OPTIMIZATION OF VARIABLE STIFFNESS COMPOSITE PLATE STRUCTURES

A Thesis
Presented to the
Faculty of
San Diego State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
in
Aerospace Engineering

by
Vishal D. Aggarwal
Summer 2013
SAN DIEGO STATE UNIVERSITY

The Undersigned Faculty Committee Approves the

Thesis of Vishal D. Aggarwal:

Optimization of Variable Stiffness Composite Plate Structures

Satchi Venkataraman, Chair
Department of Aerospace Engineering and Engineering Mechanics

Luciano Demasi
Department of Aerospace Engineering and Engineering Mechanics

Sam Kassegne
Department of Mechanical Engineering

June 28, 2012
Approval Date
DEDICATION

This work is dedicated to my father and mother, Devki and Kirti. Thank you for all your care and support. Love you.
ABSTRACT OF THE THESIS

Optimization of Variable Stiffness Composite Plate Structures
by
Vishal D. Aggarwal
Master of Science in Aerospace Engineering
San Diego State University, 2013

Over the last several decades, the trend in material choice for aerospace structures has shifted increasingly in favor of composite materials such as graphite fiber impregnated with epoxy resin. With the price of jet fuel more than doubling just in the last ten years, the trend towards composite materials is accelerating. Traditional composite structures are fabricated by stacking layers of woven fibers that are oriented in the same direction. Human operators perform the work. However, there now exist machines that can be programmed to place composite fibers in continuous paths that can vary in orientation. The present work investigates a method for determining the optimal ply angle distributions for such laminates which are also known as variable stiffness laminates. Past research in variable stiffness laminates has focused on the use of predefined fiber paths that can be mapped spatially by algebraic equations. However, it was found that these fiber paths only produce optimal designs for a limited range of load cases, boundary conditions and geometry. In this thesis, a two-level optimization method with a post optimization repair algorithm is presented. In the first level optimization, optimal lamination parameter distributions are obtained through compliance minimization using a gradient-based optimizer. Second, Genetic Algorithm (GA) based optimization is used to perform unconstrained norm minimization which returns an element-wise distribution of ply angles in each layer. In this second level optimization many of the elements do not completely converge resulting in discontinuities of fiber angles between elements. A repair algorithm that works by using the average ply angles of surrounding elements is implemented to remedy the problem, and is demonstrated to produce smooth continuous fiber paths. We find that in most cases the post repair compliance to be lower than that of the post-Genetic Algorithm compliance which will in turn yield a higher stiffness per unit mass plate.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2 LITERATURE REVIEW</td>
<td>5</td>
</tr>
<tr>
<td>2.1 Lamination Parameters</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Optimization of Orthotropic Plates</td>
<td>6</td>
</tr>
<tr>
<td>2.3 Two-Level Approach</td>
<td>6</td>
</tr>
<tr>
<td>2.4 Variable Stiffened Laminates</td>
<td>8</td>
</tr>
<tr>
<td>3 BASIC THEORY</td>
<td>12</td>
</tr>
<tr>
<td>3.1 Strain Displacement Relations</td>
<td>12</td>
</tr>
<tr>
<td>3.2 Stress-Strain Relations</td>
<td>13</td>
</tr>
<tr>
<td>3.3 Equilibrium Equations</td>
<td>14</td>
</tr>
<tr>
<td>3.4 In-Plane Response of Isotropic Layer(s)</td>
<td>14</td>
</tr>
<tr>
<td>3.4.1 Plane Stress</td>
<td>14</td>
</tr>
<tr>
<td>3.4.2 Symmetrically Laminated Layers Under In-Plane Loading</td>
<td>15</td>
</tr>
<tr>
<td>3.5 Bending Deformations</td>
<td>16</td>
</tr>
<tr>
<td>3.5.1 Kirchhoff-Love Assumptions</td>
<td>16</td>
</tr>
<tr>
<td>3.5.2 Displacements and Strains of a Single Layer</td>
<td>16</td>
</tr>
<tr>
<td>3.5.3 Bending Response of Symmetrically Laminated Layers</td>
<td>18</td>
</tr>
<tr>
<td>3.5.4 Bending-Extension Coupling of Asymmetrically Laminated Layers</td>
<td>18</td>
</tr>
<tr>
<td>3.6 Orthotropic Layers</td>
<td>19</td>
</tr>
<tr>
<td>3.7 Principle of Virtual Work and Weak Form Equations of Motion</td>
<td>23</td>
</tr>
<tr>
<td>4 FINITE ELEMENT MODEL</td>
<td>29</td>
</tr>
</tbody>
</table>
5.9 Limitation of Miki Graphical Method ................................................................. 65
5.10 Ply Angle Retrieval Using a Genetic Algorithm .................................................. 67
  5.10.1 Standard Test Cases for Ply Angle Retrieval Using a Genetic Algorithm ........... 68
  5.10.2 General Test Case 1 for Ply Angle Retrieval Using a Genetic Algorithm ........... 68
5.11 Variable Stiffened Laminates (First Level Optimization) .................................... 71
  5.11.1 Simply Supported Variable Stiffened Plate ..................................................... 72
  5.11.2 End Supported Beam .................................................................................... 74
  5.11.3 Study of Compliance vs. Load Application Point ............................................. 79
  5.11.4 Optimized Fiber Paths (Shifted Fiber Path Method) .................................... 82
    5.11.4.1 Description of Shifted Path Method ......................................................... 82
    5.11.4.2 Computational Model of Reference Path .................................................. 87
    5.11.4.3 Two-level Optimization of Fiber Paths .................................................... 88
    5.11.4.4 Optimized Fiber Paths for a Simply Supported Plate Under Distributed Load ........................................ 90
    5.11.4.5 Summary and Conclusion for Optimized Fibers Using Shifted Path Method .................................................. 98
  5.11.5 Optimized Fiber Angles For Variable Stiffness Plate Using Unconstrained Genetic Algorithm and Post Optimization Repair ..................... 101
    5.11.5.1 Example: Simply Supported Variable Stiffness Plate Under Point Load Applied at Center ........................................ 102
    5.11.5.2 Example: Clamp Supported Variable Stiffness Plate Under Point Load Applied at Center ........................................ 109
    5.11.5.3 Example: 2x1 Aspect Ratio Clamp Supported Variable Stiffness Plate Under Two Point Loads ........................................ 113
6 CONCLUSIONS AND RECOMMENDATIONS ................................................................. 121
  6.1 Summary ........................................................................................................... 121
  6.2 Conclusions ...................................................................................................... 123
  6.3 Recommendations .......................................................................................... 124
REFERENCES ............................................................................................................ 125
LIST OF TABLES

Table 4.1. Mesh Convergence Study, Square Plate in Bending ........................................44
Table 4.2. Mesh Convergence Study, Square Plate in Bending ........................................46
Table 4.3. Mesh Convergence Study, Cantilevered Plate..................................................48
Table 5.1. ABD matrix in Terms of Lamination Parameters..............................................49
Table 5.2. Point Load Application Points for a 2:1 Aspect Ratio Simply Supported Plate..................................................................................................................69
Table 5.3. Normalized Compliance Mapping for Loads Applied at Point, (ap, bp)..............81
Table 5.4. Second Level Optimization Performance Characteristics for a Variable Stiffened Plate Under Uniformly Distributed Load.........................................................95
Table 5.5. Summary of Performance for Simply Supported Variable Stiffness Plate Under Uniformly Distributed Load ..................................................................................109
Table 5.6. Summary of Performance for Clamp Supported Variable Stiffness Plate Under Centered Point Load ..................................................................................................110
Table 5.7. Summary of Orthotropic and First Level Variable Stiffness Performance for Clamp Supported 2:1 Aspect Ratio Plate Under Two Point Loads .........................115
Table 5.8. Summary of Performance for Clamp Supported 2:1 Aspect Ratio Variable Stiffness Plate under Two Point Loads ..................................................................................119
LIST OF FIGURES

Figure 1.1. (a) orthotropic laminate; (b) variable stiffness laminate. ...........................................2
Figure 2.1. Typical two-level optimization for variable stiffened laminates.................................7
Figure 3.1. Isotropic plate. ..................................................................................................................12
Figure 3.2. Laminate of Orthotropic Plies. .......................................................................................20
Figure 3.3. Ply stacking convention ...............................................................................................20
Figure 3.4. Geometry of a generic laminated plate. ..........................................................................24
Figure 3.5. Force and moment resultants on a plate element.........................................................25
Figure 4.1. Basic FEM Algorithm. ....................................................................................................32
Figure 4.2. 4 node rectangular element (in-plane displacements). ..................................................35
Figure 4.3. Thin plate in traction. .....................................................................................................37
Figure 4.4 Mesh for a thin plate in traction. .....................................................................................37
Figure 4.5. Elongation of a thin plate in traction. .............................................................................38
Figure 4.6. Compression of a thin plate in traction. ........................................................................38
Figure 4.7. Conforming laminated plate element in bending. .........................................................40
Figure 4.8. Simply supported laminated plate under uniformly distributed load. .......................41
Figure 4.9. Displacement field for a simply supported laminated plate under uniformly distributed load (30x30 mesh). .................................................................44
Figure 4.10. End supported laminated plate under uniformly distributed load. ............................45
Figure 4.11. Long plate deflection (100x10 mesh). .......................................................................47
Figure 4.12. Cantilevered plate deflection (32x16 mesh). ...............................................................48
Figure 5.1. Feasible region for flexural lamination parameters (symmetric and balanced layup). .........................................................................................................................51
Figure 5.2. Flow chart of first level optimization using method of moving asymptotes..............57
Figure 5.3. Orthotropic laminated plate subjected to uniformly distributed load (lamination parameter optimization)........................................................................................................59
Figure 5.4. Simple two-level method for optimizing an orthotropic plate. ....................................59
Figure 5.5. Convergence of lamination parameters for various starting points............................60
Figure 5.6. Objective function convergence (W1*, W3* initial =0,0).............................................61
Figure 5.7. Deflection of Plate with Optimized $W_1^*, W_3^*$. .................................................................61
Figure 5.8. End supported laminated plate under uniformly distributed load ........................................63
Figure 5.9. Compliance convergence of end supported beam (orthotropic layup)..................................63
Figure 5.10. Convergence of lamination parameters for 3 start points (end supported beam, orthotropic layup) .................................................................64
Figure 5.11. Deflection of end supported beam for optimized $W_1^*, W_3^*$. ........................................65
Figure 5.12. Angle determinant region for a 2 angle orthotropic plate in bending (equal ply thickness). .................................................................66
Figure 5.13. Angle determinant region for a 2 angle orthotropic plate in bending (4:1 ply thickness). ..............................................................................66
Figure 5.14. Two-Level Approach Using a Genetic Algorithm ..............................................................67
Figure 5.15. (a) Simply supported orthotropic plate; (b) end supported orthotropic beam ..............................................................................69
Figure 5.16. Convergence of lamination parameters for a plate defined by section 5.10.2. .........................70
Figure 5.17. (a) Deflection before optimization; (b) deflection after optimization ..................................70
Figure 5.18. (c) Convergence of compliance .........................................................................................71
Figure 5.19. (a) Simply supported variable stiffness plate under uniformly distributed load .................................................................72
Figure 5.20. (b) $W_1^*$ distribution of a simply supported variable stiffness plate under uniformly distributed load .................................................................73
Figure 5.21. $W_3^*$ distribution of a simply supported variable stiffness plate under uniformly distributed load .................................................................73
Figure 5.22. Compliance converge for a simply supported variable stiffness plate under uniformly distributed load ..............................................................................75
Figure 5.23. Optimized deflection for a simply supported variable stiffness plate under uniformly distributed load ..............................................................................75
Figure 5.24. End supported variable stiffness beam under uniformly distributed load .................................................................76
Figure 5.25. $W_1^*$ Distribution of an end supported variable stiffness beam under uniformly distributed load ..............................................................................76
Figure 5.26. $W_3^*$ Distribution of an end supported variable stiffness beam under uniformly distributed load ..............................................................................77
Figure 5.27. Compliance convergence for an end supported variable stiffness beam under uniformly distributed load ..............................................................................78
Figure 5.28. Optimized deflection for an end supported variable stiffness beam under uniformly distributed load ..............................................................................78
Figure 5.29. Simply supported variable stiffness plate under a point load

Figure 5.30. Application points for point load on square plate.

Figure 5.31. Shifted path lamina based on linearly varying reference path.

Figure 5.32. Panel (xy) and tow-path (x’y’) coordinate systems.

Figure 5.33. Linearly varying centerline.

Figure 5.34. Geometry for defining a variable-stiffness reference path.

Figure 5.35. Curvilinear-fiber reference path centerline for $T_0=0^\circ$ and $T_1=45^\circ$.

Figure 5.36. A single shifted-path centerline.

Figure 5.37. Variation of orientation angle across reference path for a constant value of $x'$. 

Figure 5.38. Algorithm for two-level optimization of fiber paths.

Figure 5.39. Simply supported variable stiffness plate under distributed (quarter plate analysis).

Figure 5.40. Deflection of simply supported variable stiffness plate under uniformly distributed load (baseline quasi-isotropic layup).

Figure 5.41. Compliance converge for a quarter symmetric simply supported variable stiffness plate under uniformly distributed load (first level optimization).

Figure 5.42. Deflection of simply supported variable stiffness plate under uniformly distributed load (based on optimized lamination parameters, $W_1^*, W_3^*$).

Figure 5.43. $W_1^*$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, first level optimization).

Figure 5.44. $W_3^*$ distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, first level optimization).

Figure 5.45. $W_1^*$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, second level optimization).

Figure 5.46. $W_3^*$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, second level optimization).

Figure 5.47. Fiber angle distributions for simply supported variable stiffness plate under uniformly distributed load (quarter plate, shifted fiber method, positive angle layers shown).

Figure 5.48. Deflection of simply supported variable stiffness plate under uniformly distributed load (based on optimized fiber paths).

Figure 5.49. Basic algorithm for ply angle optimization with post optimization repair.

Figure 5.50. Simply supported variable stiffness plate under centered point load.

Figure 5.51. Lamination parameters distributions for a simply supported variable stiffness plate under centered point load (first level optimization).
Figure 5.52. Lamination parameters distributions for a simply supported variable stiffness plate under centered point load (second level optimization) .................................................. 104

Figure 5.53. Fiber angle distributions for simply supported variable stiffness plate under centered point load (positive angle layers shown) .................................................. 105

Figure 5.54. Ply angle mapping of surrounding elements for a typical interior element ................................................................................................................................................. 107

Figure 5.55. Repaired fiber angle distributions for simply supported variable stiffness plate under centered point load (positive angle layers shown). ........................................ 108

Figure 5.56. Lamination parameters distributions for a simply supported variable stiffness plate under centered point load (repaired laminate). ........................................ 109

Figure 5.57. Clamp supported variable stiffness plate under centered point load. ............. 110

Figure 5.58. Lamination parameters distributions for a clamp supported variable stiffness plate under centered point load (first level optimization) .................................. 111

Figure 5.59. Lamination parameters distributions for a clamp supported variable stiffness plate under centered point load (second level optimization). ................................. 111

Figure 5.60. Fiber angle distributions for clamp supported variable stiffness plate under centered point load (positive angle layers shown). ........................................ 112

Figure 5.61. Lamination parameters distributions for a clamp supported variable stiffness plate under centered point load (repaired laminate). ........................................ 113

Figure 5.62. Repaired fiber angle distributions for a clamp supported variable stiffness plate under centered point load (positive angle layers shown). ......................... 114

Figure 5.63. 2x1 Aspect ratio clamp supported variable stiffness plate under two point loads ................................................................................................................................................. 115

Figure 5.64. Lamination parameters distributions for a 2x1 aspect ratio clamp supported variable stiffness plate under two point loads (first level optimization). .................................................. 115

Figure 5.65. Lamination parameters distributions for a 2:1 aspect ratio clamp supported variable stiffness plate under two point loads (second level optimization). .................................................. 116

Figure 5.66. Lamination parameters distributions for a 2:1 aspect ratio clamp supported variable stiffness plate under two point loads (repaired laminate). ........... 116

Figure 5.67. Fiber angle distributions for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (positive angle layers shown). ......................... 117

Figure 5.68. Repaired fiber angle distributions for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (positive angle layers shown). .......... 118

Figure 5.69. Deflection for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (repaired laminate). ................................................................................................................................................. 120
ACKNOWLEDGEMENTS

This paper is the product of an almost shameful number of months effort. I would not have been able to get through it without the patience and brilliance of my advisor Dr. Satchi Venkataraman. Thank you very much.
CHAPTER 1

INTRODUCTION

Over the last several decades, composite laminates have come to the forefront of the material choices available to an aerospace structural designer. One of the primary objectives in aerospace vehicle design is minimization of structural weight while maximizing the strength and/or stiffness of the structure. Conveniently, composite materials are very accommodating towards meeting this goal due to two fundamental attributes. The first is that many material fibers available on the market today are stiffer and stronger than isotropic (metallic) materials traditionally used. Secondly, the directional nature in which composites are manufactured actually allows the designer to “tailor” and effectively optimize the laminate to almost any stiffness/strength desired. In this paper, we will study this ‘tailoring’ characteristic of composites from a structural analysis and optimization standpoint. The goal of this research is to present a framework for analyzing and optimizing composite plates with curvilinear fiber paths.

Most aerospace laminates today are constructed from woven fabrics with individual plies oriented in a single direction or orthogonal directions (see Figure 1.1a). In most cases, human operators lay the fabrics into molds that represent the shape of the desired part.

Normally, fibers are placed in 0°, ±45° and 90° direction orientations for ease of manufacturing. These layups also tend to be simple to design, analyze and optimize. Many times a ‘rule of thumb’ is used for layup selection (i.e. quasi-isotropic). Analysis of the layup can be done using a variety of commercial software applications or in some cases using closed form solutions. However, given the directional limitations and manufacturing constraints of woven laminates, a true optimal stiffness laminate will likely not be achieved.

Variable stiffened laminates offer a way to essentially eliminate the directional and human operator imposed constraints of conventional laminates because variable stiffness laminates are custom manufactured with the aid of precision computer programmed layup or fiber placement machines. Unlike the human hand the machine can spatially vary the angle
of a single fiber to virtually any desired orientation. The result is a laminate with curved shaped fiber paths as shown in Figure 1.1b.

A drawback to this approach of course is that the equipment typically requires a large investment. The root of the issue might actually be that analysis techniques for variable stiffened laminates largely exist only in research circles and, for the most part, have not been incorporated into widely used commercial analysis software. Hence, variable stiffened laminates have seen limited use in commercial applications to date. However, it is the belief of this author that as analysis and optimization techniques of variable stiffened laminates become more accepted and used, competition will drive the initial costs of machinery and associated software down significantly. Also, the cost per part will decrease significantly as overhead costs can be amortized over increasingly larger production runs.

In this paper will explore analysis and optimization techniques of variable stiffness laminates with curvilinear fiber paths exposed primarily to bending loads. A ‘two-level’ optimization approach, explained in detail in the following sections, will be used. Some results of this research leave open questions as to the usefulness of curvilinear fiber path laminates. However, this work can serve as a framework for future investigations into the subject matter.

- Chapter 2 (Literature)

A synopsis of the various published papers that were read as part of the background search in this work will first look at the development of lamination
parameters then investigate how they were later applied to the optimization of composite plates. Next, works that employed a ‘two-level approach’ to optimization were studied in an effort to gain an understanding of the method for use in this research. Finally, we summarize a series of works dedicated to variable stiffness laminate research with an emphasis on works involving lamination parameters and/or the two-level approach.

- **Chapter 3 (Basic Laminate Theory)**
  This chapter describes the basic lamination theory used in this work. The chapter begins with strain displacement equations and proceeds to development of membrane and motions bending equations for orthotropic laminates. The concept of lamina invariants and how they are relevant to optimization studies are also presented. Finally, the laminate equations of motion are presented in the “weak form”. In contrast to the “strong form”, the “weak form” does not rely on strong continuity of displacements which lends itself well to a discretized method such as the finite element method.

- **Chapter 4 (Finite Element Model)**
  This chapter presents a detailed explanation of the finite element model developed for this research. First, a model for a laminated plate with in-plane loading and displacements only is developed and tested. The model is modified for bending loads; test cases are presented. Closed form solutions for simple problems are compared to finite element solutions for the same problems. Discussion is included regarding finite element mesh convergence with an emphasis of choosing a reasonable mesh size for optimization.

- **Chapter 5 (Optimization)**
  In this final chapter, the finite element codes are adapted to various optimization routines in an effort to determine minimum compliance ply angles. A ‘two-level’ approach to the problem is used. In the first level, a gradient-based optimization routine (Method of Moving Asymptotes) is used to determine optimum values of lamination parameters for both orthotropic and variable stiffened plates under various loadings and boundary conditions.

  Next, various methods for determining ply angles are explored. For the orthotropic case, we attempt to use Miki’s graphical method to determine the optimum ply angles but discover the graphical method is limited to a narrow band of loading and boundary conditions. Hence, a more robust method for determining ply angles is needed, and a second level optimization to determine ply angle is carried out with the aid of a Genetic Algorithm (GA). Finally, we present two methods for determining optimum ply angles for the variable stiffened plate. The first method involves implementation of an unconstrained gradient-based optimization based on fiber path definitions developed by Gurdal and Olmedo. The second method involves implementation of an unconstrained GA with a post-optimization
repair algorithm in which a curvature constraint is applied in an effort to enforce fiber continuity.
CHAPTER 2

LITERATURE REVIEW

This chapter presents a summary of work in the field of laminate optimization that has influenced the content of this paper. The articles referenced are by no means the only or even necessarily the first work of their kind. Rather, the idea is to highlight specific developments and tie some developments together and to indicate areas where additional research may contribute to the optimization studies.

2.1 LAMINATION PARAMETERS

Early work in addressing composite optimization can be traced back to the development of lamination parameters by Tsai et al. in the late 1960’s [1]. Their work revealed a potential advantage to lamination parameters. The discrete design variables (ply angles and thicknesses) are lumped into a set of twelve continuous parameters (4 membrane, 4 bending and 4 membrane-bending coupling) that describe the stiffness of a laminate in convex space. These lamination parameters could be used as intermediate design variables and a gradient-based optimizer could potentially be used in a compliance minimization to determine the optimal set of parameters. However, an additional step would still be required to determine the optimal layup.

When Tsai et al. published their work, no constraints were placed on the lamination parameters which made their use in optimization difficult. However, in the early 1980’s, Miki developed a simple way of constraining the design spaces of lamination parameters by assuming balanced and symmetric layups to reduce the number of lamination parameters to four: two in-plane and two out-of-plane [2]. Through the use of the trigonometric identities, Miki was able to establish a simple inequality that related each of the two sets of lamination parameters to one another. Now it was possible to run a very simple constrained optimization using any number of gradient-based solvers. Once the optimal lamination parameters were determined, closed form equations could be used to determine ply angles for simple one and two angle orthotropic laminates; however, there was a limitation in that
angles could not be obtained for all sets of lamination parameters depending on the ply thicknesses. But, for most simple load cases, an acceptable solution could be found. This limitation will be discussed in greater detail later. Further details of Miki’s method are given in Chapter 5 of this text. Chapter 5 also discusses the inequality constraint applied in optimization problems for both orthotropic and variable stiffness plates.

2.2 Optimization of Orthotropic Plates

In the years following Miki’s development, many researchers applied his techniques to the optimization of orthotropic plates. Miki himself continued to expand on his approach and applied it to plate buckling optimization using closed form objective solutions and then to more general optimization of orthotropic plates problems using closed form solution objective functions [3, 4].

Grenestedt [5] also used closed form solutions to show that for composites plates under constant pressure, optimization for free vibration frequencies, buckling loads, and deflections can be reduced to one parameter provided that the laminate was restricted to a \( \pm \theta \) layup. The parameter can be interpreted as the layup angle itself.

Haftka and Walsh [6], and Nagendra, Haftka, and Gürdal [7] applied a new twist to Miki’s approach by using integer programming techniques to optimize laminates for buckling and also added strength constraints to the problem. Ply angles were limited to \( 0^\circ \), \( \pm 45^\circ \) and \( 90^\circ \) and closed form solutions were once again used.

2.3 Two-Level Approach

About the same time that Haftka and Walsh [6] were working on integer programming techniques, genetic algorithms (GA) were beginning to be explored as an entirely new method in structural optimization. GA’s are probabilistic optimization methods that do not require gradient information of the objective function or constraints and are also insensitive to design space complexities. Many research papers have been written on the use of GA’s in composite optimization [8, 9]. The conclusion from most was that while GA’s do have a strong advantage over gradient-based methods in that a solution can almost always be obtained, they also tend to be computationally expensive when compared to gradient-based methods.
Lamination parameters did not by any means die off with the advent of discrete methods such as genetic algorithms. In fact, the computational costs of discrete solvers may have even validated the relative simplicity of using lamination parameters as design variables at least for achieving a maximized stiffness objective at low cost. This idea gave rise to the ‘two-level’ or ‘multi-objective’ technique of optimization. One of the first implementations of this approach was from Yamazaki [10] who used it to maximize the critical buckling load of a laminated plate. At the first level, lamination parameters are used as design variables, constraints are as they were in earlier work by Miki et al. and a closed-form solution is used for the buckling equation. At this point, the work mimics that done by Haftka et al. However, the departure comes when a second optimization introduces a GA to search for the optimal stacking sequence, and the optimal lamination parameters are used as constraints. The two-level approach for orthotropic laminates could probably been viewed as ”overkill” for many practical problems. Also, by today’s computing standards, a stand-alone GA may even be cheaper given the initial overhead costs of setting up the multi-level problem and solving the first level. However, the method itself has proven to be quite useful and is the foundation for much of the work that is ongoing in the area variable stiffened laminates. The two-level approach is used in this research as one means of determining ply angles of both orthotropic and variable stiffened laminated plates. A sample flow diagram of a two-level optimization is shown in Figure 2.1.

![Figure 2.1. Typical two-level optimization for variable stiffened laminates.](image-url)
2.4 VARIABLE STIFFENED LAMINATES

Work in variable stiffened laminates can be traced to the late 1980’s when Katz, Haftka, and Altus [11] and Hyer and Charette [12] published papers studying the use of curvilinear fibers for a plate with a hole in the center. In Hyer’s study, in-plane tension and buckling loads were applied to the plate, although the results were mainly focused on tension loading. Fiber angles around the hole were determined by an iterative process and were aligned to the principal stress directions within each layer. Results of the study showed that curvilinear fibers could indeed increase the failure load particularly when used in conjunction with $\pm45^\circ$ plies. Around the same time period, Katz presented a similar method for finding optimum ply angles around a hole.

Gurdal and Olmedo [13], Waldhart [14], and Tatting [15] modeled the in-plane response of laminated plates and shells with curvilinear fibers by defining fiber paths as smooth functions in x and y coordinates based on a single ‘reference path’ through the center of the laminate. Their work also focused on the condition of overlaps and gaps that exist due to curvature differences between adjacent fibers. Langley [16] published a FORTRAN 90 code for generating a Genesis® to perform a finite element analysis of variable stiffened plates. The basic algorithm presented in Langley’s paper was adopted in this research to determine optimum ply angles for the variable stiffened plate. Further details are presented in Chapter 5.

Tatting [17] conducted one of the first studies into the manufacturability of a basic variable stiffness panel. They used a seven-axis tow-placement machine to construct two panels, one with 100% overlap and one with 0% overlap. Wu, Gurdal, and Starnes [18] followed by testing the panels under compression loads. Their work showed gains of up to five times the compressive buckling load of a straight fiber panel. Jegley, Tatting, and Gurdal [19] expanded on Wu’s work with a new set of panels with centrally located holes, which were designed using a Raleigh-Ritz analysis solutions and a genetic algorithm based optimization software which was developed by Tatting and Gurdal. Jegley et al. also showed significant increases in buckling load capabilities of tow-steered laminates over straight fiber laminates. An automated finite element approach developed by Tatting and Gürdal [20] replaced the Raleigh-Ritz approach used previously and a GA was still used for the optimization of ply angles.
Blom, Abdalla, and Gürdal [21] explored a method for designing plates and shells using varying fiber orientations in which thickness build-up was predicted as a function of ply angle variation using a streamline analogy. They have also explored more three-dimensional problems such as conical shells and cylinders [22, 23]. The conical shell study focused on maximizing the fundamental frequency of truncated conical shells using sequential quadratic programming and commercial finite element analysis for frequency analysis. Improvements over the constant stiffness cone were shown. For the case of a variable stiffness cylinder, they focused on a buckling optimization for a cylinder in bending. Again, commercial software was used for the analysis portion and a surrogate model was used to minimize the number of finite element calculations. Improvements of up to 17% increase in buckling load capability were seen above the constant stiffness baseline. Finally, a variable stiffness cylinder was manufactured and tested according to one of the optimized designs and close correlation to the predicted strains was seen [24].

Setoodeh et al. were among the first to explore design of variable stiffness plates for maximum bending stiffness [25]. Local ply angles were treated as continuous design variables and their spatial distribution determined based on an optimality criterion formulation for minimum compliance using the Cellular Automata method. Geometry and loading were defined as a square plate under uniform pressure. Both simply-supported and clamped boundary conditions were examined. Compliances were normalized against a corresponding 0 deg. laminate and reductions of 25-30% were seen.

Setoodeh et al. were among the first to explore the use of lamination parameters in variables stiffness laminates. For the in-plane problem [26], they used a more general feasible domain in which all four parameters were used as design variables in a compliance minimization of an end-loaded cantilever beam with variable stiffness plies. Constant and variable stiffness laminates were studied and compared. Solutions for the reduced approach presented by Miki for balanced symmetric laminates were also presented. A 0 degree laminate was used to normalize the optimized compliances. The variable stiffness laminate showed significant reduction in compliance over the constant stiffness laminate. Also, the use of four lamination parameters over two showed about a nine percent reduction in compliance. A recursive formulation for determining the ply angles for the balanced symmetric laminate was presented.
Setoodeh, Abdalla, and Gürdal [27] followed with another publication in which both the in-plane and bending problems were addressed. Varying aspect ratios as well as five different panel layup configurations were studied (balanced constant stiffness, general constant stiffness, single layer variable stiffness, balanced variable stiffness and general variable stiffness). The in-plane problem was addressed with a cantilever beam under uniform distributed load. The general variable stiffness design displayed an approximate 36% reduction in compliance over the general constant stiffness design and about a 9% reduction over the single layer variable stiffness design. For the bending problem, a plate under uniform distributed pressure was examined under simply-supported and clamped-boundary conditions. Again, various aspect ratios were examined. In the simply-supported case, the general variable stiffness layup demonstrated an 18% reduction over the general constant stiffness layup. The values of the lamination parameters for the optimized constant stiffness laminate are given as $W_1=W_2=W_4=0$ and $W_3=-1$. Interestingly, for this geometry, boundary condition and loading, the optimal layup actually turned out to be a balanced constant stiffness layup instead of a variable stiffness one. For the clamped plate, the optimal constant stiffness lamination parameters are given as $W_1=W_2=W_4=0$ and $W_3=1$. The optimal variable stiffness compliance was reduced by 45% over the optimized general constant laminate. The results published by Setoodeh et al. were useful in helping to validate the output of the gradient-based optimizer in the present work.

Ijsselmuiden, Abdalla, and Gürdal [28], Ijsselmuiden et al. [29] addressed buckling optimization of variable stiffness laminate loads using lamination parameters. A conservative reciprocal approximation scheme was introduced to force convexity in lamination parameter space. Numerical results show improvements of 100% and more in buckling load over optimum constant stiffness designs. It should also be noted that although buckling is not the focus of the present paper, the work done by Ijsselmuiden served as a primary influence and inspiration for much of the subject matter in this thesis. For example, this thesis used the finite element equations presented in the appendix of the Ijsselmuiden paper. The sensitivity analysis presented in the appendix served as a valuable guide to developing the bending sensitivities used to optimize the lamination parameters $W_1$, $W_3$ as we shall show later in this paper.
One of the primary objections to using lamination parameters is the difficulty of incorporating strength constraints into the optimization process. However, Khani et al. [30] recently proposed a method for solving the problem. They examined the in-plane problem of a plate with a hole under uniaxial tension and a Tsai Wu failure criterion to formulate a strength constraint. The objective function was written as the minimization of the maximum failure index. A gradient-based optimization technique was used to yield optimum lamination parameter distributions. Fiber angles were plotted using the “streamline analogy” technique developed by Blom et al.

In the fall 2011, Ijsselmuiden published a PhD thesis dedicated to the study of variable stiffened structures using lamination parameters [31]. A comprehensive framework for optimizing variable stiffened laminates is used to solve several sample problems and can perhaps be considered the greatest contribution of this work. A three-step optimization framework is presented. Much like the two-level method, optimum lamination parameters are found using a gradient-based method. The second level is essentially broken up into two separate steps where the fiber angles at each element node are determined then fiber paths are constructed based on the fiber angles. To validate the method, several test problems are presented starting with simple theoretical geometries and progressing to realistic aircraft structural components.

In the current research, we use many of the same processes used in the past, but we also add some unique twists here and there. We adopt the two-level approach for both orthotropic and variable stiffened composite plates. In the first level, we obtain optimum lamination parameters using a gradient-based optimization algorithm known as the Method of Moving Asymptotes. We present a few different methods for determining ply angles. The first and simplest, based on the method developed by Miki, does not require any additional optimization. This method tends to be limited to orthotropic plates and only yields practical solutions for a limited band of lamination parameters. Second, we show how the limitation of Miki’s method can be circumvented with a simple search-based optimization such as a GA. Third, we employ a method of laying out curvilinear fiber paths based on shifted splines that was developed by Gurdal and Olmedo [13] in attempt to determine optimum fiber paths for variable stiffened plates. Finally, we use a GA to return the element ply angles of a variably stiffened plate.
CHAPTER 3

BASIC THEORY

No text concerning composite laminates would be complete without the inclusion of basic equations in plate theory and this one is no exception. To begin, the fundamental relations that govern the linear elastic response of an isotropic medium in three dimensions are presented. Next, it will be demonstrated how the equations can be reduced to two-dimensional relations when two or more thin isotropic layers are laminated together.

3.1 STRAIN DISPLACEMENT RELATIONS

An isotropic plate is shown in Figure 3.1. The key characteristic of an isotropic plate is that it’s material properties do vary with direction. Only the Young’s Modulus and Poisson’s ratio are needed to describe to fully describe an isotropic plate. Hence, the isotropic plate is an ideal starting point for developing the characteristic equations for general plates.

![Diagram of an isotropic plate with axes labeled x, y, and z, and deformation vectors w and u.]

Figure 3.1. Isotropic plate.

When load is applied, an elastic body undergoes displacement that varies spatially over the domain. For small deformations, the following linear form of the strain displacement relations holds:
Another component in describing the elastic behavior of materials is the stress-strain relation. The most general form for a three-dimensional anisotropic material can be described by Hooke’s Law as (matrix form):

\[ \sigma = C \varepsilon \]  

(3.1)

\( \sigma \) and \( \varepsilon \) represent 6 x 1 vectors for stress and strain respectively. The matrix, \( C \), is a 6x6 matrix known as the material stiffness matrix. Now, because of symmetry, the number of independent material constants reduces from 36 to 21 [32].

In the case of a three-dimensional isotropic material, the number of material constants reduces even further to just 2, \( C_{11} \) and \( C_{12} \). Equation 3.1 can be written in an expanded form as (cite):

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{C_{11} - C_{12}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{C_{11} - C_{12}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{C_{11} - C_{12}}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix}
\]  

(3.2)

The coefficients, \( C_{11} \) and \( C_{12} \), have the following relationship to the Young’s modulus, \( E \), the shear modulus, \( G \), and Poisson’s ratio, \( \nu \), by:

\[
C_{11} = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)}, C_{12} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, C_{11} - C_{12} = \frac{E}{(1 + \nu)} = G
\]

(3.3)

Note that the shear modulus is dependent on Young’s modulus and Poisson’s ratio.
3.3 Equilibrium Equations

For a body to be in equilibrium, the net internal and external forces acting on a
differential element in any given direction must be zero. The equilibrium equations can be
expressed in differential form in terms of stress and x, y and z coordinates as:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\]

From the sum of moments, the following simplification is obtained:

\[\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}\]

3.4 In-Plane Response of Isotropic Layer(s)

Thin structures such as laminated plates carry loads primarily in two ways, either
through tension and compression acting in the plane of the plate or by bending of the plate.
We first examine the former. It should be noted that although optimization studies in the
present text only involve bending loads, the in-plane FEM was used as a building block for
developing the bending FEM; and hence, is included in this discussion.

3.4.1 Plane Stress

If there is no load acting on a thin plate in the z-direction, the plate in said to be in
plane-stress. Hence, \(\sigma_z = \tau_{zx} = \tau_{zy} = 0\) at the z-normal surfaces. The equilibrium equations can be further simplified as:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]

Note that the third equation for z-direction stresses disappears completely. However,
the normal strain, \(\varepsilon_z\), is not zero. This can be seen by substituting \(\sigma_z = 0\) into Equation (3.2) and solving for \(\varepsilon_z\) in terms of in plane normal strains, \(\varepsilon_x\) and \(\varepsilon_y\), to obtain:

\[
\varepsilon_z = \frac{\nu}{1 - \nu} (\varepsilon_x + \varepsilon_y)
\]
After substituting Equation (3.4) into Equation (3.1) then substituting the values of
$C_{11}$ and $C_{12}$ from Equation (3.3) and assuming the plane stress conditions, a new material
stiffness matrix is obtained in which the coefficient Q is used instead of C:

$$
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
$$

where the Q’s are known as the reduced stiffness coefficients and are defined as:

\[
Q_{11} = \frac{E}{1 - v^2},
\]
\[
Q_{12} = \frac{vE}{1 - v^2} = \frac{\mu E}{2},
\]
\[
Q_{66} = G = \frac{E}{2(1 + v)}
\]

3.4.2 Symmetrically Laminated Layers Under In-Plane Loading

It is now possible to turn the discussion to a laminated material. As a first step, the
symmetrically laminated plate is considered. Each layer of the plate is an isotropic material
in a state of plane stress and is perfectly bonded to its adjacent layer(s). When compared to a
single layer isotropic plate under in-plane, the following can be seen:

- The perfect bonding assumption guarantees that the strain distributions are constant through the entire thickness of the laminate. The strain level of every layer is the same as its adjacent layer(s). Typically this strain is characterized as the mid-plane strain, $\varepsilon^0$.
- The stresses in adjacent layers are not necessarily the same and will likely vary from one layer to the next. This can be attributed directly to the differences in Young’s Modulus between different layers.

The in-plane stiffness matrix, $A$, can be obtained by integrating the stresses through
the thickness of the laminate of thickness, $h$ where the $N$ terms are known as stress resultants.

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\int_{-h/2}^{h/2}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\, dz
$$

(3.6)
After substituting Equation (3.5) into Equation (3.6), it can be seen that the in-plane stiffness matrix turns out to be a summation of the weighted contributions of each individual layer’s reduced stiffness:

$$A_{ij} = \sum_{k=1}^{N} Q_{ij(k)}(z_k - z_{k-1})$$

(3.7)

Once the in-plane stiffness matrix is known, the stress-resultants can be calculated by multiplying the A matrix by the mid-plane strain vector giving in matrix notation,

$$\mathbf{N} = \mathbf{A}\varepsilon$$

### 3.5 BENDING DEFORMATIONS

In addition to the perfect bonding and layer dependent stress assumptions, a plate undergoing bending is subject to assumptions commonly known as the Kirchhoff-Love assumptions which are discussed in the next section.

#### 3.5.1 Kirchhoff-Love Assumptions

The Classical Laminated Plate Theory (CLPT) can be considered an extension of classical plate theory (CPT) as applied to composite laminates. As in CPT, the Kirchhoff assumptions hold and are given as follows:

- Straight lines perpendicular to the mid-plane (transverse normals) remain straight and perpendicular after deformation.
- There is no elongation of the transverse normals. (No change in thickness)
- The transverse normals rotate so that they stay normal to the mid-plane after deformation.

The first two assumptions reveal that the transverse displacement is independent of the thickness coordinate and the transverse normal strain, $\varepsilon_{zz}$, is zero. This leads to zero transverse shear strains ($\varepsilon_{xz} = \varepsilon_{yz} = 0$).

#### 3.5.2 Displacements and Strains of a Single Layer

The displacement fields in the x, y and z directions can be described by the variables $u$, $v$ and $w$ respectively. The Kirchhoff assumption requires that the displacement fields be as follows:
Upon closer inspection of Equation (3.8), the second assumption requires some discussion since it conflicts with Equation (3.4) which implies that for the plane stress state, the out-of-plane deformation is non-zero. Research has shown that despite this inconsistency, CLPT provides an adequate base and is adequate for most engineering needs.

Based on the Kirchhoff-Love assumptions and kinematics, the bending strains in single layer can be characterized as:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = z \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = -\begin{bmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix}
\]

where the \( \kappa \)'s are the curvatures.

Unlike the in-plane case, the strain distribution for pure bending within a single layer is not constant, but rather linear, and is a function of \( z \) which is antisymmetric about the mid-plane of the layer. Note that because \( w \) is constant through the thickness, the curvatures are also constant.

Stresses within a single layer are not constant either and, like the strain distribution, are antisymmetric with respect to the mid-plane.

As in the in-plane case, it is possible to integrate the stress through the thickness to get a resultant load. This time the resultant is a moment, \( M \), and therefore, a couple exists between the stress and the distance from the mid-plane. The relation can be seen as
The matrix, $D$, is known as the flexural stiffness matrix and in a similar manner as the in-plane stiffness matrix, $A$, the values can be found by substitution and integration of Equation (3.9) to obtain an equation of the form $\mathbf{M} = \mathbf{D}\mathbf{k}$ where

$$D_{11} = D_{22} = \frac{Eh^3}{12(1 - \nu^2)}, D_{12} = \nu D_{11}, D_{66} = (1 - \nu)\frac{D_{11}}{2}$$

### 3.5.3 Bending Response of Symmetrically Laminated Layers

The flexural stiffness matrix for a symmetrically laminated plate can be obtained by applying the principles for the single layer discussed in the previous section and then summing the contributions of each layer in a manner similar to the in-plane case. The flexural stiffness $D$ can now be defined as

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} Q_{ij(k)} (z_k^3 - z_{k-1}^3)$$

### 3.5.4 Bending-Extension Coupling of Asymmetrically Laminated Layers

If a laminate were symmetrical about the mid-plane, as described in all of the discussion thus far, then in-plane and flexural stiffness would be all that is needed to describe elastic behavior of the plate. However, when the stacking sequence is asymmetric, a third matrix known as the coupling matrix, $B$, must be included. Again, although bending-extension coupling results are not presented in this paper, the basic equations are included for completeness. It should be noted that the basic Matlab and FORTRAN codes developed for this research can be easily modified to include bending-extension coupling.

When a laminate stack is asymmetric, the response to in-plane loads and pure bending is quite different than for the symmetric case. The application of in-plane loads produces bending as well as in-plane strains. Conversely, the application of bending moments creates both in-plane and bending strains.
Take, for example, an asymmetric plate subject to a moment which will create curvatures. For the symmetric case, there is no stress resultant due to the antisymmetric distribution of stress through the cross section. However, for the asymmetric case, the stress distribution through the laminate cross section is not antisymmetric. Therefore, the stress resultants do not vanish and Equation (3.10) can be written for the bending case. After substituting the stress-strain and strain-curvature relations and performing an integration over z, the coupling matrix, \( B \), is found to be

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} Q_{ij(k)} (z_k^2 - z_{k-1}^2)
\]  

(3.11)

The coupling matrix can also be found by applying the converse. That is, to assume that only an in-plane load is applied, but that the moment resultants, \( M \), are not zero.

### 3.6 Orthotropic Layers

Many laminates are composed of fiber-reinforced layers in which, unlike an isotropic layer, \( E_1 \neq E_2 \), Young’s Modulus differs depending on whether the layer is orthogonal to or parallel to the fiber direction. In addition, since it is possible to rotate the orientation of the fiber relative to a given coordinate system, the notation is changed so that, instead of \( x \) and \( y \), the subscripts 1 and 2 are used. Axis 1, defined as the first principal axis, is parallel to the fiber direction. Axis 2, the second principal axis, is orthogonal to the fiber direction (See Figure 3.2). Finally, a ply stacking convention that will be used for the remainder of this paper is given in Figure 3.3.

The governing equations and assumptions for an isotropic layer still apply, with the exception that in the material stiffness matrix, \( C \), \( C_{11} \) and \( C_{22} \) are not equal. So, by means analogous to the isotropic layer, the stress-strain relation for an orthotropic layer can be derived.

The reduced stiffness, \( Q_{ij} \)'s, are

\[
Q_{11} = \frac{E_1}{1 - \nu_{12}v_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}v_{21}}, \quad Q_{12} = \frac{v_{12}E_2}{1 - \nu_{12}v_{21}}, \quad Q_{66} = G_{12}
\]

Since an orthotropic layer can be oriented at any angle relative to \( x-y \) coordinate system, there needs to be a way to retrieve the stresses and strains of the layer relative to the \( x-y \) system. This is accomplished through a coordinate transformation matrix. The stress-strain relationship can be given as
Figure 3.2. Laminate of Orthotropic Plies.

Figure 3.3. Ply stacking convention.
\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} = \begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{16} \\
\tilde{Q}_{12} & \tilde{Q}_{22} & \tilde{Q}_{26} \\
\tilde{Q}_{16} & \tilde{Q}_{26} & \tilde{Q}_{66}
\end{bmatrix} \begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix}
\] (3.12)

where
\[
\begin{align*}
\tilde{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{11} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta, \\
\tilde{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} \sin^4 \theta + \cos^4 \theta), \\
\tilde{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta, \\
\tilde{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta, \\
\tilde{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta, \\
\tilde{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} \sin^4 \theta + \cos^4 \theta
\end{align*}
\] (3.13)

The above equations can be put into a simpler form by using trigonometric identities and defining a set of new parameters, \( U \), which are called the material invariants.

\[
U_1 = \frac{1}{8} (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})
\]
\[
U_2 = \frac{1}{2} (Q_{11} + 3Q_{22})
\]
\[
U_3 = \frac{1}{8} (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})
\]
\[
U_4 = \frac{1}{8} (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})
\]
\[
U_5 = \frac{1}{8} (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})
\] (3.14)

Substituting Equation 3.14 into Equation 3.13:
\[
\begin{align*}
\tilde{Q}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\
\tilde{Q}_{12} &= U_4 - U_3 \cos 4\theta \\
\tilde{Q}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\
\tilde{Q}_{16} &= \frac{1}{2} U_1 \sin 2\theta + U_3 \sin 4\theta \\
\tilde{Q}_{26} &= \frac{1}{2} U_2 \sin 2\theta - U_3 \sin 4\theta \\
\tilde{Q}_{66} &= U_5 - U_3 \cos 4\theta
\end{align*}
\] (3.15)

Note that both the Q’s and the U’s are independent of ply orientation in Equation (3.12). The advantage to the Equation (3.13) version of the reduced stiffness definitions is that the constant parts of \( \tilde{Q}_{11}, \tilde{Q}_{12}, \tilde{Q}_{22}, \tilde{Q}_{66} \), given in terms of \( U_1, U_4 \) and \( U_5 \) are not
dependent on ply orientation. This simplified way of writing reduced stiffness coefficients is useful in conducting design optimization studies.

The Matlab suite of codes developed for this research includes a function that contains a material library (material_library.m) with Graphite and E-glass materials included. Materials can easily be added by inputting the basic material properties, Young’s Moduli, Shear Modulus and Poisson’s ratio. The function is set up to return values of the material invariants, $U_1$, $U_2$ etc. as global variables that can be easily be made available to other functions.

For a laminated plate of orthotropic plies, Equations (3.7), (3.10), (3.11) can be used to represent the stiffness matrices simply by substituting $Q$’s for all the $Q$’s

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij(k)} (z_k^2 - z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij(k)} (z_k^3 - z_{k-1}^3)$$  \hspace{1cm} (3.16)

An early building block for this research was a Matlab function (ABD_matrix.m) that computes the $A$, $B$ and $D$ matrices, effective Young’s moduli, shear moduli, and Poisson’s ratios ($E_x$, $E_y$, $G_{xy}$, $\nu_{xy}$, $E_x^b$, $E_y^b$, $G_{xy}^b$, $\nu_{xy}^b$). When developing finite element methods to analyze a laminate of orthotropic plies, it was often necessary to compare the FEM solutions to known closed-form orthotropic plate solutions; hence, a code that could quickly return laminate properties proved to be an essential first building block to this research.

Once the $A$, $B$ and $D$ matrices are computed, the three matrices can be lumped together into a single 6x6 matrix, called the ABD matrix, that fully describes the stiffness properties of a given laminate. Furthermore, if the laminate center plane strains and curvatures are known, the resultant forces and moments can be calculated from Equation (3.17).
\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

Conversely, if the resultants are known, the inverse of the ABD matrix can be used to calculate the center plane strains and curvatures as shown in Equation (3.18).

\[
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{16} & \beta_{11} & \beta_{12} & \beta_{16} \\
\alpha_{12} & \alpha_{22} & \alpha_{26} & \beta_{12} & \beta_{22} & \beta_{26} \\
\alpha_{16} & \alpha_{26} & \alpha_{66} & \beta_{16} & \beta_{26} & \beta_{66} \\
\beta_{11} & \beta_{12} & \beta_{16} & \delta_{11} & \delta_{12} & \delta_{16} \\
\beta_{12} & \beta_{22} & \beta_{26} & \delta_{12} & \delta_{22} & \delta_{26} \\
\beta_{16} & \beta_{26} & \beta_{66} & \delta_{16} & \delta_{26} & \delta_{66}
\end{bmatrix}
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix}
\]

### 3.7 PRINCIPLE OF VIRTUAL WORK AND WEAK FORM EQUATIONS OF MOTION

The total potential energy of an elastic body, \(\Pi\), can be described as simply the sum of the internal strain energy, \(U\), and the energy due to external forces, \(V\).

\[
U + V = \Pi
\]  
(3.19)

The minimum total potential energy is found by applying the principle of virtual work to Equation (3.19):

\[
\delta U + \delta V \equiv \delta \Pi = 0
\]  
(3.20)

Where \(\delta U\) is the virtual strain energy:

\[
\delta U = \int_{\Omega} \int_{\frac{-h}{2}}^{\frac{h}{2}} \left( \sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + 2\sigma_{xy} \delta \varepsilon_{xy} \right) dz dx dy
\]

\[
= \int_{\Omega} \frac{h}{2} \left[ \sigma_{xx} (\delta \varepsilon_{xx}^{(0)} + z \delta \varepsilon_{xx}^{(1)}) + \sigma_{yy} (\delta \varepsilon_{yy}^{(0)} + z \varepsilon_{yy}^{(1)}) + \sigma_{xy} (\delta \gamma_{xy}^{(0)} + z \gamma_{xy}^{(1)}) \right] dz dx dy
\]  
(3.21)
And, $\delta V$ is the virtual work done by applied forced:

$$
\delta V = - \int_{\Omega_0} \left[ q_b(x,y) \delta w \left( x, y, \frac{h}{2} \right) + q_t(x,y) \delta w \left( x, y, -\frac{h}{2} \right) \right] dxdy
$$

$$
- \int_{\Gamma_0} \left[ \frac{h}{2} \left( \partial_{nn} \delta u_n + \partial_{ns} \delta u_s + \partial_{nz} \delta w \right) dzds
$$

$$
= - \int_{\Omega_0} \left[ q_b(x,y) + q_t(x,y) \delta w_0(x,y) \right] dxdy
$$

$$
- \int_{\Gamma_0} \left[ \frac{h}{2} \left( \partial_{nn} \left( \delta u_{0n} - z \frac{\partial \delta w_0}{\partial n} \right) + \partial_{ns} \left( \delta u_{0s} - z \frac{\partial \delta w_0}{\partial s} \right) + \partial_{nz} \delta w_0 \right) dzds
$$

(3.22)

where $q_b$ is the distributed force at the bottom of the laminate and $q_t$ is the distributed force at the top of laminate, $(\partial_{nn}, \partial_{ns}, \partial_{nz})$ are the specified stress components on the portion $\Gamma_0$ of the boundary $\Gamma$, $(\delta u_{0n}, \delta u_s)$ are the virtual displacements along the normal and tangential directions, respectively, on the boundary $\Gamma$ (Figure 3.4 [33]).


Substituting Equations (3.21) and (3.22) into Equation (3.20) and integrating through the thickness of the laminate to obtain

$$
\int_{\Omega_0} \left[ N_{xx} \delta \varepsilon_{xx}^{(0)} + M_{xx} \delta \varepsilon_{xx}^{(1)} + N_{yy} \delta \varepsilon_{yy}^{(0)} + M_{yy} \delta \varepsilon_{yy}^{(1)} + N_{xy} \delta \gamma_{xy}^{(0)} + M_{xy} \delta \gamma_{xy}^{(1)} - q \delta w_0 \right] dxdy =
$$

$$
= \int_{\Gamma_0} \left( \tilde{N}_{nn} \delta u_{0n} + \tilde{N}_{ns} \delta u_{0s} - \tilde{M}_{nn} \frac{\partial \delta w_0}{\partial n} - \tilde{M}_{ns} \frac{\partial \delta w_0}{\partial s} + \tilde{Q}_n \delta w_0 \right) ds = 0
$$

(3.23)

where $q = q_b + q_t$ is the total transverse load and
\[
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{bmatrix} = \int \frac{h}{2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \, dz,
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} = \int \frac{h}{2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \, z \, dz.
\]

\[
\begin{bmatrix}
\bar{N}_{nn} \\
\bar{N}_{ns}
\end{bmatrix} = \int \frac{h}{2} \begin{bmatrix}
\bar{\sigma}_{nn} \\
\bar{\sigma}_{ns}
\end{bmatrix} \, dz,
\begin{bmatrix}
\bar{M}_{nn} \\
\bar{M}_{ns}
\end{bmatrix} = \int \frac{h}{2} \begin{bmatrix}
\bar{\sigma}_{nn} \\
\bar{\sigma}_{ns}
\end{bmatrix} \, z \, dz,
\]

\[
\bar{Q}_n = \int \frac{h}{2} \bar{\sigma}_{nz} \, dз.
\]

\((N_{xx}, N_{yy}, N_{xy})\) are the in-plane force resultants, \((M_{xx}, M_{yy}, M_{xy})\) are the moment resultants, and \(Q_n\) is called the transverse force resultant (Figure 3.5 [33]).

---

The virtual strains in terms of the virtual displacements are

\[
\delta \varepsilon_{xx}^{(0)} = \frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial x}, \quad \delta \varepsilon_{xx}^{(1)} = -\frac{\partial^2 w_0}{\partial x^2}
\]

\[
\delta \varepsilon_{yy}^{(0)} = \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial y}, \quad \delta \varepsilon_{yy}^{(1)} = -\frac{\partial^2 w_0}{\partial y^2}
\]

\[
\delta \gamma_{xy}^{(0)} = \frac{\partial \delta u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial x}, \quad \delta \gamma_{xy}^{(1)} = -2 \frac{\partial^2 w_0}{\partial x \partial y}
\]

Substituting the virtual strains from Equation (3.24) into Equation (3.23) and then integrating by parts to relieve the virtual displacements of any differentiation, so that the fundamental lemma of variational calculus can be used to obtain

\[
\int_{\Omega_0} \left[-N_{xx,x} \frac{\partial u_0}{\partial x} - \left(N_{xx} \frac{\partial w_0}{\partial x}\right)_x \frac{\partial w_0}{\partial x} - M_{xx,xx} \frac{\partial w_0}{\partial x} - N_{yy,y} \delta v_0 - N_{xy,x} \delta v_0 - \left(N_{xy} \frac{\partial w_0}{\partial y}\right)_x \delta w_0 \right. \\
\left. - M_{xy,xy} \frac{\partial w_0}{\partial x} - M_{yy,yy} \frac{\partial w_0}{\partial y} - N_{xy,x} \delta v_0 - \left(N_{xy} \frac{\partial w_0}{\partial y}\right)_y \delta w_0 - 2M_{xy,xy} \delta w_0 - q \delta w_0 \right] dx dy \\
+ \oint_{\Gamma} \left[N_{xx} n_x \frac{\partial u_0}{\partial x} + \left(N_{xx} \frac{\partial w_0}{\partial x}\right)_x n_x \delta w_0 - M_{xx} n_x \frac{\partial \delta w_0}{\partial x} \\
+ M_{xx} n_x \delta w_0 + N_{yy} n_y \delta v_0 + \left(N_{yy} \frac{\partial w_0}{\partial y}\right)_y n_y \delta w_0 - M_{yy} n_y \frac{\partial \delta w_0}{\partial y} \\
+ M_{yy} n_y \delta w_0 - M_{xx} n_x \frac{\partial \delta w_0}{\partial y} + M_{xy} n_y \delta w_0 - M_{xy} n_y \frac{\partial \delta w_0}{\partial x} \\
+ M_{xy} n_y \delta w_0 + N_{xy} n_y \delta u_0 + N_{xy} n_x \delta v_0 + N_{xy} \frac{\partial w_0}{\partial y} n_x \delta w_0 \\
+ N_{xy} \frac{\partial w_0}{\partial x} n_y \delta w_0 \right] ds = 0
\]

where a comma follow by subscripts denotes differentiation with respect to the subscripts:

\[ N_{xx,x} = \frac{\partial N_{xx}}{\partial x}, \text{ and so on.} \quad n_x = \cos \theta \text{ and } n_y = \sin \theta. \]

Collecting the coefficients of each of the virtual displacements (\(\delta u_0, \delta v_0, \delta w_0\)) together and noting that the virtual displacements on \(\Gamma\) are zero to obtain
\[
\int_{\Omega} \left[ -(N_{xx,x} + N_{xy,y}) \delta u_0 - (N_{xy,x} + N_{yy,y}) \delta v_0 - (M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + M_{xx,xy} + M_{xy,xy} + M_{yy,yy} + M_{xy,yy}) \delta w_0 + N(w_0) + q \right] d\Omega
\]

where

\[
N(w_0) = \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right)
\]

\[
P(w_0) = \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) n_y
\]

The Euler–Lagrange equations can be obtained by setting the coefficients of \( \delta u_0, \delta v_0 \) and \( \delta w_0 \) over \( \Omega \) to zero separately:

\[
\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0
\]

\[
\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0
\]

\[
\delta w_0 : \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + N(w_0) + q = 0
\] (3.25)

Next we shall develop the weak form equations of motion. First, we will multiply Equation 3.25 \( \delta u_0, \delta v_0 \) and \( \delta w_0 \) respectively and integrating over element domain to obtain

\[
\int_{\Omega_e} \delta u_0 \left[ \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right] d\Omega = 0
\]

\[
\int_{\Omega_e} \delta v_0 \left[ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} \right] d\Omega = 0
\]

\[
\int_{\Omega_e} \delta w_0 \left[ \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x} \left( \hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( \hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) + q \right] d\Omega = 0
\]
where $\hat{N}_{xx}, \hat{N}_{xy}$ and $\hat{N}_{yy}$ are in-plane edge forces. The stress and moment resultants, $N_{xx}, N_{xy}, M_{xx}$ etc. can be found in terms of displacements $(u_0, v_0, w_0)$ from Equation (3.17).

Integration by parts to weaken the differentiability of $u_0, v_0$ and $w_0$ results in the weak form equations.

In the next chapter, we will show how the weak form equation can be converted to the finite element equations with the aid of a couple of substitutions.

\[
\int_{\Omega_e} \left[ \frac{\partial^2 \delta w_0}{\partial x^2} M_{xx} - 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} M_{xy} + \frac{\partial^2 \delta w_0}{\partial x^2} M_{yy} + \frac{\partial \delta w_0}{\partial x} \left( \hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) \right] dx dy
\]
\[
+ \frac{\partial \delta w_0}{\partial y} \left( \hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yx} \frac{\partial w_0}{\partial y} \right) + q \delta w_0 \right] dx dy
\]
\[
- \oint_{\partial \Omega_e} \left[ \left( \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \hat{N}_{xx} \frac{\partial w_0}{\partial x} + \hat{N}_{xy} \frac{\partial w_0}{\partial y} \right) n_x \right] ds
\]
\[
+ \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + \hat{N}_{xy} \frac{\partial w_0}{\partial x} + \hat{N}_{yy} \frac{\partial w_0}{\partial y} \right) n_y \right] ds
\]
\[
+ \oint_{\Gamma_e} \left[ \frac{\partial \delta w_0}{\partial x} \left( M_{xx} n_x + M_{xy} n_y \right) + \frac{\partial \delta w_0}{\partial y} \left( M_{xy} n_x + M_{yy} n_y \right) \right] ds = 0
\]

(3.26)
CHAPTER 4

FINITE ELEMENT MODEL

In this chapter, we shall discuss in detail the assumptions, theory and algorithm used to implement the FEM portion of the optimization. The chapter is organized as follows

- General finite element equations of motion
- Explanation of computer algorithm
- Two degrees of freedom (DOF) FEM with test problem (in-plane)
- Four DOF FEM with test problem (out-of-plane)

4.1 EQUATIONS OF MOTION

Before any work in variable stiffness plates could begin, a working finite element model (FEM) for a basic orthotropic plate had to be coded and tested. Element stiffness equations from classical laminated plate theory (CLPT) were obtained from the text, Mechanics of Laminated Composite Plates and Shells by J.N. Reddy [33]. Implementation of the FEM was carried out in Matlab using 4 node rectangular laminated plate elements.

4.1.1 Spatial Approximations

The displacement fields for $u_0$, $v_0$ and $w_0$ can be described using the following discretized approximations

$$
u_0(x,y) = \sum_{j=1}^{m} u_j^e \psi_j^e(x,y)$$

$$v_0(x,y) = \sum_{j=1}^{m} v_j^e \psi_j^e(x,y)$$

$$w_0(x,y) = \sum_{k=1}^{n} \Delta_k^e \varphi_k^e(x,y)$$

where $\psi_j^e, \varphi_k^e$ are the Lagrange and Hermite interpolation functions, respectively. The indices, $m$ and $n$, represent the number of DOF’s per element for each direction of displacement. For in-plane displacements, only one DOF per node is required in $u$ and $v$; and
hence, \( m = 4 \). In the case of out-of-place displacements, four DOF’s per node are required to describe \( w \) displacements with \( C^1 \) continuity; and so, \( n = 16 \).

### 4.1.2 Semi-Discrete FEM Model

The semi-discrete finite element equations are obtained by substituting the spatial approximations of Equation (4.1) into the weak form of Equation (3.26) and using the relationship of force/moment resultants to strain/curvatures given in Equation (3.17) to obtain:

\[
0 = \sum_{j=1}^{m} (K_{ij}^{11} u_j^e + K_{ij}^{12} v_j^e + M_{ij}^{11} \psi_j^e) + \sum_{k=1}^{n} (K_{ik}^{13} \Delta_k^e + M_{ik}^{13} \Delta_k^e) - F_l^1 - F_l^{T1}
\]

\[
0 = \sum_{j=1}^{m} (K_{ij}^{21} u_j^e + K_{ij}^{22} v_j^e + M_{ij}^{22} \psi_j^e) + \sum_{k=1}^{n} (K_{ik}^{23} \Delta_k^e + M_{ik}^{23} \Delta_k^e) - F_l^2 - F_l^{T2}
\]

\[
0 = \sum_{j=1}^{m} (K_{ij}^{31} u_j^e + K_{ij}^{32} v_j^e + M_{ij}^{32} \psi_j^e) + \sum_{k=1}^{n} (K_{ik}^{33} \Delta_k^e + M_{ik}^{33} \Delta_k^e) - F_l^3 - F_l^{T3}
\]

(4.2)

where \( i, j = 1 \ldots m; k, l = 1 \ldots n \) \((m = 4, n = 16)\). In this paper, thermal and inertia effects are not considered; therefore, terms with \( M \) and \( F^T \) coefficients are zero. Also, since all element ply stacking sequences are constructed in a balanced and symmetric manner, \( K^{13} \) and \( K^{23} \) are zero.

The coefficients of the in plane stiffness matrices are as follows:

\[
K_{ij}^{11} = A_{11} S_{ij}^{xx} + A_{16} (S_{ij}^{xy} + S_{ij}^{yx}) + A_{66} S_{ij}^{yy}
\]

\[
K_{ij}^{12} = A_{12} S_{ij}^{xy} + A_{16} S_{ij}^{yx} + A_{26} S_{ij}^{yy} + A_{66} S_{ij}^{yy}
\]

\[
K_{ij}^{22} = A_{66} S_{ij}^{xx} + A_{26} (S_{ij}^{xy} + S_{ij}^{yx}) + A_{22} S_{ij}^{yy}
\]

The coefficients of the bending stiffness matrix are

\[
K_{kl}^{33} = D_{11} T_{kl}^{xxx} + D_{12} (T_{kl}^{xyy} + T_{kl}^{yxy}) + 2D_{16} (T_{kl}^{xx} + T_{kl}^{xy}) + 2D_{26} (T_{kl}^{yxy} + T_{kl}^{yy}) + 4D_{66} T_{kl}^{xyy} + 2D_{22} T_{kl}^{yy} + \frac{D_{21}^2}{2} T_{kl}^{xyxy} + 2D_{22} T_{kl}^{yy} + \frac{D_{21}^2}{2} T_{kl}^{xyxy} + 2D_{22} T_{kl}^{yy} \tag{4.3}
\]

and

\[
S_{ij}^{a\beta} = \int_{\Omega^e} \frac{\partial \psi_i^e}{\partial a} \frac{\partial \psi_i^e}{\partial \beta} \, dx \, dy
\]
\[ T_{ij}^{\alpha \beta \gamma \delta} = \int_{\Omega} \left( \frac{\partial^2 \varphi_i^e}{\partial x \partial \psi} \frac{\partial^2 \varphi_j^e}{\partial y \partial \psi} \right) \, dx \, dy \]

where \( \alpha, \beta, \gamma, \delta \) can be equal to \( x \) or \( y \). \( \psi \) and \( \varphi \) are the Lagrange and Hermite interpolation function which are discussed later in this chapter.

### 4.2 Basic FEM Algorithm

A major building block needed to optimize composite plates is a working and reasonably fast finite element algorithm. Following some techniques developed by Ferreira [34], a basic algorithm that can handle orthotropic and variable stiffened laminates was developed. Both in-plane and bending loads can be input. However, for optimization, in order to reduce computation time required to develop the overall optimization framework, only out-of-plane loads can be input. Now that the framework is complete, adding design variables, load cases, boundary conditions etc. should be a relatively simple process.

At the beginning of this research project the FEM codes existed as stand-alone Matlab functions. But as we progressed into optimization, it became necessary to integrate the FEM codes into a larger optimization framework. A Matlab user has the option to just run FEM studies simply by changing a character string in the main program (comp_plate_opt.m) called ‘optonoff’ to either ‘on’ for running a lamination parameter and/or ply angle optimization and ‘off’ if just a FEM analysis is desired. Optimization is discussed in the Chapter 5.

Figure 4.1 depicts the basic flow of the finite element solution of the orthotropic laminated plate. In general, the only user intervention required is the input of a few parameters. Most inputs are defined such that character strings can be used to switch between certain inputs. The following section elaborates on some of the basic user interface portions of the FEM code and can serve as a guide for future code modification.

#### 4.2.1 Mesh Generation

This thesis investigates two basic geometries: square plates (AR=1) and long “beamlike” plates (AR=10). Square plates are defined as having 1m x 1m dimensions and long plates have 1m x .1m dimension in x and y. For convenience, several square meshes (2x2, 3x3, etc.) and one long plate mesh (100x10) are predefined and the user can switch mesh densities by simply changing the character string in the input file that corresponds to
Figure 4.1. Basic FEM Algorithm.

mesh size. For example, for a 2x2 mesh, the input is defined below as an editable character string.

    platetype='2x2';

The number of elements in the x and y directions can be calculated based on mesh density and the plate size. A function that requires plate size and element count in x and y generates nodal coordinates and maps nodes to their corresponding elements. The main program includes an ‘on/off’ switch that allows the user to control whether or not the mesh is displayed to the screen.

4.2.2 Material

Several predefined composite materials are available in the Matlab program. Material choices can also easily be switched by editing a character string in the input file. Materials
can be added to the database with little effort. For convenience, lamina invariants \((U_1, U_2\) etc.) are returned by the material database function as well. The material choice can be set by editing a simple character string. For example, all problems shown in the present work use a graphite fiber and the material is called from the database as:

```matlab
material='Graphite';
```

### 4.2.3 Ply Stacking Sequence

The stacking sequence is input into Matlab in vector form starting from the bottom ply. The user must specify the correct ply thickness in this section of Matlab code. Even if a lamination parameter based optimization study is desired, a stacking sequence should be specified. The reasons are that the Matlab code uses the overall thickness of the defined stack as a fixed parameter to determine the \(D\) matrix for the laminate, and the specified laminate serves as a baseline to determine a scaling factor for the lamination parameter optimization. Therefore, an accurate ply count and ply thickness need to be entered in order for the FEM and optimization algorithms to work correctly.

### 4.2.4 Loads

In-plane and out-of-plane loads can be specified in the input file. In-plane loads are input as evenly distributed force per unit length of a given side of the plate. Out-of-plane loads are first specified as a total pressure (force per unit area). Then the user, by changing the value of the character string called “loadtype,” may choose to distribute the load evenly or to apply the load as a series of point loads. If point load(s) are desired, the value of “loadtype” is:

```matlab
loadtype='op';
```

An evenly distributed load would be entered as

```matlab
loadtype='d';
```

If point loads are chosen, the user must edit a matrix called “qalloc” which collects separate data for each point load. The first two columns of the matrix represent \(x\) and \(y\) percentage locations of the load on the plate. The third column represents a percentage of the total pressure applied. An example entry of a two point load is as follows:

```matlab
%        x%  y% load%
qalloc= [.25  0 .5;
   .5 .75 -.5];
```
For an evenly distributed load, the value of “qalloc” does not need to be completed.

### 4.2.5 Boundary Conditions

Specifying the boundary condition is a matter of changing the value of a character string in the main program. For example, for a plate simply supported on all four sides, the user would specify the value of “boundcondition” as:

```plaintext
boundcondition='ssss';
```

Several predefined choices of boundary conditions are available:

- Simply supported on all four sides ('ssss')
- A quarter plate with symmetry boundary conditions and simply supported on the exterior edges ('quarter_sym')
- Clamped on all four sides ('cccc')
- Simply supported at the ends and free on horizontal sides ('sfsf')
- Clamped at one end (cantilevered beam or plate) ('cfff')
- Simply supported on just the corners of the plate ('4pt')
- User Defined Placement of Simple Point Supports, ('general_simple')

#### 4.2.6 S and T matrices

The values of $S_{ij}^{xx}$, $S_{ij}^{xy}$ etc. and $T_{ij}^{xxxx}$, $T_{ij}^{xxxxy}$ etc. are calculated in separate functions and their values are based on element size. Each of the S matrices is a 4x4 matrix and each T matrix is 16x16. The S matrices are computed starting from Lagrange shape functions; the T matrices from Hermite functions. Differentiation and integration over the elements are accomplished analytically in Matlab using symbolic values for the x and y variables. The T matrices can take up to one minute to generate which proved to be a very high overhead cost to pay in many smaller scale optimization studies. Therefore, for all the predefined meshes, values of the T matrices are pre-computed and are read in from separate .mat files.

### 4.3 IN-PLANE FEM

As a first step in developing a full 6 degrees of freedom FEM model, an element with in-plane degrees of freedom only was defined and tested. The in-plane model only requires two degrees of freedom per element node ($u, v$ displacements) and the linear Lagrange functions can be used to interpolate the displacement fields. In contrast, the fully conforming
out-of-plane model requires non-linear Hermite interpolation functions and four degrees of freedom per element node. Nevertheless, the basic algorithm for both cases is the same with differences imbedded in functions that generate the shape functions and stiffness matrices.

### 4.3.1 Element Geometry and Displacement Field

Bilinear rectangular elements were used to describe element geometry (See Figure 4.2). A general form for $u$ and $v$ displacement field over the element can be written as

$$u(x, y) \text{ and/or } v(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$$

(4.4)

![Figure 4.2. 4 node rectangular element (in-plane displacements).](image)

### 4.3.2 Lagrange Shape Functions

The four Lagrange shape functions can be derived from Equation (4.4) and end up being absorbed into the global stiffness matrix; and hence, linearizing the problem. In terms of natural coordinates, the shape functions are given as [35].

$$\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
(1 - \xi)(1 - \eta) \\
(1 + \xi)(1 - \eta) \\
(1 + \xi)(1 + \eta) \\
(1 - \xi)(1 + \eta)
\end{bmatrix}$$
4.3.3 Element Equations of Motion

For loading that is in-plane only with no bending-extension coupling and also
neglecting thermal and inertia effects, Equation (4.2) reduces to:

\[ 0 = \sum_{j=1}^{m} (K_{ij}^{11} u_j^e + K_{ij}^{12} v_j^e) - F_i^1 \]

\[ 0 = \sum_{j=1}^{m} (K_{ij}^{21} u_j^e + K_{ij}^{22} v_j^e) - F_i^2 \]  \hspace{1cm} (4.5)

The individual \( K \) terms are composed of contributions from the A matrix and the Lagrange
shape functions. The two \( F_i \) terms are the in-plane forces in the x and y directions specified
at each of applied boundary nodes. Finally, Equation (4.5) can be shown in a fully
discretized form as:

\[
\begin{bmatrix}
[K^{11}] & [K^{12}] \\
[K^{12}]^T & [K^{22}]
\end{bmatrix}
\begin{bmatrix}
\{u^e\} \\
\{v^e\}
\end{bmatrix}
= \begin{bmatrix}
\{F^1\} \\
\{F^2\}
\end{bmatrix}
\]

4.3.4 Thin Plate in Traction

To illustrate the functionality and accuracy of the 2 DOF FEM, the simple problem of
a long thin laminated plate in traction was examined (Figure 4.3). This was an in-plane
loading problem and only the Lagrange shape functions were required to accurately model
the displacement field in the x and y directions. A quasi-isotropic layup was chosen to allow
easy comparison of the FEM displacement field to known exact solutions for isotropic plates.

Material properties for graphite/epoxy are given as follows:

\[ E_1 = 181,000 \, MPa \]
\[ E_2 = 10,300 \, MPa \]
\[ G_{12} = 7,170 \, MPa \]
\[ v = .19 \]

Each ply is .0004 m thick and \( a = .5 \, m, \, b = .1m \). A balanced and symmetric 8 ply layup was
chosen as \([+45/-45 0/90]_s\). A uniform tension load of \( P=1 \, MPa/m \) is applied at each end of
the strip. The A matrix is calculated to be

\[
\begin{bmatrix}
244.4 & 72.3 & 0 \\
72.3 & 244.4 & 0 \\
0 & 0 & 86.0
\end{bmatrix}
\] \hspace{1cm} MPa
Figure 4.3. Thin plate in traction.

For in-plane displacements, a mesh convergence study showed that the accuracy of the solution is independent of the mesh size. Therefore, a simple 5x1 mesh with square elements as shown in Figure 4.4 was chosen.

Figure 4.4 Mesh for a thin plate in traction.

For the applied load, a maximum elongation of $u = 0.0224$ m was seen in the longitudinal axis. The compression due to Poisson’s contraction was $v = 1.33 \times 10^{-3}$ m. Figures 4.5 and 4.6 depict the $u$ and $v$ displacement fields, respectively.

In order to check the validity of the finite element solution, two closed form solutions need to be compared to one another. Conveniently, Hooke’s Law provides a simple algebraic approach to obtaining an exact solution for the elongation of a plate in traction. To start, equivalent Young’s moduli must be determined. For a homogenous orthotropic plate, equivalent longitudinal and transverse moduli can be written as:

$$E_x = \frac{1}{h\alpha_{11}}, \quad E_y = \frac{1}{h\alpha_{22}}$$
Figure 4.5. Elongation of a thin plate in traction.

Figure 4.6. Compression of a thin plate in traction.
where $\alpha_{11}, \alpha_{22}$ are in-plane compliance terms that can be found by inverting the $A$ matrix. For a balanced and symmetric laminate, $\alpha_{11} = \alpha_{22}$, and hence, $E_x = E_y$. $E_x, E_y$ are calculated to be $6.97 \times 10^4$ MPa.

Hooke’s Law, the elongation at the end of the plate is:

$$u_x = \frac{pa}{AE_x}$$

$u_x$ is calculated to be $0.0224$ m which exactly matches the FEM solution. The axial strain can be found by simply dividing by the total length:

$$\varepsilon_x = \frac{u_x}{a}$$

The transverse strain is related to the axial strain by:

$$\varepsilon_y = -\nu \varepsilon_x$$

Finally, the transverse compression can be found as:

$$\Delta b = \varepsilon_y b$$

$\Delta b$ was found to be $-1.26 \times 10^{-3}$ m, a $1.4\%$ difference from the FEM solution. Hence, a close correlation is between the FEM and exact solution is proven.

### 4.4 Out-of-Plane FEM

Now, a model is presented for an element that can handle out-of-plane loads. Like the in-plane model, the element is rectangular with 4 nodes. However, the out-of-plane model has three additional degrees of freedom—two rotational and one curvature—per node. This type of element known as a $C^1$ [36] conforming type element (See Figure 4.7), guarantees inter-element continuity of the normal slope.

#### 4.4.1 Element Geometry and Displacement Field

The general displacement field in the $z$ direction can be written as:

$$w(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3$$

$$+ a_{10} x^3 y + a_{11} x^2 y^2 + a_{12} xy^3 + a_{13} x^3 y^2 + a_{14} x^2 y^3 + a_{15} x^3 y^3$$
4.4.2 Hermite Shape Functions

There are sixteen Hermite interpolation functions for the conforming plate element. The derivation is obtained in the same manner as the Lagrange interpolation functions. They are given as:

\[
\begin{align*}
\varphi_i^e &= g_{i1}(i = 1, 5, 9, 13) \\
\varphi_i^e &= g_{i2}(i = 2, 6, 10, 14) \\
\varphi_i^e &= g_{i3}(i = 3, 7, 11, 15) \\
\varphi_i^e &= g_{i4}(i = 4, 8, 12, 16)
\end{align*}
\]

\[
\begin{align*}
g_{i1} &= \frac{1}{16} \xi + \xi_i^2(\xi_0 - 2)(\eta + \eta_i)^2(n_0 - 2) \\
g_{i2} &= \frac{1}{16} \xi_i \xi + \xi_i^2(1 - \xi_0)(\eta + \eta_i)^2(n_0 - 2) \\
g_{i3} &= \frac{1}{16} \eta_i \xi + \xi_i^2(\xi_0 - 2) \eta + \eta_i^2(1 - n_0) \\
g_{i4} &= \frac{1}{16} \xi_i \eta_i \xi + \xi_i^2(1 - \xi_0) \eta + \eta_i^2(1 - n_0)
\end{align*}
\]

\[
\begin{align*}
\xi_0 &= \xi_i; \quad n_0 = \eta_i
\end{align*}
\]
4.4.3 Element Equations of Motion

For out-of-plane loading with no bending-extension coupling and neglecting thermal and inertia effects, Equation (4.2) reduces to:

\[ 0 = \sum_{k=1}^{n} K_{ik}^{33} \Delta_{i}^{e} - F_{k}^{3} \]

\[(4.6)\]

Equation (4.16) can be written in fully discretized form as

\[ [K^{33}]\{\Delta^{e}\} = \{F^{3}\} \]

4.4.4 Bending of an Orthotropic Plate

As with the in-plane FEM model, a sample problem with known closed form solutions is presented for a plate in bending. A balanced and symmetric layup is chosen. However, the plate geometry this time is square and the plate is simply supported on all four sides. An evenly distributed transverse pressure of q=1 KPa was applied (Figure 4.8). In Matlab, the settings for boundary condition and load are as follows:

```matlab
boundcondition='ssss'
loadtype='d'
```

![Figure 4.8. Simply supported laminated plate under uniformly distributed load.](image)

The material properties for this plate were the same as the in-plane sample problem from the previous section. However, experience has shown that for bending, additional plies
are required to achieve a ‘quasi-isotropic’ type of layup. This has to do with the fact that for smaller thicknesses, the difference between the $D_{11}$ and $D_{22}$ is significant enough to make the bending behavior highly directional. For this problem, a 16 ply layup was chosen because the difference between $D_{11}$ and $D_{22}$ approaches a reasonably small percent difference of 5.4%.

Layup: $[45, 90, 0, -45]_2$

The $D$ matrix was computed from Equation (3.16) as:

$$
[D] = \begin{bmatrix}
1.472 & 0.514 & 0.263 \\
0.514 & 1.823 & 0.263 \\
0.263 & 0.263 & 0.607
\end{bmatrix} \times 10^{-3} \text{ MPa}
$$

In order to test the validity of the finite element model, two closed form solutions for the simply supported plate under constant pressure loading are presented for comparison. The first is for a quasi-isotropic plate from classical plate theory. The second solution takes the more general form of the Navier Method. Both solutions presented assume that $D_{16}$ and $D_{26}=0$ which from the above $D$ matrix is not accurate. However, the magnitudes of $D_{16}$ and $D_{26}$ are small relative to the principal directions. For the purposes of this analysis $D_{12}$ and $D_{22}$, are negligible.

For an isotropic plate under uniform pressure load, a closed form solution can be obtained from classical plate theory as [37]:

$$
w_{max} = \frac{\alpha q b^4}{E_{avg} h^3} = .0024 \text{ m}
$$

where for a plate with an aspect ratio of 1,

$$\alpha = .0444
$$

Since $D_{11}$ and $D_{22}$ are not equal, an average of the two is taken to determine the Young’s Modulus:

$$E_{avg} = \frac{12(1 - \nu^2)}{h^3} \left(\frac{D_{11} + D_{22}}{2}\right)
$$

The Poisson’s ratio is calculated from:

$$\nu = \frac{D_{12}}{D_{22}} = .28$$
As a second closed form check, the Navier Method is used. The maximum $w$ displacement using the Navier Method is given as:

$$w_{max} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin \left( \frac{m\pi}{2} \right) \sin \left( \frac{n\pi}{2} \right) = .0025 \ m$$

where

$$W_{mn} = \frac{16qb^4}{\pi^6 mn(D_{11}m^4s^4 + 2(D_{12} + 2D_{66})m^2n^2s^2D_{22}n^4)}$$

One difference between in-plane and bending deflections is that the accuracy of the bending displacement field depends on the density of the FEM mesh. However, it is important to realize that accuracy comes at the price of computation time. For a single static analysis, one may be able to accept FEM runs on the order of minutes. However, when conducting optimization studies, several dozen FEM runs may be required and would make too fine a mesh impractical. Therefore, it was important to establish a practical and reasonably accurate FEM mesh before conducting any optimization studies.

Table 4.1 shows the maximum $w$ displacement and run times for four different mesh sizes with comparisons closed form solutions. Although, the $w_{max}$ for the 40x40 mesh has an error of about 1% to the Navier and 0.7% to the plate solution, the FEM run time is about two orders of magnitude greater than the next smaller mesh of 30x30. For nearly the same accuracy, the 30x30 mesh runs in less than 5 seconds and seems to be a good choice for optimization (See Figure 4.9 for displacement plot). However, we will later see that the second level optimization can run too slowly for practical use with a 30x30 mesh size. A 20x20 mesh costs less than 1% in accuracy and runs more than 5 times faster than the 30x30. But even more importantly, the smaller number of elements tends to be much more manageable by the second level optimization.

Finally, the compliance is calculated from the strain energy equation as

$$c = \sum_{i=1}^{nelem} u_e^T k_e u_e$$

(4.7)
### Table 4.1. Mesh Convergence Study, Square Plate in Bending

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$w_{\text{max}}$</th>
<th>% Difference Plate Solution</th>
<th>% Difference Navier Solution</th>
<th>Run time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>'10x10'</td>
<td>-0.00203</td>
<td>4.6</td>
<td>4.8</td>
<td>0.078</td>
</tr>
<tr>
<td>'20x20'</td>
<td>-0.00225</td>
<td>2.0</td>
<td>2.2</td>
<td>0.827</td>
</tr>
<tr>
<td>'30x30'</td>
<td>-0.00233</td>
<td>1.1</td>
<td>1.4</td>
<td>4.503</td>
</tr>
<tr>
<td>'40x40'</td>
<td>-0.00237</td>
<td>0.7</td>
<td>1.0</td>
<td>245.547</td>
</tr>
</tbody>
</table>

**Figure 4.9.** Displacement field for a simply supported laminated plate under uniformly distributed load (30x30 mesh).
The value of \( c \) was found to be \( 9.1911 \times 10^{-7}/\text{MPa} \) and is used to normalize the optimized compliance described in the next chapter.

### 4.4.5 End Supported Beam Under Uniformly Distributed Load

As a second test case and example, we look at a simply supported laminated beam. The beam is modeled using the same elements as the plate in the previous section; however, the aspect ratio is much larger (\( a = 1\text{m} \) and \( b = .1\text{m} \)) and only the ends are supported. An evenly distributed downward pressure of \( q = 1 \text{ KPa} \) is applied (Figure 4.10) as before and the same 16 ply layup is chosen and so the \( D \) matrix is also unchanged. The compliance was calculated from Equation (4.7) as \( 6.1963 \times 10^{-7}/\text{MPa} \). In Matlab, the settings for boundary condition and load are as follows:

```matlab
boundcondition='sfsf'
loadtype='d'
```

![Figure 4.10. End supported laminated plate under uniformly distributed load.](image)

To benchmark the FEM solution, an exact solution is needed. Since the length of the plate, \( a \), is much greater than the width, \( b \), the geometry can be treated as a classical beam in bending. From basic solid mechanics, the maximum deflection for a simply supported beam under a uniformly distributed load can be found as:

\[
    w_{\text{max}} = \frac{5}{384} \frac{qa^3}{E_x I_{x}} = .0103 \text{ m}
\]

where
\[
E^b_x = \frac{12}{h^3 \delta_{11}} = 57,898 \text{ MPa}, \quad l_{xx} = \frac{bh^3}{12} = 2.1845 \times 10^{-9} m^4
\]

\(\delta_{11}\) was found from the inverse of the D matrix (see Equation 5.13).

As with the square plate in bending, a mesh convergence study was done for this geometry. Four different mesh sizes were created. The values of the maximum deflection compared to the beam theory solution found from Equation (4.8) and the results are shown in Table 4.2. Similar to the square plate mesh convergence study, the accuracy of the FEM solution increases as the number of elements increases. One interesting point is the substantial jump in run time between the 30x3, 60x6 mesh and a 100x10 mesh with only small gains in accuracy.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>(w_{\text{max}})</th>
<th>% Difference Plate Solution</th>
<th>Run time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>'10x1'</td>
<td>-0.0090</td>
<td>3.5</td>
<td>0.0039</td>
</tr>
<tr>
<td>'30x3'</td>
<td>-0.0096</td>
<td>1.7</td>
<td>.0404</td>
</tr>
<tr>
<td>'60x6'</td>
<td>-0.0098</td>
<td>1.4</td>
<td>.6901</td>
</tr>
<tr>
<td>'100x10'</td>
<td>-0.0098</td>
<td>1.2</td>
<td>52.0153</td>
</tr>
</tbody>
</table>

A plot of the deflection shape is shown in Figure 4.11.

### 4.4.6 Cantilevered Plate

To further demonstrate the capabilities of the finite element model developed for this research, we look at a cantilevered laminated plate. The same types of elements, layup and loading as the previous two examples were chosen. This time the aspect ratio of the plate is 2 to 1 (a=1 m and b=.5 m). The value of the compliance was calculated as 1.8659e-5/MPa.

The Matlab setting for boundary condition and loading are given as

```matlab
boundcondition='cfff'
loadtype='d'
```
Figure 4.11. Long plate deflection (100x10 mesh).

For this problem, the FEM solutions are compared to an exact solution from beam theory. The tip displacement for a cantilevered beam under a uniformly distributed load is given as

\[ w_{max} = \frac{aq^4}{8E_I I_{xx}} = -0.0988 \text{ m} \]

The values for \( q, a, E_b \) and \( I_{xx} \) are the same values used in the previous section. A mesh convergence study is presented in Table 4.3. The results this time are a bit curious. Unlike the two previous examples, for this case increasing the number of elements doesn’t necessarily increase the accuracy of the solution. In fact, the best solution came from the second smallest mesh size and the accuracy steadily decreased as the mesh size was increased. A sample deflection plot of a 32x16 mesh is shown in Figure 4.12.

### 4.5 SUMMARY AND CONCLUSIONS

In this chapter we presented the underlying theory, algorithms and examples for the finite element models used in this research. We started by showing how the weak form equations developed in Chapter 3 are converted into finite element equations using substitutions for spacial approximations of the nodal displacements fields and expressions for the force and moment resultants. Next, we provided the basic algorithm of how the FEM
Table 4.3. Mesh Convergence Study, Cantilevered Plate

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$w_{\text{max}}$</th>
<th>% Difference in Beam Solution</th>
<th>Run time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘4x2’</td>
<td>-.1157</td>
<td>3.9</td>
<td>.0042</td>
</tr>
<tr>
<td>‘16x8’</td>
<td>-.0992</td>
<td>.1057</td>
<td>.0610</td>
</tr>
<tr>
<td>‘24x12’</td>
<td>-.0982</td>
<td>.157</td>
<td>.3563</td>
</tr>
<tr>
<td>‘32x16’</td>
<td>-.00978</td>
<td>.27</td>
<td>1.2269</td>
</tr>
<tr>
<td>‘40x20’</td>
<td>-.0975</td>
<td>.34</td>
<td>4.1578</td>
</tr>
<tr>
<td>48x24</td>
<td>-.0973</td>
<td>.38</td>
<td>72.8029</td>
</tr>
</tbody>
</table>

Figure 4.12. Cantilevered plate deflection (32x16 mesh).

code was setup in Matlab and followed by a guide to the various settings within the main program (material, loads, boundary conditions etc.). Finally, in order to prove the accuracy and also demonstrate some the flexibility of the FEM codes, several example problems were provided. In an effort to aid in the selection of mesh size for future optimization work, mesh convergence studies were carried out for the out-of-plane sample problems.
CHAPTER 5

OPTIMIZATION STUDIES

The end goal in many optimization studies involving composites is to determine the fiber orientation angles for the elements and/or the actual continuous fiber paths. For an orthotropic laminate, the fiber angles are oriented in the same direction in any given layer. When modeled with finite elements, every element is assigned the same stiffness properties (i.e. ABD matrix). On the other hand, in variable stiffened laminates fiber angles can vary within a single layer. Hence, the elements must be formulated to allow for variation in stiffness from element to element. In this chapter we examine optimization of laminated plates using a two-level approach. Both orthotropic and variable stiffened plates are considered.

5.1 LAMINATION PARAMETERS

For optimization, it can be useful to write Equation (3.16) in a form that utilizes the material invariants of Equation (3.14) and the reduced stiffness definitions of Equation (3.15). For example, the first terms of the A, B and D matrices can be written in an integral form as

\[
\{A_{11}, B_{11}, D_{11}\} = U_1 \left\{ \frac{h^3}{12} \right\} + U_2 \int_{-h/2}^{h/2} \cos 2\theta \{1,z,z^2\} \, dz + U_3 \int_{-h/2}^{h/2} \cos 4\theta \{1,z,z^2\} \, dz
\]

Expressions from the remaining terms of A, B and D matrices are given in Table 5.1.

<table>
<thead>
<tr>
<th>Table 5.1. ABD matrix in Terms of Lamination Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A_{11}, B_{11}, D_{11}}</td>
</tr>
<tr>
<td>{A_{16}, B_{16}, D_{16}}</td>
</tr>
</tbody>
</table>
| \begin{align*}
V_{0\{A,B,D\}} & = U_1 \\
V_{1\{A,B,D\}} & = U_2 \\
V_{2\{A,B,D\}} & = 0 \\
V_{3\{A,B,D\}} & = U_3 \\
V_{4\{A,B,D\}} & = 0
\end{align*} |

\[
\begin{align*}
\{A_{11}, B_{11}, D_{11}\} & = U_1 \left\{ h, 0, \frac{h^3}{12} \right\} \\
\{A_{12}, B_{12}, D_{12}\} & = U_4 \left\{ 0, 0, 0 \right\} \\
\{A_{21}, B_{21}, D_{21}\} & = U_5 \left\{ 0, 0, 0 \right\} \\
\{A_{22}, B_{22}, D_{22}\} & = U_1 \left\{ h, 0, \frac{h^3}{12} \right\} + U_2 \int_{-h/2}^{h/2} \cos 2\theta \{1,z,z^2\} \, dz + U_3 \int_{-h/2}^{h/2} \cos 4\theta \{1,z,z^2\} \, dz
\end{align*}
\]
The $V$ terms are the lamination parameters in the integral parts and can be written in summation form:

$$V_{0(A,B,D)} = \begin{pmatrix} h, 0, \frac{h^3}{12} \end{pmatrix}$$

$$V_{1(A,B,D)} = \sum_{k=1}^{N} \cos 2\theta_{(k)} \{ t_k, t_k \bar{z}_k, t_k (z_k^2 - 2z_z z_{k-1} + z_{k-1}^2) \}$$

$$V_{2(A,B,D)} = \sum_{k=1}^{N} \sin 2\theta_{(k)} \{ t_k, t_k \bar{z}_k, t_k (z_k^2 - 2z_z z_{k-1} + z_{k-1}^2) \}$$

$$V_{3(A,B,D)} = \sum_{k=1}^{N} \cos 4\theta_{(k)} \{ t_k, t_k \bar{z}_k, t_k (z_k^2 - 2z_z z_{k-1} + z_{k-1}^2) \}$$

$$V_{4(A,B,D)} = \sum_{k=1}^{N} \sin 4\theta_{(k)} \{ t_k, t_k \bar{z}_k, t_k (z_k^2 - 2z_z z_{k-1} + z_{k-1}^2) \}$$

where $t_k = z_k - z_{k-1}$ represents the layer thicknesses and $z_k = (z_k - z_{k-1}) / 2$ are the $z$ coordinates of the mid-plane of the $k$th layer measured from the mid-plane of the laminate [38].

### 5.2 Feasible Region for Lamination Parameters

For the remainder of this text, we consider only symmetric and balanced laminates subjected to out-of-plane loads. For this special case, it has been shown that the out-of-plane lamination parameters are reduced to only two values, $V_{1D}$ and $V_{3D}$. The number of degrees of freedom per node at the element level is four ($w_0, \frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y}, \frac{\partial^2 w_0}{\partial x \partial y}$).

For symmetric laminates, flexural lamination parameters are defined as

$$W_1^* = \frac{12V_{1D}}{h^3} = \sum_{k=1}^{I} s_k \cos 2\theta_k$$

$$W_3^* = \frac{12V_{3D}}{h^3} = \sum_{k=1}^{I} s_k \cos 4\theta_k$$

where

$$s_k = \left(\frac{2z_k}{h}\right)^3 - \left(\frac{2z_{k-1}}{h}\right)^3$$

Through trigonometric identity substitution, a relationship exists between $W_1^*$ and $W_3^*$ such that
The first inequality, Equation (5.3), becomes the constraint on the lamination parameters for the first level optimization. For an orthotropic plate, the inequality is applied as single constraint to the entire plate, and so, the number of design variables is only two. For the variable stiffness plate, the inequality is applied at every element making the number of design variables equal to two times the number of elements.

The next two inequalities will become the "side constraints" or the bounding constraints on the design variables, $W_1^*$ and $W_3^*$. The relationships in Equations (5.3-5.5) can be used to construct a feasible region for the design space. A plot of the feasible region is shown in Figure 5.1.

$$W_3^* \geq 2W_1^{*2} - 1$$  \hspace{1cm} (5.3)

$$-1 \leq W_1^* \leq 1$$ \hspace{1cm} (5.4)

$$-1 \leq W_3^* \leq 1$$ \hspace{1cm} (5.5)

![Figure 5.1. Feasible region for flexural lamination parameters (symmetric and balanced layup).](image-url)
5.3 Compliance Optimization

A common objective function in structural optimization is the minimization of compliance, which is also referred to as the strain energy [39]. The compliance of a structure is essentially the inverse of stiffness, and by minimizing compliance, one can maximize the stiffness. There are other ways to maximize stiffness such as minimizing the size of the displacement vector or simply directly maximizing the stiffness matrix. However, normally compliance is chosen for stiffness maximization because it is a convex function of the design variables, in this case, $W_1^*$ and $W_3^*$. Convexity makes finding the sensitivities of the design variables feasible by analytical means, and hence a gradient-based optimizer can be used efficiently to find a solution.

The compliance minimization problem [40, 41] can be formulated in a form suitable for finite element analysis as

$$\min: c(x) = u^T K u = \sum_{i=1}^{nelem} u_e^T k_e u_e$$  \hspace{1cm} (5.6)

subject to

$$W_3^* \leq 1 - 2W_1^{*2}$$  \hspace{1cm} (5.7)

$$K u = F$$  \hspace{1cm} (5.8)

$$-1 \leq W_1^* \leq 1$$  \hspace{1cm} (5.9)

$$-1 \leq W_3^* \leq 1$$

where $K$ is the global stiffness matrix and $u$ and $F$ are the global displacement and force vectors. $u_e$ and $k_e$ are the element displacement vector and stiffness matrix, respectively. Equation (5.8) is the finite element equation, which is evaluated at every iteration so that the displacement vector, $u$, can be updated for the objective function.

5.4 Method of Moving Asymptotes

The Method of Moving Asymptotes (MMA) by K. Svanberg [42] is now a well-known gradient optimization algorithm for convex functions and has been applied to wide variety of structural problems. The MMA is used in the “first level” optimization to find optimized $W_1^*$ and $W_3^*$ values for both orthotropic and variable stiffened plates.
Here we provide a brief description of the method. Consider an optimization problem of the form:

\[ \text{P: minimize } f_0(x) \quad (x \in \mathbb{R}^n) \]

subject to

\[ f_i(x) \leq \bar{f}_i \quad \text{for } i = 1 \ldots m \]

bounded by

\[ x_l \leq x_j \leq x_u \]

where \( x = (x_1 \ldots x_N)^T \) is the vector of design variables which will be \( W_1^* \) and \( W_3^* \) in this text. The objective function, given as \( f_0(x) \), is equal to the compliance objective given in Equation (5.6). Likewise, the inequality constraint, \( f_i(x) \), and bounds, \( x_j \) are given in Equation (5.7) and Equation (5.8 and 5.9), respectively.

The MMA basically follows a well-established approach for solving this type of optimization problem. The approach involves solving a sequence of explicit subproblems according to the following iterative scheme:

- **Step 0.** Choose a starting point \( x^{(0)} \), and let the iteration index, \( k = 0 \).
- **Step 1.** Given an iteration point, \( x^{(k)} \), calculate \( f_i(x^{(k)}) \) and the gradients \( \nabla f_i(x^{(k)}) \) for \( i = 0, 1 \ldots m \).
- **Step 2.** Generate a subproblem \( P^{(k)} \) by replacing, in \( P \), the (usually implicit) functions \( f_i^{(k)} \), based on the calculations from step 1.
- **Step 3.** Solve \( P^{(k)} \) and let the optimal solution of this subproblem be the next iteration point \( x^{(k+1)} \). Let \( k = k + 1 \) and go to step 1.

The process is stopped when a pre-determined convergence criteria is fulfilled (max iteration count) or when the user is satisfied with the solution.

The MMA may be viewed as a further generalization of a "dual method" developed by Fleury and Braibant (1984). In the Fleury method each \( f_i^{(k)} \) can be obtained by a linearization of \( f_i \) using ‘mixed variables’ which are either \( x_j \) or \( \frac{1}{x_j} \) depending on the sign of \( \frac{\partial f_i}{\partial x_j} \) at \( x^{(k)} \). The MMA extension introduces parameters \( L_j \) and \( U_j \) and the previous linearization is rewritten in the form \( \frac{1}{x_j-L_j} \) or \( \frac{1}{U_j-x_j} \), depending on the sign of the derivative.
of the constraint equations at the current iteration. The values of $L_j$ and $U_j$ are normally changed between iterations and are referred to as "moving asymptotes."

Svanberg has made the MMA available through a series of Matlab functions, which can be relatively easily integrated with a variety of objective functions. The computer implementation of the MMA is discussed in detail in Section 5.6.

### 5.5 Objective Function Gradient and Sensitivity Analysis

In order to use the MMA algorithm, gradients of the objective function must be derived and coded. Using the chain rule of differentiation, the first derivative (gradient) with respect to the design variables, $W_i^k$ ($i=1$ and $3$), of the compliance objective function from Equation (5.6) is:

$$
\frac{\partial f}{\partial W_i^k} = \sum_{i=1}^{\text{nelem}} \left( \frac{\partial u_e}{\partial W_i^k} \right)^T k_e u_e + u_e^T \left( \frac{\partial k_e}{\partial W_i^k} \right) u_e + u_e^T k_e \frac{\partial u_e}{\partial W_i^k}
$$

Grouping terms one and three yields:

$$
\frac{\partial f}{\partial W_i^k} = \sum_{i=1}^{\text{nelem}} 2 \left( \frac{\partial u_e}{\partial W_i^k} \right)^T k_e u_e + u_e^T \left( \frac{\partial k_e}{\partial W_i^k} \right) u_e
$$

Since it is difficult to evaluate $\frac{\partial u_e}{\partial W_i^k}$ directly, an adjoint variable, $\lambda$, is introduced [43].

The objective function can be re-written as:

$$
f(x) = \sum_{i=1}^{\text{nelem}} u_e^T k_e u_e + \lambda (k_e u_e - f_e)
$$

The derivative of the new objective function can be written as:

$$
\frac{\partial f}{\partial W_i^k} = \sum_{i=1}^{\text{nelem}} 2 \left( \frac{\partial u_e}{\partial W_i^k} \right)^T k_e u_e + u_e^T \left( \frac{\partial k_e}{\partial W_i^k} \right) u_e + \lambda \left( \frac{\partial k_e}{\partial W_i^k} \right) u_e + \lambda k_e \frac{\partial u_e}{\partial W_i^k} \tag{5.10}
$$

Upon inspection of Equation (5.10), it can be see that for the substitution $\lambda = -2u_e$, the first and last terms cancel out leaving only derivatives of the design variables, $W_i^k$.

Grouping the two middle terms yields:

$$
\frac{\partial f}{\partial W_i^k} = \sum_{i=1}^{\text{nelem}} -u_e^T \left( \frac{\partial k_e}{\partial W_i^k} \right) u_e
$$

where,
\[
\frac{\partial k_e}{\partial w_i^k} = \sum_{\alpha} \sum_{\beta} \frac{\partial k_e}{\partial D_{\alpha\beta}} \left( \frac{\partial D_{\alpha\beta}}{\partial w_i^k} \right)
\]

and \(\alpha, \beta = 1, 2, 6\).

For the variable stiffness problem, the design sensitivities will vary from element to element and the sensitivity can be written as:

\[
\frac{\partial f}{\partial w_i^k} = -u_e^T \left( \frac{\partial k_e}{\partial w_i^k} \right) u_e \tag{5.11}
\]

For a four DOF plate bending element, the displacement field can be described by a 16 term vector as:

\[
u_e^T = \left\{ \frac{\partial w_1}{\partial x}, \frac{\partial w_1}{\partial y}, \frac{\partial^2 w_1}{\partial x \partial y}, \frac{\partial w_2}{\partial x}, \frac{\partial w_2}{\partial y}, \frac{\partial^2 w_2}{\partial x \partial y}, \frac{\partial w_3}{\partial x}, \frac{\partial w_3}{\partial y}, \frac{\partial^2 w_3}{\partial x \partial y}, \frac{\partial w_4}{\partial x}, \frac{\partial w_4}{\partial y}, \frac{\partial^2 w_4}{\partial x \partial y} \right\}
\]

From Equation (4.3), the element stiffness matrix can be written as:

\[
k_e = K_{kl}^{33} = D_{11} T_{kl}^{xxx} + D_{12} (T_{kl}^{xyy} + T_{kl}^{yyy}) + 2D_{16} (T_{kl}^{xxy} + T_{kl}^{yxy}) + 2D_{26} (T_{kl}^{xyy} + T_{kl}^{yxy}) + 4D_{66} T_{kl}^{xyy} + 2D_{22} T_{kl}^{yxy}
\]

The derivative of the element stiffness matrices can simply be taken analytically as:

\[
\frac{\partial k_e}{\partial D_{11}} = T_{kl}^{xxx}, \quad \frac{\partial k_e}{\partial D_{12}} = T_{kl}^{xyy}, \quad \frac{\partial k_e}{\partial D_{16}} = (T_{kl}^{xxy} + T_{kl}^{yxy}), \quad \frac{\partial k_e}{\partial D_{66}} = 4T_{kl}^{xyy}, \quad \frac{\partial k_e}{\partial D_{22}} = T_{kl}^{xyy} + T_{kl}^{yxy}
\]

From Table 5.1, the terms of the flexural stiffness matrix, \(D\), can be written in terms of laminate invariants and lamination parameters as:

\[
D_{11} = U_1 V_{0D} + U_2 V_{1D} + U_3 V_{3D}
\]

\[
D_{22} = U_1 V_{0D} - U_2 V_{1D} + U_3 V_{3D}
\]

\[
D_{12} = U_4 V_{0D} - U_3 V_{3D}
\]

\[
D_{66} = U_5 V_{0D} - U_3 V_{3D}
\]

\[
D_{16} = U_2 V_{2D} + 2U_3 V_{4D}
\]

\[
D_{26} = U_2 V_{2D} - 2U_3 V_{4D}
\]

From Equation (5.1), flexural lamination parameters, \(W_1^*\) and \(W_3^*\), that depend only on \(V_{1D}\) and \(V_{3D}\) are defined as:
\[ W_1^* = \frac{12V_{1D}}{h^3} \]
\[ W_3^* = \frac{12V_{3D}}{h^3} \]

Taking the derivatives of the terms in the \( D \) matrix:

\[ \frac{\partial D_{11}}{\partial w_1} = \frac{u_2 h^3}{12}, \quad \frac{\partial D_{22}}{\partial w_1} = \frac{-u_2 h^3}{12}, \quad \frac{\partial D_{12}}{\partial w_1} = 0, \quad \frac{\partial D_{66}}{\partial w_1} = 0, \quad \frac{\partial D_{11}}{\partial w_3} = 0 \]
\[ \frac{\partial D_{11}}{\partial w_3} = \frac{u_3 h^3}{12}, \quad \frac{\partial D_{22}}{\partial w_3} = \frac{u_3 h^3}{12}, \quad \frac{\partial D_{12}}{\partial w_3} = \frac{-u_3 h^3}{12}, \quad \frac{\partial D_{66}}{\partial w_3} = \frac{-u_3 h^3}{12}, \quad \frac{\partial D_{16}}{\partial w_3} = 0, \quad \frac{\partial D_{26}}{\partial w_3} = 0 \]

Summing all terms and simplifying, the final sensitivities of the design variables are:

\[ \frac{\partial k_e}{\partial w_1^k} = \frac{u_2 h^3}{12} \left( T_{kl}^{xxx} - T_{kl}^{yyy} \right) \]
\[ \frac{\partial k_e}{\partial w_3^k} = \frac{u_3 h^3}{12} \left( T_{kl}^{xxx} + T_{kl}^{yyy} - T_{kl}^{xyy} + T_{kl}^{yxy} - 4T_{kl}^{xyy} \right) \quad (5.12) \]

From inspection of Equation 5.12, it can be seen that the derivatives of the design variables, \( W_1^* \) and \( W_3^* \), are independent of ply orientation and are the same for every element. Therefore, the element sensitivities will vary only with the displacement field. Substituting Equation (5.11) into Equation (5.25) the objective function sensitivities for each are as follows:

\[ \frac{\partial f}{\partial w_1^k} = -u_e^T \left( \frac{u_2 h^3}{12} \left( T_{kl}^{xxx} - T_{kl}^{yyy} \right) \right) u_e \]
\[ \frac{\partial f}{\partial w_3^k} = -u_e^T \left( \frac{u_3 h^3}{12} \left( T_{kl}^{xxx} + T_{kl}^{yyy} - T_{kl}^{xyy} + T_{kl}^{yxy} - 4T_{kl}^{xyy} \right) \right) u_e \]

**5.6 MATLAB IMPLEMENTATION OF MMA**

The previous sections discussed some background information for building the first level optimization routine. This section focuses on computer implementation of the optimization. A flow chart of the basic algorithm is given in Figure 5.2. The finite element algorithm developed in Chapter 4 (Section 4.2) has now been integrated into the larger optimization framework and is called every time the objective function is evaluated or a displacement plot is desired.

Many of the settings were needed to use the Matlab code discussed in Section 4.2. Here, additional settings relevant to the first level optimization are described.
Figure 5.2. Flow chart of first level optimization using method of moving asymptotes.
Once the user chooses the desired FEM settings, the optimization can be turned on and off with the parameter `optonoff`. The user may also select whether an orthotropic plate or a variable stiffness plate is desired by changing the parameter `varornot`. For an orthotropic plate:

```
varornot='not'
```

For a variable plate:

```
varornot='var'
```

Initial guesses for the lamination parameters can be by modifying parameters $W_{1i}$ and $W_{3i}$. Output file generation can be turned on and with the parameter `outputfiles`.

If output files are turned on, the code will write a new folder in the working directory that includes a variety of output files depending on the type of plate (orthotropic or variable).

### 5.7 Orthotropic Plate Lamination Parameter Optimization

We continue our study of the orthotropic plate from Section 4.4.4, but now with the goal of seeking optimal values of the flexural lamination parameters, $W_1^*$ and $W_3^*$, then determining the ply angles of the stacking sequence (Figure 5.3). A simple two-level approach is used (Figure 5.4). In the first level, the Method of Moving Asymptotes algorithm is used to determine the optimum flexural lamination parameters. In the second level analytical equations developed by Miki [2] are used to determine the ply angles.

The ply thickness and number of plies remain at .004 m and 16 respectively; however, the flexural stiffness matrix, $D$, will be allowed to vary. From Table 5.1, the $D$ Matrix can be written in a compact form as

$$
D = \frac{h^3}{12} (\Gamma_0 + W_1^* \Gamma_1 + W_3^* \Gamma_3)
$$

where

$$
\Gamma_0 = \begin{bmatrix}
U_1 & U_4 & 0 \\
U_4 & U_1 & 0 \\
0 & 0 & U_{51}
\end{bmatrix},
\Gamma_1 = \begin{bmatrix}
U_2 & 0 & 0 \\
U_4 & -U_2 & 0 \\
0 & 0 & 0
\end{bmatrix},
\Gamma_3 = \begin{bmatrix}
U_3 & -U_3 & 0 \\
-U_3 & U_3 & 0 \\
0 & 0 & -U_{31}
\end{bmatrix}
$$
Figure 5.3. Orthotropic laminated plate subjected to uniformly distributed load (lamination parameter optimization).

Figure 5.4. Simple two-level method for optimizing an orthotropic plate.

The objective function, constraint and sensitivities are defined in sections 5.3 and 5.5. To check the robustness of the optimized $W_1^*$ and $W_3^*$, three sets of starting values were chosen, (0,0), (-1,1), (1,1). The maximum number of outer iterations allowed by the MMA solver was set to 7 and the convergence to solution of $W_1^* = 0$ and $W_3^* = -1$, for all three starting points (Figure 5.5). The solution matches the one given in Setoodeh et al. [28] and provides a level of confidence that the MMA algorithm has been adapted correctly for this problem.
Figure 5.5. Convergence of lamination parameters for various starting points.

The normalized compliance, \( \frac{c}{c_{qp}} \), is shown in Figure 5.6 and the normalization factor, \( c_{qp} \), is the quasi-isotropic compliance found in Section 4.4.4. From the figure, it can be seen that after four iterations the normalized compliance converges to about 78% of the quasi-isotropic layup. Hence, a significant gain (22%) in stiffness over the quasi-isotropic layup can be made from optimizing lamination parameters. The theoretical deflection is plotted in Figure 5.7. As expected, the deflection is reduced from the quasi-isotropic layup. From Section 4.4.4, we found that the maximum deflection (at the center of the plate) was approximately 0.0022 m. We see that the deflection is reduced to approximately 0.0017 m given we can find a layup that satisfies the optimized values of the lamination parameters.

For a laminate with two distinct orientation angles, it is possible to find a large number of optimal ply angles. Equation 5.1 can be written as:

\[
W_1^* = s_1 \cos 2\theta_1 + s_2 \cos 2\theta_2, \quad W_3^* = s_1 \cos 4\theta_1 + s_2 \cos 4\theta_2
\]  

(5.14)
Figure 5.6. Objective function convergence ($W_1^*, W_3^*$ initial =0,0).

Figure 5.7. Deflection of Plate with Optimized $W_1^*, W_3^*$. 
From Equation (5.2), $s_1$ and $s_2$ were found to be .875 and .125 respectively. Solving Equation (5.14) for $\theta_1$ and $\theta_2$ yields:

$$\theta_1 = \frac{1}{2} \cos^{-1} T_1, \quad \theta_2 = \frac{1}{2} \cos^{-1} T_2$$

where

$$T_1 = \frac{2s_1 W_1^* \pm \sqrt{4s_1^2 W_1^*^2 - 2s_1 (2W_1^*^2 - s_1 W_3^* - s_2)}}{2s_1}, \quad T_2 = \frac{W_3^* - s_1 T_1}{s_2} \quad (5.15)$$

For $W_1^* = 0$ and $W_3^* = -1$, $\theta_1$ and $\theta_2$, are both found as $\pm 45$ degrees. The normalized compliance is determined by inputting the layup $[45, -45, 45, -45]_2s$ into a stand alone Matlab function for determining compliance. The value for normalized compliance returned is 0.78 which matches the expected value determined from the first level MMA optimization for flexural lamination parameters. Hence, Miki’s graphical method for determining ply angles yields a true minimum compliance layup for this geometry and load case.

At this point, it should be noted that there is a limitation on Equation (5.15) in that for values of $|T_1| > 1$, it is not possible to determine ply angles. However, for the present problem, the limitation does not apply and it is possible to determine the optimum ply angles from Equation (5.15). This limitation becomes a larger issue when optimizing variable stiffness laminates and will be discussed further in Section 5.9.

5.8 Orthotropic End Supported Beam Optimization

The optimization method from the previous section is now performed on the end-supported beam for which the quasi-isotropic FEM solution was discussed in Section 4.4.5 (Figure 5.8).

Again, the compliance is normalized against the quasi-isotropic solution (Figure 5.9). For this case, the optimized compliance is only about 33% of the quasi-isotropic solution. Figure 5.10, shows the progression of the lamination parameters through the design space for three different start points. All three solutions converge to a final value of $W_1^* = 1$ and $W_3^* = 1$. Therefore, we have added confidence that the optimization algorithm is functioning as expected and the sensitivity analysis was done correctly.
Figure 5.8. End supported laminated plate under uniformly distributed load.

Figure 5.9. Compliance convergence of end supported beam (orthotropic layup).
The ply angles for a two-angle laminate are calculated from Equation (5.14). Both $\theta_1$ and $\theta_2$, are found to be 0 degrees. The normalized compliance is determined by inputting the layup $[0\ 0\ 0\ 0]_2s$ into a stand alone Matlab function for determining compliance. The value for normalized compliance returned is 0.33, which matches the expected value determined from the first level MMA optimization for flexural lamination parameters. The layup also correlates to the well known practice of orienting the outermost fibers in a laminated structure to be aligned with the principal axis; hence, the solution matches an expected result.

In Figure 5.11 we see the deflection of the beam based on the optimal $W_1^*$ and $W_3^*$. The maximum deflection at the center of the beam is -0.0032 m. In section 4.4.5 we found that the maximum deflection of the quasi-isotropic beam was -0.0098 m, and so we see that the reduction in deflection has a one to one correlation with reduction in compliance.
Figure 5.11. Deflection of end supported beam for optimized $W_1^*$, $W_3^*$.

5.9 LIMITATION OF MIKI GRAPHICAL METHOD

As discussed in Section 5.6, it is not always possible to determine ply angle using the procedure by Miki. In fact, if the ply thickness if kept constant, there exists a relatively narrow band in the design space for which ply angles $\theta_1$ and $\theta_2$ can be determined (Figure 5.12).

The plot in Figure 5.12 was generated by evaluating Equation (5.14) for all values of $W_1^*$ and $W_3^*$ in the feasible region using an increment value of 0.01. Any combination of lamination parameters that yielded an indeterminate value of $T_1$ were tossed out.

Combinations for which ply angles could be determined are plotted as small squares the size of the increment value.

From inspection of Equation (5.14) and Equation (5.2), it can be seen that the angle determinant space can be expanded by varying the thickness of each ply in the two ply stack. Using an iterative method, we find that by increasing the ratio of ply thickness to approximately 4:1 it is possible to occupy the entire design space (Figure 5.13). However, we also know that from a practical perspective it is generally desirable to have plies of equal thickness. So, we need a more robust method to determine individual ply angles that will for work laminates of equal ply thickness.
Figure 5.12. Angle determinant region for a 2 angle orthotropic plate in bending (equal ply thickness).

Figure 5.13. Angle determinant region for a 2 angle orthotropic plate in bending (4:1 ply thickness).
5.10 Ply Angle Retrieval Using a Genetic Algorithm

As discussed in the previous section, for equal ply thickness laminates, there exists only a small band within the design space for which individual ply angles can be retrieved using the Miki graphical procedure. In this section the ‘simple’ two-level approach used in Sections 5.8 and 5.9 will be modified by replacing the Miki method with a Genetic Algorithm (GA) (Figure 5.14).

![Figure 5.14. Two-Level Approach Using a Genetic Algorithm.](image)

Genetic Algorithms are well known means of solving a wide range of optimization problems. They are useful for solving discrete problems such as ours where we are trying to determine the ideal stacking sequence for a laminate under a given loading condition. GAs are known to be expensive in terms of computation time. However, for problems with relatively few numbers of variables and short objective function computation time, a GA can be quite handy. Our orthotropic plate problem falls into this category.

The optimized values, $W_1^*$ and $W_3^*$, obtained from the first level of optimization can be used in a new objective function which can be stated as follows:

$$\min \left( \sum_{i=1}^{2} \left( f_i^p \right)^{\frac{1}{p}} \right)$$

where

$$f_i = (\Omega_i^{opt} - \Omega_i)$$

(5.16)
and \( p \) is an even numbered positive integer. The \( 2 \times 1 \) vector, \( \Omega^{opt}_1 \) and \( \Omega^{opt}_3 \), is obtained from the first level optimization. The \( 2 \times 1 \) vector, \( \Omega_i \), of \( \Omega^{opt}_1 \) and \( \Omega^{opt}_3 \) are at the current iteration. We know from Equation (5.1) that there is a relationship between \( \Omega^{opt}_1 \) and \( \Omega^{opt}_3 \) and the laminate ply angles. So now it is possible to substitute the layer ply angles as design variables into our objective function Equation (5.16).

A built in GA function in Matlab was used to implement the solution for the unconstrained minimization problem given in Equation (5.16). For convenience, the function was integrated into the overall optimization algorithm and can be initiated by turning the switch parameters \texttt{optonoff} and \texttt{ply_angles} to the ‘on’ position. The switch, \texttt{varornot}, should be set to ‘not’ for an orthotropic plate. Results are generally returned in less than 45 seconds and in most cases on the order of 25 seconds. The code is set up to handle variable plates, but as shown in the coming sections, the greatest value comes from the orthotropic optimization.

### 5.10.1 Standard Test Cases for Ply Angle Retrieval Using a Genetic Algorithm

To demonstrate the effectiveness of the GA, two test problems from Sections 5.7 and 5.8 are revisited (Figure 5.15). For the simply supported orthotropic plate, the GA yields a slightly different optimal layup, \([35, -35, 45, -45, 45, -45, 45, -45]\), in that the first two plies from the centerline are not \( \pm 45^\circ \). The normalized compliance is slightly lower than that of the Miki optimized layup, but by a very small amount. When rounded to the nearest hundredth, the value is the same (0.78). For the end supported orthotropic beam, the GA returns an all \( 0^\circ \) layup when the results are rounded to the nearest degree. The normalized compliance is slightly lower than the Miki result, but again when rounded to the nearest hundredth, becomes the same (0.33).

### 5.10.2 General Test Case 1 for Ply Angle Retrieval Using a Genetic Algorithm

We have shown that GA works well for our two standard sample problems and that equally valid solutions can be obtained using the Miki graphical method, which does not require any optimization at the second level. So, why use a GA at all? As discussed in Section 5.9, the Miki method is limited to when the plotted values of the optimized flexural
lamination parameters lie outside the small band of usable design space. For problems with different loading, geometry and boundary condition scenarios the plotted flexural lamination parameters frequently lie outside the usable design space for a two angle laminate. As an example, consider a rectangular plate with a 2:1 aspect ratio and four supports defined at x and y percentages in Table 5.2.

Table 5.2. Point Load Application
Points for a 2:1 Aspect Ratio Simply Supported Plate

<table>
<thead>
<tr>
<th></th>
<th>Percent X</th>
<th>Percent Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Support 1</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>Support 2</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>Support 3</td>
<td>60</td>
<td>90</td>
</tr>
<tr>
<td>Support 4</td>
<td>90</td>
<td>10</td>
</tr>
</tbody>
</table>

Two equal point loads are applied at 25% and 75% of the length (x direction) and both at 50% of the width (y direction). Optimized values of $W_1^*$ and $W_3^*$ from the first level optimization (MMA) come to -1.17 and -1.18 respectively and do indeed lie outside the small band for a 2 angle laminate shown in Figure 5.16.

Plots of the deflected shapes before and after the first level optimization are shown in Figure 5.17. As expected the optimized deflection is less than the non-optimized shape. The normalized compliance converges to 95% of the orthotropic layup defined in Section 4.4.4 (Figure 5.18).
Figure 5.16. Convergence of lamination parameters for a plate defined by section 5.10.2.

Figure 5.17. (a) Deflection before optimization; (b) deflection after optimization.
In the second level optimization, the GA converges so that the percent difference between the first and second level compliance is virtually zero. The GA returns the optimized layup of $[20 \ -20 \ 20 \ -20 \ 30 \ -30 \ 65 \ -65]_s$.

### 5.11 Variable Stiffened Laminates (First Level Optimization)

We will now turn our focus to variably stiffened laminates. Again, a two-level approach is applied. As with the orthotropic laminate, the Method of Moving Asymptotes will be used in the first level to return optimal flexural lamination parameters. This time, however, the lamination parameters will be optimized at each element. In the second level optimization, a norm minimization objective will be used, but, unlike that used for the orthotropic place, two different ways of retrieving fiber paths will be explored. The first method is based on fiber path definitions developed by Olmedo and Gurdal; the second will use the GA that developed for the orthotropic plate, with the GA run on a per element level. The limitations of each method are discussed.

Very few modifications to the Matlab implementation of the MMA algorithm were made to obtain the flexural lamination parameters, $W_1^*$ and $W_3^*$ for the orthotropic plate.
Equation (5.13) was modified simply by adding a superscripted ‘$e$’ to indicate that the design variables and $D$ matrix are stored at the element level.

$$D^e = \frac{h^3}{12} (\Gamma_0 + W_1^e \Gamma_1 + W_3^e \Gamma_3)$$

The values of $\Gamma_0, \Gamma_1, \Gamma_3$ are directly related to the laminate invariants and are constant for each element. For each element, the $D^e$ matrix is used to calculate the value of the element stiffness matrix which for the case of pure bending is $K^{33}$ from Equation (4.3). In Matlab, each element stiffness matrix is stored in a single cell array, which is passed back to the objective function calculation for compliance.

### 5.11.1 Simply Supported Variable Stiffened Plate

The optimization of lamination parameters for the variable stiffness supported square plate (Figure 5.19) takes quite a bit longer than that of the orthotropic plate. For the 20x20 plate, the variable stiffness optimization takes approximately 211 seconds whereas the orthotropic plate converges in about 13 seconds.

![Figure 5.19. (a) Simply supported variable stiffness plate under uniformly distributed load.](image)

Plots of the optimized $W_1^*$ and $W_3^*$ are shown in Figures 5.20 and 5.21. Figure 5.20 shows a large majority of the elements carry a $W_1^*$ value of 0, and from Figure 35, most element $W_3^*$ values are -1. In Section 5.4, it was found that the optimal $W_1^*$ and $W_3^*$ values for an orthotropic plate with the same loading and boundary conditions are 0 and -1,
Figure 5.20. (b) $W_1^*$ distribution of a simply supported variable stiffness plate under uniformly distributed load.

Figure 5.21. $W_2^*$ distribution of a simply supported variable stiffness plate under uniformly distributed load.
respectively. Hence, the results in the “field” area of the variable plate correlate well to the orthotropic plate.

The values of $W_1^*$ transition in a symmetrical fashion about the x and y center-planes of the part to +0.66 and - 0.66, and the transition from 0 appears to occur at sharp gradient, but is still smooth and continuous (Figure 5.20). In comparison to the $W_1^*$ distribution, the distribution of $W_3^*$ in Figure 5.21 follows a symmetrical pattern, but is truly symmetric about both the x and y and axes. Unlike the $W_1^*$ distribution, near the center of the plate, the values of $W_3^*$ abruptly transition from -1 to about -.3. The value furthest from the field values are located in narrow bands around both x and y centerlines. Despite the fact that no conclusion can be drawn of these observations; however it may be that the isolated nature at the extreme values of the $W_1^*$ and $W_3^*$ will increase the difficulty of finding continuous fiber paths that satisfy the distributions.

Figure 5.22 plots the normalized compliance of the plate versus the outer iteration number of the MMA algorithm. The optimization converges to a final normalized compliance of 0.78 in about four iterations. In the case of the orthotropic plate (Section 5.4), a normalized compliance of 0.78 was found as well. And so, there is no reduction in compliance for the variable over the orthotropic plate for this load case and boundary condition. This leads to the conclusion that the orthotropic layup in this case is the best solution.

The optimized deflected shape is plotted in Figure 5.23. The maximum deflection at the center of the plate is 0.0018 m.

### 5.11.2 End Supported Beam

The optimization of lamination parameters for the end-supported variable stiffness beam (Figure 5.24) takes quite a bit longer than that of the orthotropic plate. For the 60x6 plate, the variable stiffness optimization takes approximately 211 seconds whereas the orthotropic plate converges in about 4.5 seconds.
Figure 5.22. Compliance converge for a simply supported variable stiffness plate under uniformly distributed load.

Figure 5.23. Optimized deflection for a simply supported variable stiffness plate under uniformly distributed load.
Figure 5.24. End supported variable stiffness beam under uniformly distributed load.

Plots of the optimized $W_1^*$ and $W_3^*$ are shown in Figures 5.25 and 5.26. Figure 5.25, shows that a large majority of the elements carry a $W_1^*$ value of close to 1. Figure 5.26 indicates that most element $W_3^*$ values are close to 1 as well. In Section 5.8, it was shown that the optimal $W_1^*$ and $W_3^*$ values for an orthotropic plate with the same loading and boundary conditions were 1. Hence, results in the “field” area of the variable plate results correlate well to the those for the orthotropic plate.

Figure 5.25. $W_1^*$ Distribution of an end supported variable stiffness beam under uniformly distributed load.
The values of $W_1^*$ transition at each end from a value of .15 to 1 within just a few elements in a smooth and continuous fashion (Figure 5.25). Similarly, the values of $W_3^*$ (Figure 5.26) transition quickly at each end from a value of 0.19 to 1. Until a second level optimization to determine fiber angle is run, it is not possible to draw any conclusion about these observations. However, it is possible to hypothesize given the steep gradients in lamination parameters at the ends, it may be more difficult to find continuous fiber paths that satisfy the distributions of lamination parameters.

Figure 5.27 shows a plot of the normalized compliance of the plate versus outer iteration number of the MMA algorithm. The optimization converges to a final normalized compliance of 0.35 in about four iterations. For the orthotropic plate (Section 5.8), the normalized compliance was 0.33. The compliance is slightly worse for the variable plate than for the orthotropic plate. This might be explained by the fact that the variable plate lamination parameter distributions close to the ends of the beam are being influenced by edge effects (i.e, plies near the end of the plate are angle plies). Whereas, the orthotropic plate can only have monotonous lamination parameters for the entire length of the plate and therefore cannot account for edge effects. For that reason, the variable plate lamination parameter distribution is probably a more accurate model.

Figure 5.28 shows a plot of the deflection of the beam based on the optimized lamination parameter distribution. The maximum deflection at the center of the beam is -0.0033 m a slight increase in deflection for the variable plate as compared with the optimized orthotropic plate that deflected at -0.0032 m (Section 5.8). But, as discussed in the previous
Figure 5.27. Compliance convergence for an end supported variable stiffness beam under uniformly distributed load.

Figure 5.28. Optimized deflection for an end supported variable stiffness beam under uniformly distributed load.
paragraph the compliance and deflection of the variable plate more closely mimics a "real world" beam. For example, a typical wing spar is designed with mostly 0° plies in the caps. However, cross plies (typically ±45°) are added in regions where there might be fasteners as in the case of wing spar at the root of the beam.

5.11.3 Study of Compliance vs. Load Application Point

The previous two sections examined two different variable stiffness plate geometries under uniform loading only to find that the end performance turns out to be no better or even slightly worse than the orthotropic cousin. So, why would one design a laminate containing curvilinear fibers? The ideal layup for a square plate under uniform loading can be achieved with straight fibers; the use of curvilinear fibers is unnecessary. Likewise, in the case of the end-supported beam, variable plate compliance was actually slightly higher than that for the straight fiber version. This might be explained by the theory that the ideal layup for this loading, boundary condition and geometry is a straight fiber layup, with all fibers oriented in the 0° direction.

Of course the problem could be defined using infinite combinations of geometries, boundary conditions and loadings. Gains in performance might be seen when designing with curvilinear fibers. This hypotheses is tested in a controlled manner by conducting a study in which a point load is systematically applied to locations on a square plate compliance is computed for both the orthotropic and variable plates (Figure 5.29) for the given load application points. The intent here is to generate a mapping of the relative compliances between the orthotropic and variable plate versus the x and y location of the load application. Because the plate has 4 planes of symmetry (2 diagonal, horizontal and vertical), only 1/8 of the geometry needs to be mapped (See Figure 5.30).

The results of the study are summarized in Table 5.3. A compliance value for the quasi-isotropic layup, [45, 90, 0, -45]_2s, is given in the third column and is used as a normalization factor for the compliances of the orthotropic and variable plates given in the fourth and fifth columns. The percent difference between the optimum compliance of the orthotropic plate and the variable plate is given in the last column. The greatest potential gain in performance occurs when the load is applied at a point closest to a corner of the plate.
Figure 5.29. Simply supported variable stiffness plate under a point load.

Figure 5.30. Application points for point load on square plate.
Table 5.3. Normalized Compliance Mapping for Loads Applied at Point, \((a_p, b_p)\)

<table>
<thead>
<tr>
<th>(a_p)</th>
<th>(b_p)</th>
<th>Compliance quasi-orthotropic layup (Normalization Factor)</th>
<th>Normalized Compliance Optimized Orthotropic Plate</th>
<th>Normalized Compliance Optimized Variable Plate</th>
<th>Percent Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>7.185E-07</td>
<td>0.79</td>
<td>0.66</td>
<td>17.9</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1</td>
<td>1.201E-06</td>
<td>0.82</td>
<td>0.68</td>
<td>17.7</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>1.438E-06</td>
<td>0.83</td>
<td>0.69</td>
<td>17.7</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>1.540E-06</td>
<td>0.84</td>
<td>0.71</td>
<td>16.5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>1.552E-06</td>
<td>0.86</td>
<td>0.72</td>
<td>17.7</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>2.549E-06</td>
<td>0.81</td>
<td>0.67</td>
<td>17.7</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2</td>
<td>3.339E-06</td>
<td>0.82</td>
<td>0.70</td>
<td>15.2</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>3.721E-06</td>
<td>0.83</td>
<td>0.72</td>
<td>13.9</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>3.791E-06</td>
<td>0.84</td>
<td>0.73</td>
<td>12.6</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>4.749E-06</td>
<td>0.80</td>
<td>0.7</td>
<td>13.1</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>5.462E-06</td>
<td>0.81</td>
<td>0.72</td>
<td>11.7</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>5.634E-06</td>
<td>0.82</td>
<td>0.73</td>
<td>11.6</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>6.501E-06</td>
<td>0.80</td>
<td>0.73</td>
<td>9.1</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>6.810E-06</td>
<td>0.80</td>
<td>0.73</td>
<td>9.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>7.166E-06</td>
<td>0.80</td>
<td>0.74</td>
<td>7.8</td>
</tr>
</tbody>
</table>

(17.9%). The least gain comes when the load is applied at the center of the plate (7.8%). This correlates to the findings of the previous sections in which a uniform distributed load was applied which can be idealized as a point load applied at the center of the plate.

Many other methods could be used to attempt to widen the compliance gap between the orthotropic and variable plates. Different aspect ratios of the plate, different boundary conditions, or combinations could be tried. The point is that it may be possible to improve performance by using a variable stiffness plate.
5.11.4 Optimized Fiber Paths (Shifted Fiber Path Method)

Lamination parameter distributions are of little practical use unless one can find a way to determine optimized ply angles. Several methods have been proposed and tested over the years and many were touched on in Chapter 2. A FORTRAN 90 code that computes fiber angles for 2-D finite element meshes using the “shifted path” method originally developed by Gürdal and Olmedo [13] was published as part of a Masters thesis by Langley [16]. The method relies on a defined reference path that passes through the center of the plate and is then shifted vertically to define the rest of the paths as shown in Figure 5.31.

![Figure 5.31. Shifted path lamina based on linearly varying reference path.](image)

The original code was designed to generate an input file to be used with the GENESIS finite element package. The various overlap and gap conditions that exist between adjacent paths can be accounted for by the F90 code. For the current work, the algorithm was simplified to neglect overlaps and gaps and the sections of the code that involved writing the GENESIS output file were omitted.

5.11.4.1 Description of Shifted Path Method

The first step of the shifted path method is to define a reference path. For simplicity, a square plate will be used for the remainder of this thesis. The reference path always passes
through the center of the plate, which is also the origin of the plate. The reference path can be rotated about the origin by defining a new coordinate system defined by orthogonal axes x’ and y’. The x’y’-coordinate system are rotated counter-clockwise from the panel xy-coordinate system by a tow-path rotation angle, \( \phi \) as shown in Figure 5.32.

![Figure 5.32. Panel (xy) and tow-path (x’y’) coordinate systems.](image)

Transforming between coordinate systems can be accomplished by using the simple equations below

\[
x' = x \cos \phi + y \sin \phi \\
y' = x \sin \phi + y \cos \phi
\]

or

\[
x = x' \cos \phi + y' \sin \phi \\
y = x' \sin \phi + y' \cos \phi
\]

A fiber-orientation angle, \( \theta \), along the tow-path center is defined as a linear function of position along the x’ axis. A ‘tow-placement-head’ angle, \( \beta \), is defined as the sum of the tow-path rotation angle, \( \phi \), and the varying fiber-orientation angle \( \theta(x') \).

\[
\beta = \phi + \theta(x')
\]

An example of a linearly varying tow-path centerline is shown in Figure 5.33.
The fiber-orientation angle varies linearly between two angles along the $x'$ axis. The first of the two angles, $T_0$, is the fiber-orientation angle at the center of the panel. The second angle, $T_1$, is the fiber-orientation angle at a distance $a/2$ from the center of the plate, where $a$ represents the width of the panel. This convention simplifies the tow path definition process and allows for a large degree of design flexibility. Figure 5.34 is a schematic of where $T_0$ and $T_1$ are defined along the $x'$ axis. The result of a linearly varying tow path is a smooth and continuous reference-path centerline. An example of a curvilinear path centerline where $T_0 = 0^\circ$, $T_1 = 45^\circ$ and $\phi = 0^\circ$ is shown in Figure 5.35.

The reference path in Figure 5.35 is a smooth and continuous function, composed of four piecewise continuous functions Equation (5.17).

$$
\theta(x') = \begin{cases} 
\frac{2}{a} (T_1 - T_0) x' + T_0 - 2(T_0 - T_1), & \text{for } -a \leq x' \leq -a/2 \\
\frac{2}{a} (T_0 - T_1) x' + T_0, & \text{for } -a/2 \leq x' \leq 0 \\
\frac{2}{a} (T_1 - T_0) x' + T_0, & \text{for } 0 \leq x' \leq a/2 \\
\frac{2}{a} (T_0 - T_1) x' + T_0 - 2(T_0 - T_1), & \text{for } a/2 \leq x' \leq a
\end{cases}
$$

(5.17)
Figure 5.34. Geometry for defining a variable-stiffness reference path.

Figure 5.35. Curvilinear-fiber reference path centerline for $T_0=0^\circ$ and $T_1=45^\circ$. 
In order to conduct finite element analysis and structural optimization, the \( x \) and \( y \) coordinates of any point along the centerline path must be mapped. Like the fiber-
orientation-angle equations, the \( y'(x') \) position equations are piecewise continuous.

\[
y'(x') = \frac{a}{2(T_0 - T_0)} \left\{ \ln[\cos T_0] + \ln \left[ \cos \left( -T_0 + 2T_1 + \frac{2(T_1 - T_0)}{a} x' \right) \right] \right\}, \quad \text{for } -a \leq x' \leq -a/2
\]

\[
\frac{a}{2(T_0 - T_0)} \left\{ -\ln[\cos T_0] + \ln \left[ \cos \left( T_0 + \frac{2(T_0 - T_1)}{a} x' \right) \right] \right\}, \quad \text{for } -a/2 \leq x' \leq 0
\]

\[
\frac{a}{2(T_0 - T_0)} \left\{ -\ln[\cos T_0] + \ln \left[ \cos \left( T_0 + \frac{2(T_0 - T_0)}{a} x' \right) \right] \right\}, \quad \text{for } 0 \leq x' \leq a/2
\]

\[
\frac{a}{2(T_0 - T_0)} \left\{ \ln[\cos T_0] + \ln \left[ \cos \left( -T_0 + 2T_1 + \frac{2(T_0 - T_1)}{a} x' \right) \right] \right\}, \quad \text{for } a/2 \leq x' \leq a
\]

Once all the necessary parameters have been defined for a single reference path, tow
placement geometry for the remainder of the plate must be defined. To accomplish this, a
method that shifts the reference path a set distance along the \( y' \) axis is employed. Typically,
the shift distance is equal to the width of the head of the tow placement machine. This work
assumes a 2 inch wide head and hence, a 2 inch shift distance. The reference path and single
shifted path are shown in Figure 5.36.

Figure 5.36. A single shifted-path centerline.
5.11.4.2 Computational Model of Reference Path

The first step in developing a discretized model of the plate is to create a discrete reference path. For consistency and in order to use the same algorithm, a quarter plate will continue to be used. The reference path is defined by determining which elements lie within the area of the tow path. The distances between the element centers and the reference path must be determined. Langley determined the distance using a Golden Section search, which minimizes the distance between the element center and the reference fiber path. Once the distance has been computed, if the value is less than half the tow width, then the thickness of that element is increased to a single thickness. The Golden Section search is also used to determine the angle of each element. The minimum distance between the center of an element and the reference path results in a line perpendicular to the reference-path centerline. All points along the perpendicular line have the same orientation angle as the point the line crosses on the reference path. We see from Figure 5.37 that for a constant value of $x'$ the fiber-orientation angle will change across a single tow path.

![Figure 5.37. Variation of orientation angle across reference path for a constant value of $x'$.](image)
Once the discretized reference path has been determined, the geometry for the rest of the ply can be developed. To accomplish this, the reference path is shifted and duplicated along the y’ axis. The shift distance is discretized by determining the number of elements in the x’ and y’ directions that are required in order to place an adjacent tow path.

5.11.4.3 TWO-LEVEL OPTIMIZATION OF FIBER PATHS

In Section 5.10, a norm minimization was used in conjunction with a Genetic Algorithm to produce optimal ply angles for an orthotropic plate in a second level optimization. Now a similar approach is employed for the variable plate. The optimized $W_1^*$ and $W_3^*$ from the first level of optimization can be used in a very similar objective function to Equation (5.15), which can be stated as follows:

$$\min \left( \frac{1}{p} \sum_{i=1}^{2n_e} (f_i^p) \right)$$

where

$$f_i = (\Omega_{\text{opt}} - \Omega_i)$$

$p$ is an even numbered positive integer ($p=12$ for this problem), and $n_e$ is the number of elements. The $2n_e \times 1$ vector, $\Omega_{\text{opt}}$, is a vector of the $W_1^*$ and $W_3^*$ obtained from the first level optimization (see Sections 5.11.1 and 5.11.2). The size of $\Omega_{\text{opt}}$ is $i = 2n_e$, and the vector is arranged by alternating $W_1^*$ and $W_3^*$ values. The matrices $W_1^*$ and $W_3^*$ are converted into vector form one column at a time starting with column one. The vector $\Omega$ is obtained by evaluating the design variables to obtain the ply angle distribution in each layer. In this case, the design variables are the values $T_0, T_1$ and $\phi$ in each layer, and a single layer can be described by notation of the form $\phi(T_0|T_1)$. Since the layup must be balanced and symmetric, the number of design variables is equal to 3 times the number of layers divided by 4. For all sample problems presented in this text there are 16 layers, (8 per side of the symmetry plane and the possibility of 4 unique sets of +/- layers) so the number of design variables is 12. The notation used to describe the entire laminate, will denote two consecutive +/- layers as $\pm \phi(T_0|T_1)$.
Once the algorithm computes the fiber angle in each element, the $W_1^*$ and $W_3^*$ matrices at the current iteration for each layer can be calculated from Equation (5.1). The $W_1^*$ and $W_3^*$ matrices at the current iteration can then be combined and reshaped into the $\Omega$ and the final value of the objective function can be computed. To evaluate the objective function, the FORTRAN 90 program written by Langley was transcribed into a FORTRAN 77 format and converted into a subroutine. The subroutine accepts the vector of design variables as input and outputs the matrices $W_1^*$ and $W_3^*$ which can be fed into a separate function.

Figure 5.38 is a flow chart that provides a general overview of the two-level algorithm used for this optimization. As discussed in the previous paragraphs, the Method of Moving Asymptotes (MMA) is employed in the first level optimization to obtain the optimal lamination parameter distribution for the plate.

Figure 5.38. Algorithm for two-level optimization of fiber paths.

The MMA algorithm is also used to determine the optimal lamination parameters for the orthotropic cousin, then a second level norm minimization routine is conducted using a GA to determine the ideal orthotropic layup. The ideal orthotropic layup is written to a text file that will be read in by the DOT code as initial guesses for rotation, $\phi$, of the $x'y'$. The logic is that at the origin of the plate, if $T_0$ and $T_1$ are set to zero the layup will closely match
its orthotropic cousin and will be an exact match throughout the plate. The objective
function is noisy and discontinuous; and therefore, it is likely that optimization for fiber paths
will settle into a local minima. So, attempts to produce initial guesses for $\phi$, $T_0$ and $T_1$ are as
close to the global optimum as possible. Close initial guesses will also reduce necessary
computational time. Testing of the code has shown that each objective function calculation
takes on average 0.4 seconds. This is a relatively long time considering that in order to reach
a global optimum, most optimization problems of this scale require several thousand
evaluations.

5.11.4.4 Optimized Fiber Paths for a
Simply Supported Plate Under
Distributed Load

As a first test of the performance of the two-level fiber path optimization, the classic
simply supported plate under uniformly distributed loads is examined. The first level
optimization compliance of the variable plate is no less than its orthotropic cousin (See
Section 5.10). Therefore, the second level optimization can at best match the performance of
the orthotropic plate and will in all likelihood be worse than the orthotropic plate. A
benchmark for the algorithm is established using a simple example.

Since the geometry required by the second level optimization is a quarter symmetric
plate (Figure 5.39), compliance normalization factors used in previous sections must be
rescaled. This is done by computing the compliance of the standard quasi-isotropic layup,
$[45, 90, 0, -45]_{2s}$. The compliance of this layup is $1.808 \times 10^{-8}$/MPa and this equates to a
normalized compliance of 1. The maximum deflection for this geometry is at the origin, and
for the quasi-isotropic layup, the maximum deflection is $-0.00072$ m (Figure 5.40).

Following the first level optimization procedure for variable stiffness plate discussed
in previous sections, an optimal normalized compliance value of 0.67 is found (Figure 5.41);
and hence, a near 40% reduction in compliance appears possible. Likewise, maximum
deflection of the plate is reduced by 40% (Figure 5.42). The optimized lamination parameter
distributions, $W_1^*$ and $W_3^*$, for the quarter plate are shown in Figures 5.43 and 5.44. The
distributions for the quarter plate match the distributions for the whole plate (Figures 5.19
and 5.21).
Figure 5.39. Simply supported variable stiffness plate under distributed (quarter plate analysis).

Figure 5.40. Deflection of simply supported variable stiffness plate under uniformly distributed load (baseline quasi-isotropic layup).
Figure 5.41. Compliance converge for a quarter symmetric simply supported variable stiffness plate under uniformly distributed load (first level optimization).

Deflection (Ideal $W_1^*, W_3^*$ distribution from 1st Level Opt)

Figure 5.42. Deflection of simply supported variable stiffness plate under uniformly distributed load (based on optimized lamination parameters, $W_1^*, W_3^*$).
Figure 5.43. $W_1$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, first level optimization).

Figure 5.44. $W_3$ distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, first level optimization).
A two-level optimization on the orthotropic cousin is conducted to assist the second level norm minimization for the variable stiffness plate. The optimized layup of [35 -35 45 -45 45 -45 45 -45], is used to initialize the coordinate system rotation angles, $\phi$, for each layer. The final normalized compliances from both levels of optimization are 0.67, which as expected matches the first level optimization compliance for the variable plate.

Now all the necessary inputs are in place to run the second level optimization to obtain optimal fiber paths. To do this, a FORTAN 77 optimization program, DOT, developed by Vanderderplaats et al. is used. The first level optimization Matlab codes are set up to generate various output files (geometry, $W_1^*$ and $W_3^*$ vectors, initial $\phi$ values) that are utilized by the FORTRAN 77 DOT program.

The DOT program is a well-known and versatile set of optimization codes, selected for this research because of its speed and because it allows for unconstrained optimization without requiring the user to supply gradient information.

Experimentation using the DOT optimizer with different starting values for the design variables showed that the objective function settles into a local minimum after a few iterations, and in general, there is very little movement in the design variables from their initial values. Various settings, including termination criteria and finite difference step values, were adjusted to enlarge the scope of the search space. However, these produced little effect leading to the strong belief that the objective function is noisy and prone to settling into a local minimum in the vicinity of the initial values to the design variables.

In an effort to find a global minimum, the initial values of $T_0$ and $T_1$ were systematically varied while keeping the initial value of $\phi$ in each layer set to ply angles of the optimized orthotropic plate. The DOT optimization program was run for each starting case.

The optimized values for the 12 design variables and the objective function value were output by the DOT program. However, to determine the true performance of the optimization, the normalized compliance must be determined. Since there is no readily available method to compute compliance in FORTRAN, the vector of optimal design variables is input into a Matlab function that returns the normalized compliance value.

A summary of the results is presented in Table 5.4. None of the solutions match the ideal compliance of 0.67. However, the first set of initial $T_0$ and $T_1$ values 5 -5 degrees
Table 5.4. Second Level Optimization Performance Characteristics for a Variable Stiffened Plate Under Uniformly Distributed Load

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>$T_1$</th>
<th>Objective Function Value</th>
<th>Normalized Compliance Optimized Variable Plate</th>
<th>Number of Function Evaluations</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-5</td>
<td>1.19</td>
<td>.70</td>
<td>121</td>
</tr>
<tr>
<td>10</td>
<td>-10</td>
<td>1.22</td>
<td>.71</td>
<td>89</td>
</tr>
<tr>
<td>15</td>
<td>-15</td>
<td>1.21</td>
<td>.71</td>
<td>124</td>
</tr>
<tr>
<td>20</td>
<td>-20</td>
<td>1.20</td>
<td>.72</td>
<td>90</td>
</tr>
<tr>
<td>25</td>
<td>-25</td>
<td>1.21</td>
<td>.73</td>
<td>90</td>
</tr>
<tr>
<td>45</td>
<td>-45</td>
<td>1.33</td>
<td>.88</td>
<td>150</td>
</tr>
<tr>
<td>60</td>
<td>-60</td>
<td>1.07</td>
<td>.92</td>
<td>342</td>
</tr>
<tr>
<td>90</td>
<td>-90</td>
<td>3.16</td>
<td>.84</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>.90</td>
<td>.75</td>
<td>204</td>
</tr>
<tr>
<td>-5</td>
<td>-5</td>
<td>.89</td>
<td>.78</td>
<td>465</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>.98</td>
<td>.79</td>
<td>198</td>
</tr>
<tr>
<td>-5</td>
<td>5</td>
<td>.98</td>
<td>.76</td>
<td>331</td>
</tr>
<tr>
<td>-15</td>
<td>15</td>
<td>1.20</td>
<td>.72</td>
<td>200</td>
</tr>
<tr>
<td>-15</td>
<td>-15</td>
<td>.87</td>
<td>.75</td>
<td>350</td>
</tr>
</tbody>
</table>

produced the best result of 0.70 normalized compliance (4.4% difference from the ideal).

The complete laminate is $[\pm 30(5|−5), \pm 47(5|−5), \pm 52(5|−5), \pm 57(5|−5)]$.  

In general, the normalized compliance value increases as the initial $T_0$ and $T_1$ deviate further and further from $(5|−5)$. This is expected since the minimum compliance of 0.67 is actually for the orthotropic layup. If plotted in curvilinear coordinates, each layer of the orthotropic layup would have a $T_0$ and $T_1$ of 0.

The final value of the objective function does not necessarily correlate to the value for normalized compliance. Since the objective function essentially works to minimize the difference between ideal and current value for lamination parameters for every element, it
might be expected that the smaller the value of the objective function, the smaller the value for compliance. But, this is not exactly the case. The objective function value for the minimum normalized compliance of 0.70 is 1.19. However, the objective function value is lower for several instances with different starting values of $T_0$ and $T_1$. The counterintuitive phenomenon is explained by the fact there are more instances of elements in which the GA did not completely converge. However, the number of anomalous elements weren’t enough to impact the overall performance of the second level optimization.

Plots of the $W_4^*$ and $W_3^*$ distributions for the best second level optimization solution ($T_0 = 5$ and $T_1 = -5$) are shown in Figures 5.45 and 5.46. The $W_4^*$ distribution values found in the first level optimization (Figure 5.41), range from -0.7 to 0.7. Figure 5.45 shows that the $W_4^*$ values for second level optimization range from -0.44 to -0.16; and hence, the extreme ends of the ideal range of $W_4^*$ are not completely captured by the second level optimization. More importantly, however, is the distinct difference in the patterns of $W_4^*$ distributions between the two plots. The predominant $W_4^*$ value from the first level optimization is actually zero (Figure 5.41). However, Figure 5.45 shows very little area covered by values close to zero. Hence, there is little match in $W_4^*$ distributions between the first and second level optimizations.

The $W_3^*$ distribution values from the first level optimization (Figure 5.44), range from -1 to 0.27. Values from the second level optimization of $W_3^*$ range of -0.93 to -0.56 (Figure 5.46). Hence, the lower bound of the $W_3^*$ range is covered in the second level, but the upper bound of the range is significantly truncated. The ideal $W_3^*$ value for the orthotropic cousin is -1, and a large percentage of the plate area from the first level variable plate optimization is covered by the value of -1 (Figure 5.44). In contrast to the second level distribution for $W_4^*$, a larger percentage of the second level $W_3^*$ distribution is covered by the predominant distribution value of -1. However, the pattern of distribution is still quite different than that of the first level optimization. Figure 5.44 shows that the $W_3^*$ value of -1 covers most of the plate area except that for that around the left and bottom corners of the quarter plate. Figure 5.46, shows that large area covered by $W_3^*$ values close to -1 sit mostly in the vertical center of the plate. There is little match in the values between the first and second level optimizations.
Figure 5.45. $W_1^*$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, second level optimization).

Figure 5.46. $W_3^*$ Distribution of a simply supported variable stiffness plate under uniformly distributed load (quarter plate, second level optimization).
Figures 5.45 and 5.46 show that there is a vertically repeating pattern in the $W^*_1$ and $W^*_3$ distributions. This phenomenon can be explained by the fact that the tow in any given layer is shifted along the $y'$ axis. The width of the tow path is two units (elements), and in both plots for $W^*_1$ and $W^*_3$, there 5 distinct repeating bands in the vertical direction. Each band can be broken into two smaller bands that are 2 units wide. Plots of the first level $W^*_1$ and $W^*_3$ distributions follow patterns that are more or less symmetric about both the horizontal and vertical axis (for a full plate). Hence, it seems that the nature of the shifted fiber path method will impose a limitation that will make matching the $W^*_1$ and $W^*_3$ distributions found in the first level very difficult.

The fiber angle distributions for positive $\phi$ value layers are shown in Figure 5.47. To enforce condition of +/- balance, even layers are represented by negative values of $\phi$. Figure 5.48 plots the deflection of the plate. The maximum deflection is $-5.04 \times 10^{-4}$ m which is 1.6% greater than that of the ideal compliance layup (Figure 5.40). Since the percent difference in compliance between first and second level optimization is 4.4%, there is not an exact one to one correlation between deflection and compliance in this problem.

Several more problems were examined as part of this research. As the loading and boundary conditions were made increasingly general, the solutions become worse.

5.11.4.5 SUMMARY AND CONCLUSION FOR OPTIMIZED FIBERS USING SHIFTED PATH METHOD

In the previous sections, a procedure was laid out for optimizing fiber path angles for compliance using method that involves offsetting a series of curvilinear fiber paths from a known reference path. Many difficulties were found with this approach. First, the second level optimization tended to settle into different minimums depending on the initial guesses for the design variables. This led to the conclusion that the objective function itself is noisy meaning that the search space is filled with many local minima. A variety of techniques were examined, including the use of a Matlab Genetic Algorithm, in an effort to find a global minima. However, the time required for one evaluation of the objective function in Matlab was found to be extremely long and not practical for optimization. Ultimately, an empirical approach that involved systematically choosing initial values of $T_0$ and $T_1$ was used. There is
Figure 5.47. Fiber angle distributions for simply supported variable stiffness plate under uniformly distributed load (quarter plate, shifted fiber method, positive angle layers shown)
still some uncertainty that this approach guarantees a global minimum since it is impossible
to search every possible start value of \( T_0 \) and \( T_1 \). However, for the problem of a simply
supported plate under uniform distributed load, the number of starting values \( T_0 \) and \( T_1 \) was
limited to a small handful, and using trend analysis, it was possible to narrow down a
solution. As more general problems were examined, it was found that the number of starting
guesses for \( T_0 \) and \( T_1 \) needs to be expanded.

Perhaps a more telling indication of the difficulty in obtaining truly optimized fiber
paths is the almost complete lack of match between the first and second level lamination
parameter distribution patterns. The differences can be attributed to natures of the shifted
path method, which essentially forces a repeating pattern of fiber angles along the rotated
vertical axis, \( y' \).

As mentioned previously, solving this two-level optimization problem requires that
the user switch between the Matlab and FORTRAN language platforms. The first level
optimization for ideal lamination parameters is conducted in Matlab. The user must transfer
files with lamination parameter distributions and other input parameters into an appropriate
location for use by the FORTRAN program. Once the second level optimization is complete,
the user must compute the compliance, which requires inputting the vector of design variables back into a Matlab function.

### 5.11.5 Optimized Fiber Angles For Variable Stiffness Plate Using Unconstrained Genetic Algorithm and Post Optimization Repair

Many difficulties and limitations were imposed by the shifted path method. The objective function is noisy, and a search-based method would be better able to provide solutions; However, the length of time required for one function evaluation was too long to be able to reasonably use a search based method such as a Genetic Algorithm or Particle Swarm. The lamination parameter distributions from the second level optimization don’t match well to the ideal distributions returned by the first level optimization. This issue stems from the inherent nature of shifting fiber paths along a directional axis. Finally, the software architecture of the previous level optimization was limiting and time consuming to use. So, a more robust, reliable, accurate, user-friendly and faster method for obtaining optimal ply angle distributions is necessary.

Section 5.10 discussed determination of optimal ply angles for an orthotropic laminate by using a least square minimization as the objective function and a Genetic Algorithm as the solver. For an orthotropic laminate, the two lamination parameters $W_1^*$ and $W_3^*$ are constant for every element. But, for the variable stiffness plate the $W_1^*$ and $W_3^*$ values can vary from element to element.

What would happen if the same method used to find the ideal layup for an orthotropic laminate were applied to the variable plate where the number of objective functions is equal to the number of elements. For the variable plate, Equation (5.16) can be modified and written as:

$$ for \ k = 1:ne, \ \min \left( \sum_{i=1}^{2} \left( f_{k_i}^P \right)^{\frac{1}{p}} \right) $$

where

$$ f_{k_i} = (\Omega_{k_i}^{opt} - \Omega_{k_i}) $$
As before, the value of $p$ is set to 12. For this problem, $\Omega^{opt}$ is two column array containing the ideal $W_1^*$ and $W_3^*$ values from the first level optimization, and the number of rows is equal to the number of elements. The $ne \times 2$ array, $\Omega$, contains the current iteration values for $W_1^*$ and $W_3^*$. Equation (5.1) is used to establish a relationship between $W_1^*$ and $W_3^*$ and the laminate ply angles.

Since the second level optimization for ply angles is without constraints, there is no way to guarantee continuity of fiber paths between elements. In previous research, applying a curvature constraint enforced continuity. However, this method can be computationally expensive as it forces many more iterations in the optimization. What was found through experimentation was that by not applying any constraints to the second level optimization and allowing the GA to search for the best global solution, fairly continuous fiber angle distributions were generated. The degree of continuity varies depending on initial conditions (geometry, boundary conditions and loading). A simple repair function was developed to correct for high curvatures and/or discontinuities. It corrects any element ply angles that deviate drastically from its neighboring elements. The basic algorithm for the two-level optimization and subsequent repair is illustrated in Figure 5.49.

### 5.11.5.1 Example: Simply Supported Variable Stiffness Plate Under Point Load Applied at Center

To demonstrate the post-optimization repair, we will first look at a simply supported square plate with a point load applied at the center (Figure 5.50).

Figure 5.51 shows the ideal $W_1^*$ and $W_3^*$ returned by the first level optimization. Figure 5.52 shows the lamination parameter distributions returned by the second level optimization. For the most part, the second level optimization returns $W_1^*$ and $W_3^*$ distributions that match the first level optimization closely. However, several elements along the horizontal axis of symmetry don’t match the $W_1^*$ and $W_3^*$ from the first level optimization very well at all. It appears that the GA cannot find a good solution for the lower bound of $W_1^*$ or the upper bound of $W_3^*$.

Plots of the fiber angle distributions are shown in Figure 5.53. Only the odd numbered layers are shown; the even numbered layers are merely the negative angle opposites of the odd layers. As expected, there are some sharp discontinuities along the
Figure 5.49. Basic algorithm for ply angle optimization with post optimization repair.

Figure 5.50. Simply supported variable stiffness plate under centered point load.
Figure 5.51. Lamination parameters distributions for a simply supported variable stiffness plate under centered point load (first level optimization).

Figure 5.52. Lamination parameters distributions for a simply supported variable stiffness plate under centered point load (second level optimization).
Figure 5.53. Fiber angle distributions for simply supported variable stiffness plate under centered point load (positive angle layers shown).
horizontal centerlines of all four layers. In general, there is good continuity of fiber angles between elements. There are several areas where there are some less than desirable changes in angles between adjacent elements.

As mentioned earlier, previous research has focused on applying a curvature limitation constraint within the optimization to enforce fiber angle continuity between elements. In this research, a pseudo-curvature constraint is created which is enforced after the second level optimization. The idea is to correct misfit element layer ply angles by modifying them to the average of the surrounding ply angles. A diagram of a typical interior element layer ply and its surrounding eight elements is shown in Figure 5.54. Elements along the edges of the plate are surrounded by five elements,

\[
\theta_{avg_{ij}} = \frac{\theta_{i+1,j} + \theta_{i+1,j-1} + \theta_{i,j-1} + \theta_{i-1,j-1} + \theta_{i-1,j} + \theta_{i-1,j+1} + \theta_{i,j+1} + \theta_{i+1,j+1}}{8}
\]

and the four corner elements are surrounded by three elements so, for these special case elements the sum of the surrounding elements is divided by five and three, respectively.

Once the average angle of the surrounding elements is obtained, the absolute value of the difference between the current element ply angle and the average angle of the surrounding elements (Equation 5.18) is computed. The delta angle is then compared to a user defined maximum allowable delta angle Equation (5.19). If the difference is greater than the allowable limit, then the value of the element layer ply angle is changed to the average angle of the surrounding elements; if it is less than or equal to the maximum allowable delta, then the angle is unchanged.

The repaired fiber angle distributions are shown in Figure 5.55 and the repaired lamination parameter distributions are shown in Figure 5.56. For this problem, the value for \(\Delta \theta_{allowable}\) is set to 1° and the plate is 1 meter square in dimension. Looking at the \(\Delta \theta_{allowable}\) constraint in terms of curvature, the following relation can be written as:

\[
\kappa_{allowable} = \frac{\Delta \theta_{allowable}}{\Delta x} = \frac{\Delta \theta_{allowable}}{\Delta y}
\]

(5.20)
Figure 5.54. Ply angle mapping of surrounding elements for a typical interior element.

\[ \theta_{i-1,j+1} \quad \theta_{i,j+1} \quad \theta_{i+1,j+1} \]

\[ \theta_{i-1,j} \quad \theta_{i,j} \quad \theta_{i+1,j} \]

\[ \theta_{i-1,j-1} \quad \theta_{i,j-1} \quad \theta_{i+1,j-1} \]

\[ \Delta x \text{ and } \Delta y \text{ are length and width dimensions for one element. The element geometry is square, so } \Delta x = \Delta y. \text{ There are 20 elements in both } x \text{ and } y \text{ directions, so } \Delta x \text{ and/or } \Delta y \text{ will be 1 meter divided by 20, or 0.05 m. The } \kappa_{\text{allowable}} \text{ is equal to } 1/0.05=20 \text{ degrees/meter. To input the constraint in the actual code, the value for } \Delta \theta_{\text{allowable}} \text{ must be entered. If the allowable curvature is known, the user simply needs to solve Equation (5.20) for } \Delta \theta_{\text{allowable}}. \]

Table 5.5 is a summary of the normalized compliances from the first and second level optimizations of both the orthotropic and variable stiffness versions of the simply supported plate with a center point load. The normalized compliance of the repaired laminate is shown in the last column, which shows just over a 1% loss in performance as compared with the normalized compliances from both the first and second level optimizations. If it is possible to manufacture the repaired laminate, it is possible to achieve just over a 6% gain in performance over an orthotropic layup.
Figure 5.55. Repaired fiber angle distributions for simply supported variable stiffness plate under centered point load (positive angle layers shown).
5.11.5.2 EXAMPLE: CLAMP SUPPORTED VARIABLE STIFFNESS PLATE UNDER POINT LOAD APPLIED AT CENTER

This section examines a square plate with a center point load where the plate is restrained by clamp supports on all sides (Figure 5.57). The performance summary is given in Table 5.6. For this problem, the best orthotropic solution is not much better than the baseline quasi-isotropic layup. However, when the variable plate is examined, the first level optimization returns a 44% lower compliance than its orthotropic cousins. A large proportion of the performance is gain is lost in the second level GA optimization, but in contrast to the simply supported problem, compliance is actually reduced after being run through the angle repair algorithm.
Figure 5.57. Clamp supported variable stiffness plate under centered point load.

Table 5.6. Summary of Performance for Clamp Supported Variable Stiffness Plate Under Centered Point Load

<table>
<thead>
<tr>
<th>O₁</th>
<th>O₂</th>
<th>V₁</th>
<th>V₂</th>
<th>V₂R</th>
</tr>
</thead>
<tbody>
<tr>
<td>.97</td>
<td>.99</td>
<td>.64</td>
<td>.84</td>
<td>.80</td>
</tr>
</tbody>
</table>

Plots of the $W₁^*$ and $W₃^*$ distributions from the first and second level optimizations are shown in Figures 5.58 and 5.59. There are far more elements than from the previous example for the simply supported plate for which the lamination parameters, $W₁^*$ and $W₃^*$ don’t match, resulting in a 22% loss in efficiency (normalized compliance equal .84). In comparison to the simply supported version, the difficulty in matching lamination parameters occurs when $W₁^*$ approaches -1 and $W₃^*$ approaches 1. As expected, in the areas for which there are poor matching of lamination parameters, there is also poor angle continuity (Figure 5.60).

However, the repair algorithm remedies angle continuity and reduces compliance by 5%. Figure 5.61 shows plots of repaired $W₁^*$ and $W₃^*$ in which the averaging algorithm effectively creates continuous bands for the values of 0 and -1 for $W₁^*$ and $W₃^*$, respectively.

The plots of the repaired ply angle distributions are given in Figure 5.62. The curvature constraint, $Δθ_{allowable}$, is set to 1°.
Figure 5.58. Lamination parameters distributions for a clamp supported variable stiffness plate under centered point load (first level optimization).

a) $W_1^*$ Distribution  

b) $W_3^*$ Distribution

Figure 5.59. Lamination parameters distributions for a clamp supported variable stiffness plate under centered point load (second level optimization).

a) $W_1^*$ Distribution  

b) $W_3^*$ Distribution
Figure 5.60. Fiber angle distributions for clamp supported variable stiffness plate under centered point load (positive angle layers shown).
5.11.5.3 Example: 2x1 Aspect Ratio
Clamp Supported Variable Stiffness Plate Under Two Point Loads

A 2 to 1 aspect ratio clamped supported plate is considered as a final example. Two equal point loads are applied at \( x=.25, y=.5 \) and \( x=.75, y=.5 \). The \( \Delta \theta_{\text{allowable}} \) is set to 1° (Figure 5.63). Results from ten independent runs of the optimization are considered to illustrate the random nature of the GA and the consequently varying range of performance.

The first level orthotropic plate optimization results indicate the possibility of a 10% reduction in compliance (Table 5.7). However, the second level orthotropic optimization that the GA has difficulty returning an optimal layup: the compliance value is more than twice that of the baseline quasi-isotropic layup. Hence, based the method we outlined in Section 5.10, an optimized orthotropic solution is not available.

Plots of first level, second level and repaired lamination parameter distributions based on trial number 8 are given in Figures 5.64, 5.65 and 5.66, respectively. Fiber angle distributions for post-GA and the repaired laminate are given in Figures 5.67 and 5.68 respectively.

The first level variable stiffness optimization returns a normalized compliance of 0.65, which appears to be promising. There is a considerable loss in performance after the second level variable stiffness optimization (See Table 5.8, trial 1). In fact, the performance approaches that of the baseline orthotropic plate. However, there is an approximate 3%
Figure 5.62. Repaired fiber angle distributions for a clamp supported variable stiffness plate under centered point load (positive angle layers shown).
Figure 5.63. 2x1 Aspect ratio clamp supported variable stiffness plate under two point loads.

Table 5.7. Summary of Orthotropic and First Level Variable Stiffness Performance for Clamp Supported 2:1 Aspect Ratio Plate Under Two Point Loads

<table>
<thead>
<tr>
<th>O₁</th>
<th>O₂</th>
<th>V₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>.90</td>
<td>2.32</td>
<td>.65</td>
</tr>
</tbody>
</table>

Figure 5.64. Lamination parameters distributions for a 2x1 aspect ratio clamp supported variable stiffness plate under two point loads (first level optimization).

a) $W₁^*$ Distribution

b) $W₃^*$ Distribution
Figure 5.65. Lamination parameters distributions for a 2:1 aspect ratio clamp supported variable stiffness plate under two point loads (second level optimization).

Figure 5.66. Lamination parameters distributions for a 2:1 aspect ratio clamp supported variable stiffness plate under two point loads (repaired laminate).
Figure 5.67. Fiber angle distributions for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (positive angle layers shown).

reduction in compliance after the angle repair is performed. This table also shows values for the total number of function evaluations and the total elapsed time required by the GA.

Since it is the nature of GAs to involve random number generation it is likely that a different solution could be attained for the same problem. To test this idea, the optimization was run nine more times (see Table 5.8, trials 2 thru 10). The final compliance of the repaired laminate values vary from a low of 0.93 to a high of 1.00. Given a large enough sampling of results, it is possible to achieve better performance.

Plots of first level, second level and repaired lamination parameter distributions based on trial number 8 are given in Figures 5.64, 5.65 and 5.66, respectively. Fiber angle distributions for post-GA and the repaired laminate are given in Figures 5.67 and 5.68, respectively. Deflection plots for four cases are shown in Figure 5.69. The first case (Figure
Figure 5.68. Repaired fiber angle distributions for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (positive angle layers shown).

5.69a) is the baseline quasi-isotropic layup, described in Section 4.4.4 that establishes the compliance normalization scale. Figure 5.69b is the deflection of an orthotropic plate based on optimal lamination parameters where see that the maximum deflection is 76% lower than that of the baseline layup. In Figure 5.69c shows the plotted deflection of the variable plate based on the fiber paths returned from the unconstrained GA. In contrast to Figures 5.69a and 5.69b, which have each have two distinct valleys coincident with the load application point, the deflection of the post-GA plate has been smoothed to a single valley with shallow asymmetrical troughs at each end. The maximum deflection is about 1% greater than the baseline layup even though the actual compliance of 0.97 is slightly lower than the baseline compliance of 1. The deflection of the repaired (constrained) variable stiffness laminate is shown Figure 5.69d. The shape resembles that of Figure 5.69c; however, the maximum deflection is about 5% less than the post-GA layup.
Table 5.8. Summary of Performance for Clamp Supported 2:1 Aspect Ratio Variable Stiffness Plate under Two Point Loads

<table>
<thead>
<tr>
<th>Trial Number</th>
<th>V2</th>
<th>V2R</th>
<th>Total # Function Evaluations</th>
<th>Total GA time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.99</td>
<td>.96</td>
<td>11,729,840</td>
<td>4,007</td>
</tr>
<tr>
<td>2</td>
<td>.98</td>
<td>.96</td>
<td>11,930,960</td>
<td>4,136</td>
</tr>
<tr>
<td>3</td>
<td>1.03</td>
<td>.99</td>
<td>11,951,860</td>
<td>4,157</td>
</tr>
<tr>
<td>4</td>
<td>1.04</td>
<td>1.00</td>
<td>12,003,520</td>
<td>4,147</td>
</tr>
<tr>
<td>5</td>
<td>.96</td>
<td>.94</td>
<td>11,794,380</td>
<td>4,057</td>
</tr>
<tr>
<td>6</td>
<td>1.01</td>
<td>.97</td>
<td>11,933,700</td>
<td>4,509</td>
</tr>
<tr>
<td>7</td>
<td>1.01</td>
<td>.96</td>
<td>12,058,620</td>
<td>4,675</td>
</tr>
<tr>
<td>8</td>
<td>.97</td>
<td>.93</td>
<td>12,037,600</td>
<td>4,681</td>
</tr>
<tr>
<td>9</td>
<td>1.01</td>
<td>.97</td>
<td>11,981,100</td>
<td>4,673</td>
</tr>
<tr>
<td>10</td>
<td>1.03</td>
<td>.96</td>
<td>11,882,820</td>
<td>3,428</td>
</tr>
</tbody>
</table>
Figure 5.69. Deflection for clamp supported 2:1 aspect ratio variable stiffness plate under two point loads (repaired laminate).
CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 SUMMARY

This research focuses on the directional characteristic of fibrous composites, which allows tailoring and effectively optimization of the stiffness of a laminate for a given loading condition. Traditionally, composite structures have been manufactured by stacking layers of woven fibers into a laminate, which are known as orthotropic layups. Optimization of orthotropic layups is possible by allowing the orientation angle of each layer to vary. So, in addition to the inherent material stiffness gain over many metal materials, further gains in stiffness can be achieved by tailoring the direction of each layer in composite layup. This principle can be extended further by allowing the fiber angle orientations to vary within a single layer. Instead of using woven fabrics, these types of laminates are manufactured by machines that place fibers in curved paths.

Various methods for optimizing composite plates have been used in the past. In this research, a two-level approach is adopted, which obtains optimal lamination parameters in a first level compliance minimization, then use the first level values in a second level norm minimization to obtain the optimal ply angles for each layer. The finite element analysis is imbedded into the first level compliance minimization objective function. In this research, the two-level approach is used in several different frameworks in which each successive optimization builds on the previous one.

In the first framework (Section 5.7), the first level optimization is devised to determine the optimal lamination parameters for an orthotropic plate subject to out-of-plane loads. As with all first level optimizations in this text, compliance is chosen as the objective function and the Method of Moving Asymptotes is adopted as a suitable gradient-based optimization algorithm. The first level optimization is constrained by assuming a symmetric and balanced layup which reduces the number of lamination parameters to two, $W_1^*$ and $W_3^*$. Through the use of trigonometric identities, Miki was able to show that an inequality relationship exists between $W_1^*$ and $W_3^*$. Once the optimal lamination was obtained, the graphical method developed by Miki was used to determine the optimal ply angles. For this
framework, the second level optimization does not require an optimization; rather, it is solved using algebraic equations for ply angles using the lamination parameters from the first level optimization as inputs. For simple load cases, boundary conditions and geometries (i.e., simply supported square plate under uniform load), the optimal lamination parameters, $W_1^*$ and $W_3^*$, tend to be on or near the parabolic edge of the feasible region. However, for more unique loading, boundary conditions and geometry in general, no ply angles can be retrieved using the Miki method without having to vary ply thicknesses because the optimal lamination parameters tend to lie outside of the feasible region. Hence, a more robust solution is sought for determining ply angle for the orthotropic plate.

The second two-level optimization framework (Section 5.10) uses the first level MMA algorithm with a second level Genetic Algorithm designed to perform norm minimization to generate the optimal ply layups. The simply supported square plate and the simply supported long plate under uniform loads are again used. Results show a close matching of compliance and layups. A more complex example is also carried out in which two points loads and discrete supports are placed at four interior locations. In this case, the optimal lamination parameter values lie in the part of the feasible region that is not covered by the Miki graphical procedure, requiring incorporation of the GA. From the first level optimization, it appears that there is a 5% possibility of reduction in compliance; the second level GA returns a layup that achieves the same 5% reduction in compliance. However, although none is presented here., there were results in which the second level optimization failed to return a layup with a compliance lower than 1.0. One instance was when $W_1^*$ and $W_3^*$ values reached -1 and 1, respectively. In conclusion, when compared to the Miki graphical method, norm minimization expands the range of available solutions without being limited to a two angle laminate.

In the third two-level optimization framework (Section 5.11.4), a variable stiffness plate was considered. Like the first level optimization for the orthotropic plate, the MMA is used to return the ideal $W_1^*$ and $W_3^*$ values where the values are allowed to vary from element to element. In the second level optimization, the shifted fiber path method developed by Gurdal and Olmedo is used to map fiber paths. Like the second level optimizations for the orthotropic plate, a norm minimization objective function is used only for the variable stiffness plate. The minimization is carried out for every element. To implement the second
level optimization, a Matlab Genetic Algorithm was attempted, but convergence time was prohibitive. A gradient-based optimizer written in FORTRAN 77 called DOT was implemented. It did reduce convergence time; however, several initial guess sets were needed to increase the chance of reaching the global minimum. In the end, the very nature by which fiber paths are mapped by this method is limiting and prohibits good matching to the lamination parameter distributions of the first level optimization. Hence, a more robust solution for obtaining ply angle distributions for the variable stiffness plate were sought.

For the fourth two-level optimization framework, the optimal lamination parameter distributions are obtained as before using the MMA. In the second level optimization, a norm minimization objective function is used again, but this time the same GA used to solve for the ply angles of an orthotropic layup (second framework), was run for every element.

No constraints are placed on the optimization and many of the elements don’t completely converge to a zero value objective function which results in discontinuities within the layer ply angle distributions. To correct the discontinuities a post-optimization repair algorithm that applies a curvature constraint was used. Three examples are used to help illustrate the validity of this framework.

6.2 CONCLUSIONS

The goal of this research was to establish a framework by which a composite plate with curvilinear fiber paths can optimized for minimum compliance. To meet the goal an analysis method using finite elements was first developed. Optimization difficulties: mapping load paths, efficient optimization. By building on a series of two-level optimization frameworks, we ultimately arrive at a new framework that adds an algorithm that corrects discontinuities that result from an unconstrained optimization for ply angles. The optimization framework algorithm has been shown to reduce the compliance value relative to an orthotropic plate for several test problems; however, there are also cases for which the framework does improve on an orthotropic plate solution. In conclusion, there is the potential to improve the stiffness to weight ratio of a composite plate using curvilinear fiber paths versus straight fiber path fabrics for certain combinations of geometries, load cases and boundary conditions. However, the performance improvement comes at the price of
additional computation time for analysis and optimization; several manufacturing difficulties will need to be overcome as well.

6.3 RECOMMENDATIONS

This research can be considered a foundation upon which to study a variety of problems. For example, the finite element and optimization codes could be modified to perform a buckling optimization. As we showed in Chapter 4, both in-plane and out-of-plane finite element codes were developed and tested. Now that a framework exists, the task of debugging will become less arduous. The number of degrees of freedom per node required by the finite element analysis will increased from four to six, and the number of lamination parameters per element increases from two to four. A second extension of this research could be to optimize the composite laminate for multiple load cases. This would more closely simulate real world analysis of aircraft structures where several dozen load cases may have to be considered for a single piece of structure. Finally, a logical extension of this research would be to map continuous fiber paths based on the element wise angle distributions found in the two-level variable stiffness optimization and subsequent repair algorithm. This would be the next logical step in attempting to manufacture an actual laminate.
REFERENCES


