = A^{*}_{2n} \) so \( \Gamma_{2n} \) and \( \Gamma_1 \) create the left and right sides of the boundary of the stability region. The above lemma shows that the two corners determining the stability boundary of the region are symmetric with respect to \( C \) axis.

All of the above lemmas deal with the boundary curves for the stability region. Lemma 4.1 shows that when \( R \to \frac{1}{2n+1} \) from below, there is a transition \( A^{*}_{2n} \to \infty \), which results in a degeneracy line at \( \omega = \frac{2\pi}{1-R} \) that approaches \( B + C = A^{*}_{2n} \), and is parallel to \( \Lambda_0 \) and lies on the opposite side of the MRS. Lemma 4.2 shows that as \( R \to \frac{1}{2n+1} \) from below and \( A^{*}_{2n} \to +\infty \), the point \((B^{*}_{2n}, C^{*}_{2n})\) tends toward an ordered pair which has odd symmetry to the point of intersection of \( \Gamma_0 \) and \( \Gamma_1 \), given by Eqn. (3.6). Finally, Lemma 4.3 shows the other ends of the segments on the boundary of the stability region composed of \( \Gamma_1 \) and \( \Gamma_{2n} \).
approach the MRS at the point \((A_{2n}^*, 0)\) and \((-A_{2n}^*, 0)\), respectively. These facts provide strong evidence that as \(R \to \frac{1}{2n+1}\), the stability region is bounded by just four bifurcation curves, \(\Gamma_0, \Gamma_1, \Gamma_{2n}, \) and \(\Delta_{2n}\), as seen in Fig 4.2.

This thesis provides more analytic details to support the simplicity of the boundary of the stability region for \(R \to \frac{1}{2n+1}\) at \(A_{2n}^*\), given generically in Fig 4.2. Our proofs are designed to show how transitions cause bifurcation curves to become ordered outside the stability region, so that only the first member of the families with \(\Gamma_1\) and \(\Gamma_{2n}\) fall on the boundary of the stability region.

![Figure 4.8: A summary of boundary changes, for \(R \in [0.2, 0.4]\) and \(A \leq 200\) (Busken[8]).](image)

This chapter began with a detailed discussion of different events near \(A_{2n}^*\) for \(R = \frac{1}{3}\). Fig. 4.8 summarizes the ordering of key events; transitions, tangencies, and transferrals for a range of \(R\) values. We observed that transitions occurred with a distinct ordering, and these transitions pulled bifurcation curves outside the stability region via reverse tangency or reverse transferral. This sequential ordering of events and distinctive patterns of the families of bifurcation curves gives us confidence in our conjecture that the stability region reduces to just four bifurcation curves a \(A_{2n}^*\) for \(R \to \frac{1}{2n+1}\).
In this thesis we analytically prove several lemmas to further support the conjecture of the simple boundary for \( R \to \frac{1}{2n+1} \) at \( A_{2n}^* \). We concentrate on the case \( R = \frac{1}{3} \). Our first lemma proves the ordering of transitions, which allows sequential consideration of events ultimately resulting in the simplified boundary of the stability region. Next we prove the ordering of the key points \((B_{2n}^*, C_{2n}^*)\) at a transition and demonstrate how this aligns families of bifurcation curves outside the region of stability. As shown earlier in this chapter, these transition points alternatively draw bifurcation curves outside the region of stability through reverse tangencies or a reverse transferral. Furthermore, we observe that the families of bifurcation curves align in a particular order, leaving only the first member on the boundary of the stability region, as seen in Fig 4.1.

Our first lemma shows the ordering of the transitions from the left as \( R \to \frac{1}{3} \). This lemma provides the motivation for our sequential study of important events occurring on the boundary of the stability region.

**Lemma 4.4.** Fix \( N (\varepsilon) \), a positive even integer. Choose \( \varepsilon \) such that for \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \),

\[
A_N^* (R) = - \left( \frac{N \pi}{1 - R} \right) \cot \left( \frac{NR \pi}{1 - R} \right)
\]

is continuous in \( R \) with \( \cot \left( \frac{NR \pi}{1 - R} \right) < 0 \). From Eqn (3.7) with \( j = 1, 2, \ldots \) and \( 2j \leq N \) we have

\[
A_2^* (R) > A_4^* (R) > \cdots > A_{2j}^* (R)
\]

for each \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \).

**Proof.** By Eqn. (3.7),

\[
A_{2j}^* (R) = - \left( \frac{2j \pi}{1 - R} \right) \cot \left( \frac{2j R \pi}{1 - R} \right).
\]

We show that \( A_{2j}^* \) is a decreasing sequence for \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \). Let \( t = \frac{R \pi}{1 - R} \) or \( R = \frac{t}{\pi + t} \). It follows that

\[
A_{2j}^* (t) = -2j (\pi + t) \cdot \cot (2tj).
\]

For \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \), it follows that \( t \in \left( \frac{(1 - 3\varepsilon) \pi}{2 + 3\varepsilon}, \frac{\pi}{2} \right) \). We prefer to analyze the cotangent fraction near zero, so we let \( k = \frac{\pi}{2} - t \) with \( k > 0 \). Thus, we have

\[
A_{2j}^* (k) = -2j \left( \frac{3\pi}{2} - k \right) \cdot \cot \left( \left( \frac{\pi}{2} - k \right) 2j \right) = -j (3\pi - 2k) \cdot \cot (\pi j - 2k j).
\]
Let \( C(k) = \frac{1}{A_{2j}} \) then
\[
C(k) = -\frac{1}{j(3\pi - 2k)} \cdot \tan(\pi j - 2kj) \\
= \frac{1}{j(3\pi - 2k)} \cdot \tan(2kj) \\
= \frac{2k}{3\pi - 2k} \cdot \tan(2kj).
\]

If we define \( f(x) = \frac{\tan(x)}{x} \), then \( f'(x) = \frac{x - \sin(x) \cos(x)}{x^2 \cos^2(x)} \). For \( 0 < x < \frac{\pi}{2}, x > \sin(x) \) and \( \cos(x) < 1 \). It follows that \( f'(x) > 0 \). Thus, \( C(j) \) is positive and increasing for \( k < \frac{3\pi}{2} \) and \( 0 < 2kj < \frac{\pi}{2} \) or \( 0 < j < \frac{\pi}{4k} \).

From the calculation above, \( A_{2j}^* (R) \) is decreasing in \( j \) provided \( j < \frac{\pi}{4k} \).
Since \( k = \frac{\pi}{2} - t \) and \( t = \frac{R\pi}{1-R} \),
\[
j < \frac{\pi}{4(\frac{\pi}{2} - t)} \frac{1 - R}{2(1 - 3R)}.
\]
For \( R \in (\frac{1}{3} - \varepsilon, \frac{1}{3}) \),
\[
\frac{1 - R}{2(1 - 3R)} > \frac{1 - (\frac{1}{3} - \varepsilon)}{2(1 - 3(\frac{1}{3} - \varepsilon))} = \frac{1}{9\varepsilon} + \frac{1}{6}.
\]
It follows that \( A_{2j}^* (R) \) is decreasing in \( j \), if \( j < \frac{1}{9\varepsilon} + \frac{1}{6} \) or equivalently
\[
\varepsilon < \frac{2}{3} \cdot \frac{1}{6j - 1}
\]
Thus, given \( N \), if we take \( \varepsilon < \frac{2}{3} \left( \frac{1}{3N - 1} \right) \), then for each \( R \in (\frac{1}{3} - \varepsilon, \frac{1}{3}) \), \( A_{2j}^* (R) > \cdots > A_{2j}^* (R) \) with \( 2j \leq N \).

Fig. 4.1a shows the organization of the 1st and 4th families outside the region of stability. As noted before, the intersection of \( \Gamma_0 \) and \( \Gamma_1 \) provides the lowest point at \( A_{2j}^* \) for the region of stability as \( R \to \frac{1}{3} \) from below. The previous lemma shows the sequential ordering of transitions, which create this significant bulge in the stability region. Below we provide two lemmas, which prove the ordering of the 1st and 4th families outside the region of stability for \( R \in (\frac{1}{3} - \varepsilon, \frac{1}{3}) \) at \( A_{2j}^* \), for at least the sequence of transition points \( (B_{4n}^*, C_{4n}^*) \)
Lemma 4.5. Fix $N$ $(\varepsilon)$, a positive integer. Choose $\varepsilon$ such that $R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right)$, $A_N^*(R)$ is continuous in $R$ with $\cot \left(\frac{N \pi R}{1 - R}\right) < 0$. From Eqn. (3.8) with $j = 1, 2, \cdots$ and $4j \leq N$, we have

$$B^*_4(R) < B^*_8(R) < B^*_{12}(R) < \cdots < B^*_{4j}(R)$$

for each $R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right)$.

Proof. From Eqn. (3.8),

$$B^*_j(R) = (-1)^j \cdot \frac{(1 - R) \cos \left(\frac{j \pi R}{1 - R}\right) - j R \pi \csc \left(\frac{j \pi R}{1 - R}\right)}{(1 - R)^2}.$$ 

Let $t = \frac{R \pi}{1 - R}$ or $R = \frac{t}{\pi + t}$. It follows that

$$B^*_j(t) = (-1)^j \cdot \frac{(1 - \frac{t}{\pi + t}) \cos (j t) - j \frac{t}{\pi + t} \pi \csc (j t)}{(1 - \frac{t}{\pi + t})^2}$$

$$= (-1)^j \left[\frac{(\pi + t) \cos (j t) - (\pi + t)jt \csc (j t)}{\pi}\right].$$

We need only $B^*_4(R)$ for $j = 1, 2, \cdots$, so

$$B^*_4(t) = \frac{(\pi + t) \cos (4jt) - (\pi + t)4jt \csc (4jt)}{\pi}.$$  

As easy calculation shows that $t$ is slightly less than $\frac{\pi}{2}$, so let $k = \frac{\pi}{2} - t$ or $t = \frac{\pi}{2} - k$, then

$$B^*_4(k) = \left(\frac{3\pi}{2} - k\right) \cos \left(4j \left(\frac{\pi}{2} - k\right)\right) - \left(\frac{3\pi}{2} - k\right)4j \left(\frac{\pi}{2} - k\right) \csc \left(4j \left(\frac{\pi}{2} - k\right)\right)$$

$$= \left(\frac{3}{2} - \frac{k}{\pi}\right) \left[\cos (4jk) + (2j \pi - 4jk) \frac{1}{\sin (4jk)}\right].$$

Since $k \approx 0$, $B^*_4(k)$ becomes

$$B^*_4(k) \approx \frac{3}{2} \left[\cos (4jk) + \frac{\pi}{2k \sin (4jk)}\right].$$

However, $k$ is small, so the second term dominates. We know $\frac{x}{\sin (x)}$ is monotonically increasing for $0 < x < \pi$, so $\frac{4jk}{\sin (4jk)}$ is monotonically increasing for $0 < 4jk < \pi$. Thus, $B^*_4(k)$ is increasing for $0 < j < \frac{\pi}{4k}$. It follows that

$$0 < j < \frac{\pi}{4k} = \frac{\pi}{4 \left(\frac{\pi}{2} - t\right)} = \frac{\pi}{2 \pi - 4 \left(\frac{R \pi}{1 - R}\right)} = \frac{1 - R}{2(1 - 3R)}.$$
For \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \),
\[
\frac{1 - R}{2(1-3R)} > \frac{1 - \left( \frac{1}{3} - \varepsilon \right)}{2 \left( 1 - 3 \left( \frac{1}{3} - \varepsilon \right) \right)} = \frac{1}{9\varepsilon} + \frac{1}{6}.
\]
Thus, \( B_j^* (R) \) is increasing if \( j < \frac{1}{9\varepsilon} + \frac{1}{6} \) or equivalently
\[
\varepsilon < \frac{2}{3} \cdot \frac{1}{6j - 1}.
\]

Given \( N \), if we take \( \varepsilon < \frac{2}{3} \left( \frac{2}{3N-2} \right) \), then for each \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \), \( B_4^* (R) < \cdots < B_{4j}^* (R) \) with \( 4j \leq N \).

\[ \square \]

**Lemma 4.6.** Fix \( N (\varepsilon) \), a positive integer. Choose \( \varepsilon \) such that \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \), \( A_*^N (R) \) is continuous in \( R \) with \( \cot \left( \frac{N\pi R}{1-R} \right) < 0 \). From Eqn. (3.8) with \( j = 1, 2, \cdots \) and \( 4j \leq N \), we have
\[
C_4^* (R) > C_8^* (R) > C_{12}^* (R) > \cdots > C_{4j}^* (R)
\]
for each \( R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right) \).

**Proof.** From Eqn. (3.8),
\[
C_j^* (R) = -(-1)^j \cdot \left[ \frac{(1 - R) \cos \left( \frac{j\pi}{1-R} \right) - j\pi \csc \left( \frac{j\pi}{1-R} \right)}{(1 - R)^2} \right].
\]
Let \( s = \frac{\pi}{1-R} \) or \( R = \frac{s-\pi}{s} \). It follows that
\[
C_j^* (s) = -(-1)^j \cdot \left[ \frac{s \cos (js) - j\pi \csc (js)}{\left( \frac{s}{s} \right)^2} \right] = -(-1)^j \cdot \left[ \frac{s \cos (js) - s^2 j \pi \csc (js)}{s^2 \pi} \right].
\]
We need only \( C_{4j}^* (R) \) for \( j = 1, 2, \cdots \) so
\[
C_{4j}^* (R) = -\left[ \frac{s}{s} \cos (4js) - \frac{4s^2 j \pi \csc (4js)}{s} \right].
\]
As easy calculation shows that \( s \) is slightly less than \( \frac{3\pi}{2} \), so let \( z = \frac{3\pi}{2} - s \) or \( s = \frac{3\pi}{2} - z \), then
\[
C_{4j}^* (z) = -\left[ \frac{1}{\pi} \left( \frac{3\pi}{2} - z \right) \cos (6\pi j - 4zj) - \frac{4j}{\pi} \left( \frac{3\pi}{2} - z \right)^2 \csc (6\pi j - 4zj) \right]
\]
\[ = \left( \frac{3}{2} - \frac{z}{\pi} \right) \cdot \left[ \cos (4zj) + (6j\pi - 4jz) \frac{1}{\sin (4zj)} \right].\]
Since $z \approx 0$, $C_{4j}^* (z)$ becomes

$$C_{4j}^* (z) \approx -\frac{3}{2} \cdot \left[ \cos (4jz) + \frac{3\pi}{2z} \cdot \frac{4z}{\sin (4jz)} \right].$$

However, $z$ is small, so the second term dominates. Since $\frac{x}{\sin(x)}$ is monotonic increasing in $0 < x < \pi$, it follows that $\frac{4z}{\sin (4jz)}$ is monotonic increasing in $0 < 4jz < \pi$. Thus, $C_{4j}^* (z)$ is monotonically decreasing in $0 < j < \frac{\pi}{4z}$, with

$$0 < j < \frac{\pi}{4z} = \frac{\pi}{4 \left( \frac{3\pi}{2} - t \right)} = \frac{\pi}{6\pi - 4 \left( \frac{\pi}{1-\tau} \right)} = \frac{1 - R}{2 \left( 1 - 3R \right)}.$$

For $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right),$

$$\frac{1 - R}{2 \left( 1 - 3R \right)} > \frac{1 - \left( \frac{1}{3} - \varepsilon \right)}{2 \left( 1 - 3 \left( \frac{1}{3} - \varepsilon \right) \right)} = \frac{1}{9} \varepsilon + \frac{1}{6}.$$

It follows that $C_j^* (R)$ is monotonically decreasing in $j$ if $j < \frac{1}{9} \varepsilon + \frac{1}{6}$ or equivalently

$$\varepsilon < \frac{2}{3} \cdot \frac{1}{6j - 1}.$$

Thus, given $N$, if we take $\varepsilon < \frac{2}{3} \left( \frac{2}{3N-2} \right)$, then for each $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right), C_1^* (R) > \cdots > C_{4j}^* (R)$ with $4j \leq N$.

The previous two lemmas show the ordering of the $1^{\text{st}}$ and $4^{\text{th}}$ families outside the lower corner of the region of stability at transitions. This demonstrates how this families of bifurcation curves align outside the region of stability as we show in Fig. 4.1a. Fig. 4.1c. shows how the $2^{\text{nd}}$ and $3^{\text{rd}}$ families align for the upper corner of the stability region. Below we provide two lemmas, which prove the ordering of $2^{\text{nd}}$ and $3^{\text{rd}}$ families outside the region of stability for $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right)$ at $A_2^*$ for the sequence of transition points $(B_{4n-2}^*, C_{4n-2}^*)$.

**Lemma 4.7.** Fix $N (\varepsilon)$, a positive integer. Choose $\varepsilon$ such that $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right)$, $A_N^* (R)$ is continuous in $R$ with $\cot \left( \frac{N\pi}{1-R} \right) < 0$. From Eqn. (3.8) with $j = 1, 2, \cdots$ and $4j - 2 \leq N$, we have

$$B_{2}^* (R) > B_{6}^* (R) > B_{10}^* (R) > \cdots > B_{4j-2}^* (R)$$

for each $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right)$. 

\[\square\]
Proof. From Eqn. (3.8),
\[ B_j^* (R) = (-1)^j \cdot \frac{(1 - R) \cos \left(\frac{j \pi R}{1 - R}\right) - j R \pi \csc \left(\frac{j \pi R}{1 - R}\right)}{(1 - R)^2}. \]

Let \( t = \frac{R \pi}{1 - R} \) or \( R = \frac{t}{\pi + t} \). It follows that
\[
B_j^* (t) = (-1)^j \cdot \frac{(1 - \frac{t}{\pi + t}) \cos (jt) - \frac{t}{\pi + t} \pi \csc (jt)}{(1 - \frac{t}{\pi + t})^2}
= (-1)^j \left[ \left(\frac{\pi}{\pi + t}\right) \cos (jt) - (\pi + t) \frac{jt \csc (jt)}{\pi} \right].
\]

We need only \( B_{4j-2}^* (R) \) for \( j = 1, 2, \ldots \), so
\[
B_{4j-2}^* (t) = \frac{\pi + t}{\pi} \left[ \cos (4j - 2) t - t (4j - 2) \csc (4j - 2) t \right].
\]

As easy calculation shows that \( t \) is slightly less than \( \frac{\pi}{2} \), so let \( k = \frac{\pi}{2} - t \) or \( t = \frac{\pi}{2} - k \), then
\[
B_{4j-2}^* (k) = - \left(\frac{3}{2} - \frac{k}{\pi}\right) \left[ \cos (4jk - 2k) + (\frac{\pi}{2} - k) (4j - 2) \frac{1}{\sin(4jk - 2k)} \right].
\]

Since \( k \approx 0 \), \( B_{4j-2}^* (k) \) becomes
\[
B_{4j-2}^* (R) \approx - \frac{3}{2} \left[ \cos (4jk - 2k) + \frac{\pi \csc (4jk - 2k)}{2k} \right].
\]

However, \( k \) is small, so the second term dominates. Since \( \frac{x}{\sin(x)} \) is monotonically increasing on \( 0 < x < \pi \), it follows that \( \frac{4jk - 2k}{\sin (4jk - 2k)} \) is monotonically increasing for \( 0 < 4jk - 2k < \pi \). Thus, \( B_{4j}^* (k) \) is decreasing for \( 0 < j < \frac{\pi}{4k} + \frac{1}{2} \). However, we obtain
\[
0 < j < \frac{\pi}{4k} + \frac{1}{2} = \frac{\pi}{4 (\frac{\pi}{2} - t)} + \frac{1}{2} = \frac{\pi}{2 \pi - 4 (\frac{R \pi}{1 - R})} + \frac{1}{2} = \frac{1 - R}{2 (1 - 3R)} + \frac{1}{2}
\]

For \( R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right) \),
\[
\frac{1 - R}{2 (1 - 3R)} + \frac{1}{2} > \frac{1 - \left(\frac{1}{3} - \varepsilon\right)}{2 (1 - 3 \left(\frac{1}{3} - \varepsilon\right))} + \frac{1}{2} = \frac{1}{9 \varepsilon} + \frac{2}{3}.
\]

It follows that \( B_j^* (R) \) is decreasing for \( j < \frac{1}{9 \varepsilon} + \frac{2}{3} \) or equivalently
\[
\varepsilon < \frac{1}{3} \cdot \frac{1}{j - 2}.
\]

Given \( N \), if we take \( \varepsilon < \frac{1}{3} \left(\frac{1}{N-6}\right) \), then for each \( R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right) \), \( B_2^* (R) > \cdots > B_{4j-2}^* (R) \) with \( 4j - 2 \leq N \).

\[\square\]
Lemma 4.8. Fix $N$ ($\varepsilon$), a positive integer. Choose $\varepsilon$ such that $R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right)$, $A_N^* (R)$ is continuous in $R$ with $\cot \left(\frac{NR\pi}{1-R}\right) < 0$. From Eqn. (3.8) with $j = 1, 2, \cdots$ and $4j - 2 \leq N$, we have

$$C_2^* (R) < C_6^* (R) < C_{10}^* (R) < \cdots < C_{4j-2}^* (R)$$

for each $R \in \left(\frac{1}{3} - \varepsilon, \frac{1}{3}\right)$.

Proof. From Eqn. (3.8),

$$C_j^* (R) = -(-1)^j \cdot \left[\frac{(1-R) \cos \left(\frac{j\pi}{1-R}\right) - j\pi \csc \left(\frac{j\pi}{1-R}\right)}{(1-R)^2}\right].$$

Let $s = \frac{\pi}{1-R}$ or $R = \frac{s-\pi}{t}$. It follows that

$$C_j^* (s) = -(-1)^j \cdot \left[\frac{\frac{s}{\pi} \cos \left(\frac{j\pi}{s}\right) - j\pi \csc \left(\frac{j\pi}{s}\right)}{\left(\frac{s}{\pi}\right)^2}\right]$$

$$= -(-1)^j \cdot \left[\frac{s}{\pi} \cos \left(\frac{j\pi}{s}\right) - \frac{s^2 j}{\pi} \csc \left(\frac{j\pi}{s}\right)\right].$$

We need only $C_{4j-2}^* (R)$ for $j = 1, 2, \cdots$ so

$$C_{4j-2}^* (R) = -\left[\frac{s}{\pi} \cos \left(\frac{4j\pi}{s} - 2s\right) - \frac{s^2}{\pi} \cdot (4j - 2) \csc \left(\frac{4j\pi}{s} - 2s\right)\right].$$

As before, $s$ is slightly less than $\frac{3\pi}{2}$, so let $z = \frac{3\pi}{2} - s$ or $t = \frac{3\pi}{2} - z$, then

$$C_{4j-2}^* (z) = -\left[\frac{3\pi}{2} - z}{\pi} \cos \left(\left(4j - 2\right) \left(\frac{3\pi}{2} - z\right)\right) - \frac{(4j-2)\left(\frac{3\pi}{2} - z\right)^2}{\pi} \csc \left(\left(4j - 2\right) \left(\frac{3\pi}{2} - z\right)\right)\right]$$

$$= \left(\frac{3}{2} - \frac{z}{\pi}\right) \left(\cos \left(2z (2j + 1)\right) + (6j\pi - 4jz - 3\pi + 2z) \frac{1}{\sin \left(2z (2j - 1)\right)}\right).$$

However, $z$ is small, $C_{4j-2}^* (z)$ becomes

$$C_{4j-2}^* (z) \approx \frac{3}{2} \left(\cos \left(2z (2j - 1)\right) + \frac{2z (2j-1)}{2z (2j - 1)} \right).$$

With $z$ is small, the second term dominates. However, $\frac{x}{\sin (x)}$ is monotonically increasing for $0 < x < \pi$. It follows that $\frac{2z (2j-1)}{2z (2j - 1)}$ is monotonically increasing for $0 < j < \frac{\pi}{4z} + \frac{1}{2}$. We find
$$0 < j < \frac{\pi}{4z} + \frac{1}{2} = \frac{\pi}{4 \left( \frac{\pi}{2} - s \right)} + \frac{1}{2} = \frac{\pi}{2\pi - 4 \left( \frac{R\pi}{1-R} \right)} + \frac{1}{2} = \frac{1 - R}{2(1 - 3R)} + \frac{1}{2}.$$ 

For $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right)$,

$$\frac{1 - R}{2(1 - 3R)} + \frac{1}{2} > \frac{1 - \left( \frac{1}{3} - \varepsilon \right)}{2 \left( 1 - 3 \left( \frac{1}{3} - \varepsilon \right) \right)} + \frac{1}{2} = \frac{1}{9\varepsilon} + \frac{2}{3}.$$ 

It follows that $C^*_j(R)$ is increasing $j$ if $j < \frac{1}{9\varepsilon} + \frac{2}{3}$ or equivalently

$$\varepsilon < \frac{1}{3} \cdot \frac{1}{j - 2}.$$ 

Thus, given $N$, if we take $\varepsilon < \frac{4}{3} \left( \frac{1}{N-6} \right)$, then for each $R \in \left( \frac{1}{3} - \varepsilon, \frac{1}{3} \right)$, $C^*_2(R) < \cdots < C^*_j(R)$ with $4j - 2 \leq N$. 

\[\square\]
CHAPTER 5
CONCLUSION AND FUTURE WORK

Many fields of study take advantage of delay differential equations (DDEs), including biology, control theory, physiology, epidemiology, and economics. Delay problems are infinite dimensional, so a stability analysis of DDEs with delays can be very complex. This thesis examined a first order scalar linear differential equation with two delays. We concentrated on analytic proofs about the region of stability for Eqn. (3.1), which extend the numerical results of Busken [8]. We built upon the systematic approach of Busken [8] in locating the stability region of Eqn. (3.1), and applied an analytical approach to changes in the stability region as the parameters of Eqn. (3.1) vary. This thesis provides additional support for the idea that a rational delay gives a larger region of stability.

This thesis concentrates its analytical results for (3.1) near the specific rational delay, $R = \frac{1}{3}$. The thesis begins with a summary of key definitions, theorems, and various numerical results shown by Busken [8] and Mahaffy and Busken [27]. The analysis centers on the transcendental characteristic equation (3.2), which has infinitely many eigenvalues, $\lambda$.

Eqn. (3.1) is stable if and only if (3.2) has only eigenvalues with nonpositive real parts. It follows that letting $\lambda = i\omega$, $\omega$ real, and inserting these values into the characteristic equation, we can determine the boundary of the stability region in the parameter space. Busken [8] provided a valuable numerical framework for visualizing how the stability region changes.

The main results of this thesis are presented in Chapter 4, which contains two sections. Section 4.1 gives details for the evolution of the stability surface at $R = 0.33$. With the help of Busken’s Matlab program [8], we determined all changes to the boundary of the stability region from $A_0$ to $A_2^*$. Table 4.1 summarizes all the events that changed the bifurcations curves composing the boundary of the stability region for $R = 0.33$. As $A \rightarrow A_2^*$, the numerical computations confirmed the conjectures of Mahaffy and Busken [27] that the stability region reduces to just four bifurcation curves at $A_2^*$ as $R \rightarrow \frac{1}{3}$. This geometric study provided the motivation for our analytical results in Section 4.2. Mahaffy and Busken [27] observed that transitions caused distortions in the stability surface, which enlarged the stability region. We noticed that even transitions, $A_{2n}^*$, occurred with a distinct ordering, and these transitions pulled bifurcation curves outside the region of stability. This resulted in reverse tangencies and eventually a reverse transferral, which caused the boundary of the stability region to contain fewer bifurcation curves. Numerical studies of other delays with $0.33 < R < \frac{1}{3}$ showed a repeatable pattern as $A \rightarrow A_2^*$ with the stability region changing with
the same sequence of reverse tangencies and a reverse transferral. The stability region always
reduced at \( A = A^*_2 \) to the same generic shape seen in Fig. 4.2 with a stable region that is
44.3\% larger than the MRS and is bounded by only four bifurcation curves. The continuity of
the characteristic equation (3.2) indicates this pattern should persist until the singularity at
\( R = \frac{1}{3} \) with \( A^*_2 \to \infty \).

Section 4.2 began with a summary of results from Mahaffy and Busken [27] for
\( R \to \frac{1}{2n+1} \). These results show how the boundary of the stability region is composed of just
the four bifurcation curves \( \Gamma_0, \Gamma_1, \Gamma_{2n}, \) and \( \Delta_{2n} \) at \( A^*_{2n} \) as \( R \to \frac{1}{2n+1} \). Geometrically, this
region is similar to the one seen in Fig. 4.2. This thesis analytically proves several lemmas
that support the conjecture of this simple boundary for the stability region near \( R = \frac{1}{3} \) at \( A^*_{2n} \).
Lemma 4.4 proves that transitions near \( R = \frac{1}{3} \) have a distinct ordering over a limited range.
This sequence of transitions directs the ordering of bifurcation curves leaving the boundary of
the stability region as it is reduced to a simplified region bounded by only four curves (along
with a tiny segment from a fifth bifurcation curve). Our next four lemmas show detailed
proofs for the ordering of key points \( (B^*_{2n}, C^*_{2n}) \) at transitions \( A^*_{2n} \). We demonstrate how this
aligns families of bifurcation curves outside the region of stability and draws these bifurcation
curves outside the region of stability. We show \( (B^*_{2n}, C^*_{2n}) \) moving away from the region of
stability with increasing \( n \). This method determined that family members of bifurcation
curves are pulled out in the same sequence we observed from the graphs in Section 4.1. Our
lemmas establish that for \( R \to \frac{1}{3} \) only the bifurcation curves \( \Gamma_0, \Gamma_1, \Gamma_{2}, \) and \( \Delta_{2} \) are on the
boundary of the stability region near the corners furthest from the MRS with all higher
bifurcation curves lying outside these corners of the stability region.

Our results prove that for \( R < \frac{1}{3} \), but sufficiently close to \( \frac{1}{3} \), the region of stability
extends out 50\% more in the linear direction beyond the MRS. This increases the area of the
stability region by more than 44.3\% over the MRS at \( A^*_2 \). The proofs show a very simplified
structure with only four bifurcation curves on the boundary at \( A^*_2 \). The proofs also help
explain how the bifurcation surface evolves as it approaches this simplified shape at \( A^*_2 \) as
\( R \to \frac{1}{3} \). For \( R \to \frac{1}{3} \) and \( A > A^*_2 \), this structure collapses, and the stability region more closely
approaches the MRS, which demonstrates the high sensitivity of models with a linear form of
(3.1) with respect to the delay near \( R = \frac{1}{3} \).

There are several interesting directions for future work, which could follow from this
thesis. We suspect that only minor variation in the proofs to Lemma 4.4-4.8 should give
similar results for \( R = \frac{1}{2n+1} \). Numerical studies show that as \( R \to \frac{1}{2n+1} \), then at \( A^*_{2n} \), the
stability region becomes bonded by four bifurcation curves, \( \Gamma_0, \Gamma_1, \Gamma_{2n}, \) and \( \Delta_{2n} \). Our
analytic results for \( R = \frac{1}{3} \) should extend to the extreme corners of the stability region for
delays \( R \to \frac{1}{2n+1} \) at \( A^*_{2n} \).
The studies of Mahaffy and Busken [27] for $R \to \frac{1}{2n}$ showed a slightly different shaped stability region. However, this region was again enlarged with the boundary primarily composed of the four bifurcation curves $\Gamma_0$, $\Gamma_1$, $\Gamma_{2n-1}$, and $\Delta_{2n-1}$ at $A_{2n-1}^*$. The methods developed in this thesis should extend to show the same types of sequential changes to prove the extension of the corners for this simplified stability region.

This thesis analytically showed the sequential ordering of the points $(B_{2n}^*, C_{2n}^*)$ for $R$ near $\frac{1}{3}$. The thesis did not examine analytically the ordering of the complete bifurcation curves seen in the numerical studies. More analytical studies in the future are needed to show that the families of curves have a specific ordering, which would prove the conjecture that only four bifurcation curves compose the boundary at $A_2^*$ for $R$ near $\frac{1}{3}$.

Both the numerical and analytical studies concentrated on the behavior of the stability region for $R < \frac{1}{3}$. Details of the limit as $R \to \frac{1}{3}$ need further investigation. We anticipate the enlarged region persists as $A_2^* \to +\infty$, but more analytical results should be pursued.

Finally, the original conjecture is that all rational delays $R$ result in larger regions of stability. We provided details of the evolution of the larger region for the specific case $R = \frac{1}{3}$, and the numerical studies of Mahaffy and Busken [27] showed the asymptotic region of stability for large $A$ was 44.3% larger. Future studies similar to ours should provide more details for large $A$ when $R = \frac{1}{n}$. However, it remains an open question as to how the stability surface evolves and how much larger it is when $R = \frac{k}{n}$, $1 < k < n$. It follows that many open problems remains in the stability analysis of Eqn. (3.1).
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