POLARIZATION CONTROL OF LIGHT WITH A LIQUID CRYSTAL DISPLAY SPATIAL LIGHT MODULATOR

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   Polarization Control of Light with a
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When you know that all is light, you are enlightened.

-Anonymous

A strangely appropriate quote from the tag of some tea I was drinking while writing this
In this work, we use a programmable liquid crystal display spatial light modulator to provide nearly complete polarization control of the undiffracted order for the case where the beam only makes a single pass through the liquid crystal element. This is done by programming and modifying a diffraction grating on the liquid crystal display, providing the amplitude and phase control necessary for polarization control. Experiments show that for the undiffracted order we can create linearly polarized light at nearly any angle, as well as elliptically polarized light. Furthermore, the versatility of the liquid crystal display allows for the screen to be sectioned, which we utilize for the creation of radially polarized-type beams. Such polarization control capabilities could be useful to applications in optical communications or polarimetry.

Through the experiments, we also uncover the disadvantages of the single pass system, which include some limitations on the range of linear polarization angles, large intensity variations between different polarization angles, and the inability to create a pure radially polarized beam. These experiments provide a foundation for future work where greater polarization control could be obtained through the use of a dual-pass system, including control of other diffracted orders rather than just the undiffracted order.
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CHAPTER 1

INTRODUCTION

Polarization is an important characteristic of electromagnetic radiation, which is a result of the relative phase and amplitude relationship between the $x$ and $y$ components of the electric field. The interaction of light with materials can result in distinct polarization states, which have become increasingly important in areas such as remote sensing and imaging, fiber optics, biomedical sensing and imaging, and industrial applications. Polarized light can be created with a number of optical elements and techniques, including dichroic materials, reflection, and birefringent materials, yet the number of devices for adaptive polarization control are very few.

The majority of polarization controllers are manufactured for fiber use only, and generally have manual controls. Liquid crystal devices are included in the few adaptive polarization control devices that exist, yet many simply adjust the phase retardation of the electric field components. Without the combination of amplitude and phase control, the control over polarization is limited. One device provides polarization control of a single beam with the use of two consecutive liquid crystal elements, but we will explore previous work that has shown that a single element can provide control with the use of diffractive optical elements.

This thesis examines the use of a single pass through a transmissive, parallel aligned, liquid crystal display spatial light modulator (LCD-SLM) for polarization control of light. This is done with the unique use of a diffractive optical element programmed onto the LCD for amplitude control, allowing for phase and amplitude control over the DC order. Experiments show that light of nearly any polarization state can be created from a linearly polarized input beam, with some limitations. The results of these experiments are then applied to show that more complex beams, such as radially polarized-type beams, can be created with the LCD-SLM by sectioning the display.
First, we use Maxwell’s equations to introduce the wave equation for the electric and magnetic fields of light, along with the origins of the index of refraction. The solution to the electric field wave equation provides the basis for understanding different polarization states of light, along with ways in which the polarization can be changed. Additionally, the index of refraction is used to explain the principles of birefringent materials, which are capable of further manipulating polarization by inducing phase shifts between the electric field components. Jones matrices and Jones calculus are discussed as the primary method for performing calculations of electric field components, polarization, and intensity in experimental optical systems.

Next, the LCD-SLM is introduced as the main component in the experimental setup. The underlying principles of the display are explained, and the device is calibrated to determine the induced phase shift as a function of the gray level on the pixels of the LCD.

Diffraction theory is used to introduce the optical Fourier transform and imaging setups used in the experiments. Additionally, we use Fourier analysis to write diffraction gratings as an infinite series of spatial frequencies representing the diffracted orders, where further phase and amplitude information is calculated. This leads to the analysis of the blazed grating, which is the grating used and modified for the polarization control experiments.

Jones calculus, diffraction theory, and Fourier analysis are used to theoretically determine the relative amplitude and phase of the electric field components for the diffracted orders of the blazed grating as a function of several parameters. Experiments with a blazed grating show that adjusting the parameters of the grating allows for the polarization state and the linear polarization angle of the undiffracted order to be very well controlled, matching theoretical predictions.

Finally, the LCD is sectioned off, where each section is programmed to produce a different polarization state. We show that a quasi-radially polarized beam can be produced using the single pass system. The control capabilities are further demonstrated with a more complex grating where the polarizations of the beam sections are rotated through the sections. Experimental results show excellent agreement with predicted polarization states and angles, as well as intensities. The limitations of the single-pass system are then discussed, along with future possible research for this method.
CHAPTER 2

POLARIZATION

Polarization is a characteristic of light and electromagnetic radiation that describes the orientation of the electric field. While generally light is not uniformly polarized in one state, interactions of light with some objects and materials will result in polarization. Knowing the polarization can be advantageous, like in the common use of polarized sunglasses to reduce the glare of reflected light, which becomes primarily polarized parallel to the reflecting surface.

Factors such as the amplitude and phase relationship of the electric field affect the polarization, and the polarization state constantly changes over time due to the oscillation of the electric field. This chapter will provide background on polarization and how the polarization can be altered, particularly with the use of birefringent materials. In addition to providing this background, Jones calculus will be introduced as a method for performing calculations involving electric fields and their interactions with different polarization manipulation devices.

THE WAVE EQUATION AND THE INDEX OF REFRACTION

Electromagnetic radiation is made up of a set of electric and magnetic fields, defined by Maxwell’s equations as the electric field, \( \vec{E} \), the electric displacement, \( \vec{D} \), the magnetic field, \( \vec{H} \), and the magnetic displacement, \( \vec{B} \). These are defined in terms of one another as

\[
\vec{D} = \epsilon \vec{E} \\
\vec{B} = \mu \vec{H}
\]

where \( \epsilon \) is the permittivity of the material the field is in and \( \mu \) is the permeability of the material. The constants \( \epsilon_0 \) and \( \mu_0 \) represent the permittivity and permeability in a vacuum. Maxwell’s four equations are
Faraday’s Law: \[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
Ampere’s Law: \[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]
Gauss’ Law for Electric Fields: \[ \nabla \cdot \vec{D} = \rho \]
Gauss’ Law for Magnetic Fields: \[ \nabla \cdot \vec{B} = 0 \]

(2)

where \( \vec{J} \) is the current density and \( \rho \) is the charge density.

Taking the curl of Faraday’s Law, and using some mathematical manipulation, a differential equation is derived resembling a standard wave equation. Taking the curl of the Ampere’s Law produces a similar result, and thus

\[ \nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \]

(3)

and

\[ \nabla^2 \vec{B} = \mu \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} \]

(4)

The solutions to these equations produce the expressions for the electric and magnetic fields.

A plane-wave solution of the form \( e^{i(\omega t - kz)} \) is assumed, where \( \omega \) is the angular frequency \( (2\pi f) \) of the wave, \( k \) is the wave number \( (2\pi / \lambda) \), \( \lambda \) is the wavelength, and \( f \) is the frequency of a wave traveling along the +z axis. The speed of light in a vacuum is defined as \( c = 1/\sqrt{\varepsilon_0 \mu_0} = 299 722 458 \) m/s. It is more useful to define the velocity of light in other materials with permittivity \( \varepsilon \) and permeability \( \mu \) as

\[ v = \frac{\sqrt{\varepsilon_0 \mu_0}}{\sqrt{\varepsilon \mu}} c = \frac{c}{n} \]

(5)

Equation 5 introduces the index of refraction, \( n \), which is defined as \( \sqrt{\varepsilon \mu}/\sqrt{\varepsilon_0 \mu_0} \). The values of \( \varepsilon \) and \( \mu \) vary for different materials and therefore so does the speed of light in different materials. It should also be noted that these values also depend on the frequency of the wave propagating through. The index of refraction will be used to understand birefringence, which is the underlying principle behind the operation of the LCD used in the experiments of this thesis.

The solutions to the wave equations for the electric and magnetic fields (Equations 3 and 4) describe transverse waves that oscillate perpendicular to each other. The field equations are
\[
\vec{E}(x, y, z, t) = (E_x e^{i\phi_x} \hat{i} + E_y e^{i\phi_y} \hat{j}) e^{i(\omega t - kz)}
\]

(6)

and

\[
\vec{B}(x, y, z, t) = (B_x e^{i\phi_x} \hat{i} + B_y e^{i\phi_y} \hat{j}) e^{i(\omega t - kz)},
\]

(7)

where the \(E\) and \(B\) coefficients and \(e^{i\phi}\) terms represent the amplitude and phase of the \(x\) and \(y\) components for each wave. Maxwell’s equations can be used to show that the magnetic field is weaker than the electric field by a factor of \(c/n\) in any material, so the remainder of this thesis will focus on the dominant electric field. Equation 6 will be used to define and understand the concepts of polarization, but first we return to the index of refraction.

**Birefringence and the Index Ellipsoid**

In the previous section it was found that the index of refraction affects the velocity of a propagating electromagnetic wave in a given material. Birefringence occurs when a material exhibits a varying index of refraction,\(^{12,13}\) depending on which direction the electromagnetic wave propagates through the material. This generally occurs in crystals such as quartz or calcite, but also occurs in semi-crystalline materials such as polymers.

The oscillation of the electric field induces vibration in a crystal or molecule along the direction of the oscillation, known as induced polarization, which in turn affects the permittivity along that direction. The dielectric response along the crystal’s principal axes can be written in terms of the permittivity and the electric field along each axis as

\[
D_x = \epsilon_x E_x
\]

\[
D_y = \epsilon_y E_y
\]

\[
D_z = \epsilon_z E_z
\]

The crystal structure has a great impact on the induced polarization. If the crystal structure is the same along each axis, the induced polarization is the same along each axis. Many crystals have anisotropic structure, such that the induced polarization is different on each axis. This means the permittivity along each axis is different, such that \(\epsilon_x \neq \epsilon_y \neq \epsilon_z\), and therefore the index is different along each axis. In a crystal like this, the indices are defined as \(n_x, n_y, \) and \(n_z\).
Considering the three indices of refraction along the crystal axes, an ellipsoid can be constructed where the shape is determined by the indices of refraction. This is illustrated in Figure 1a, which is constructed with the equation

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \quad (8)$$

It is important to reiterate that a propagating wave will experience the index along the axis parallel to the oscillation of the electric field. For example, in Figure 1a, a wave oscillating along the $z$ axis experiences an index of refraction of $n_z$ whether it propagates along the $x$ or $y$ axis.

![Figure 1. (a) Example of an index ellipsoid, where the values of the index of refraction along the $x$, $y$, and $z$ axis define the shape. (b) The cross-section of the ellipsoid formed by the plane normal to the direction of the wave propagation defines the effective $x$ and $y$ indices the wave will encounter.](image-url)

Determining the index of refraction a wave will experience becomes more complicated if the wave enters the crystal along an arbitrary direction. In this case, the index of refraction can be determined by considering the cross-section of the ellipsoid created by a plane perpendicular to the propagation direction of the wave. This is shown in Figure 1b. The cross section created is in the shape of an ellipse, and the major and minor axes of the ellipse represent the new indices of refraction, $n'_x$ and $n'_y$. 

The electric field $x$ and $y$ components from Equation 6 now encounter the indices of refraction $n_x'$ and $n_y'$ as they propagate through a birefringent crystal, and therefore the components travel at different velocities. This means the phase relationship between the $x$ and $y$ components is altered after propagating through such a material. As stated in the beginning of this chapter, the phase relationship of the electric field components is one of the defining factors of the polarization of the wave. Polarization will now be explained in more detail, and we will see how birefringent materials can be used to manipulate the polarization of an electromagnetic wave.

**Polarization States of Electromagnetic Radiation**

Figure 2 shows the electric field of light as it propagates, described by

$$\vec{E}(x, y, z, t) = (E_x e^{i\phi_x} \hat{i} + E_y e^{i\phi_y} \hat{j}) e^{i(\omega t - kz)}.$$  

The resultant of the $x$ and $y$ vectors at any point give the resultant electric field. The polarization of the wave is described by the magnitude and direction of the resultant.

For the field in the Figure 2, note that the resultant direction is always in one plane, with the direction and magnitude alternating sinusoidally. This is referred to as linear polarization, which occurs when the electric field components are in phase such that $|\phi_x - \phi_y|$
is a multiple of $\pi$. The direction of linear polarization is given as an angle with respect to the vertical $y$ axis, which is found from the amplitudes of the components as $$\theta_{\text{pol}} = \arctan(E_x/E_y).$$

Figure 3 shows two cases of linearly polarized light. For linearly polarized light, if a phase shift of a half-wavelength occurs to either component, the polarization angle flips to become polarized at the negative of the initial angle.

Another form of polarization is elliptical polarization, which occurs if $e^{i\phi_x} \neq e^{i\phi_y}$. In this case, the $x$ and $y$ components are out of phase, and the resultant electric field rotates as the wave propagates. The term “elliptical” comes from the ellipse traced out by the resultant vector when viewed down the propagation axis. The ellipticity depends on the phase difference and the difference in amplitude between the components.

Figure 4 illustrates the special case where the amplitudes of the components are equal ($E_x = E_y$) and their relative phase shift is a quarter-wavelength ($|\phi_x - \phi_y| = \pm \pi/2$). In this case, the ellipticity is 0, so the magnitude of the resultant is constant and traces a circle as it propagates. Each case, a $+\pi/2$ or $-\pi/2$ phase shift, produces a different rotation referred to as the “handedness”. The handedness is defined as “right” or “left” by looking against the propagation direction, where left-circularly polarized (LCP) rotates clockwise over time, and right-circularly polarized (RCP) rotates counter-clockwise over time.

Most light sources emit randomly polarized light, where every ray may have a different polarization state than the other. Some sources, like lasers and light emitting diodes,
Figure 4. Creating (a) right circularly polarized light with a $\pi/2$ phase delay and (b) left circularly polarized light with a $3\pi/2$ phase delay to the horizontal component of Fig. 2.

may have a higher degree of one polarization state compared to others due to the nature of the source, but they are still not completely one state. Linearly polarized light may be created in a number of ways, but the most common is probably the polarizing filter, which utilizes a dichroic material to absorb polarizations along one axis, and pass polarizations perpendicular to that axis, which is referred to as the transmission axis. The orientation of the filter or laser determines the angle of linear polarization.

The creation of elliptically polarized light is more complicated, as a specific phase shift must be applied to one component of the electric field with respect to the other. Recalling the previous section, a birefringent crystal allows for such a phase shift to occur. The different velocities of the $x$ and $y$ components of the electric field as they propagate through the crystal causes the components to become out of phase. The exact phase difference is a function of the indices of refraction for each component ($n_x$ and $n_y$), the thickness of the crystal ($d$), and the wavelength of the light ($\lambda$), and can be calculated as

$$\Delta \phi = \frac{2\pi d}{\lambda} (n_x - n_y) \quad (9)$$

To achieve the specific phase shifts of $\pi$ or $\pi/2$ in Figures 3 and 4, the index of refraction each component encounters as it propagates must be known, and the propagation path length in the material must be tailored such that the difference in the phase shifts between the components is precisely one-half or one-quarter of a wavelength, respectively. Quarter-wave plates and half-wave plates are optical components that are specifically manufactured to create these phase shifts for a specific wavelength of light. It is important to
note that to have an output of only one polarization from these waveplates, the input beam
must be linearly polarized such that the electric field components are in phase before the light
enters the waveplate.

A typical optical system will have many components, and the resulting polarization of
light traveling through the components can be determined. However, each component of the
electric field must be treated separately and as the optical system becomes more complicated,
so do the calculations. Fortunately, a very useful system of matrices and vectors was
developed by R.C. Jones to specifically deal with polarization and components that
manipulate the electric fields, making calculations much simpler.

**Jones Matrices and Jones Calculus**

Jones calculus\textsuperscript{14} treats the propagation of polarized light through linear optical
components that manipulate the polarization. Polarization states are expressed as vectors
constructed from the electric field $x$ and $y$ components, and optical elements are represented
by $2 \times 2$ matrices. Through the process of matrix multiplication, the relative phase and
magnitude of the electric field components can be tracked at every step of the system. To
begin, from Equation 6, an input beam vector is written as

$$E_0 = \begin{pmatrix} E_{0x} e^{i\phi_x} \\ E_{0y} e^{i\phi_y} \end{pmatrix}$$

(10)

The exponential terms display the relative phase shift between the components, and the $E_{0-}$
coefficients display the relative amplitudes. Generally, the vector is normalized and a common
phase term is factored out.

The intensity of the beam may be determined fairly easily with the Poynting vector,
calculated by taking the cross product of the electric field vector with the magnetic field
vector ($\vec{S} = \vec{E} \times \vec{H}$). The magnetic field can be written in terms of the electric field, such that
the result of the cross product is proportional to the electric field squared. For a Jones vector
representing the electric field, the intensity is found by

$$I = E_x E_x^* + E_y E_y^*$$

(11)

where the asterisk represents the complex conjugate.
Noting that the electric field vector may be complex, several common polarization states are then defined by

\[
\text{Linearly Polarized along } x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
\text{Linearly Polarized along } y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\text{Right Circularly Polarized} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
\text{Left Circularly Polarized} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}
\]

One disadvantage of Jones calculus is there is no easy way to deal with unpolarized light. Most optical systems utilize linear polarizers as one of the first optical elements, so the initial polarizer is ignored and the resulting linearly polarized beam is used for the initial electric field vector.

The Jones matrices represent optical elements that manipulate the electric field and polarization. While common matrices exist for common optical elements like linear polarizers, a more complicated element may require some calculations to derive the Jones matrix. The linear polarizer is the simplest, given by

\[
LP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

This matrix is for a polarizer oriented along the \( x \) axis, while a polarizer along the \( y \) axis has the 1 in the lower right element, and the upper left element is a 0.

Any birefringent material is represented by a waveplate, where the matrix is given by

\[
WP = \begin{pmatrix} e^{-i \frac{2\pi}{\lambda} d} & 0 \\ 0 & e^{i \frac{2\pi}{\lambda} d} \end{pmatrix}
\]
The exponential terms represent the phase shifts that occur for each axis, which is a factor of the $x$ and $y$ indices, $n_x$ and $n_y$, the wavelength of light, $\lambda$, and the thickness of the crystal, $d$. Factoring some components produces

$$ WP = e^{-\frac{2\pi}{\lambda} n_x d} e^{-\frac{i\phi}{2}} \begin{pmatrix} e^{2} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix} $$

where now $\phi = 2\pi d(n_x - n_y)/\lambda$, which represents the relative phase shift between the components of the electric field. For a quarter-wave plate, $\phi = \pi/2$, and for a half-wave plate, $\phi = \pi$. Note that from the definition of intensity, the terms on the outside of the matrix will drop out after multiplying by their complex conjugate.

Many times, an optical component is rotated at an angle with respect to the vertical axis. To treat this component appropriately, the electric field vector must undergo a basis transformation into the basis of the optical component. This is easily done with the rotation matrix, given as a function of the angle $\theta$ at which the component is rotated

$$ R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} $$

Likewise, after the electric field has been treated through the optical component in its basis, the electric field must be rotated back to its original basis with $R(-\theta)$.

To treat an optical system made up of many components, the initial electric field vector is multiplied by the matrices of the optical components in their order. The initial electric field vector is placed on the far right, and the elements are multiplied from right to left in order, with the last optical component on the far left. For example, vertically polarized light passing through a horizontal polarizer is represented by

$$ E_{out} = (LP)(E_{in}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$

The result shows that the output electric field has no horizontal or vertical component, meaning the horizontal polarizer completely blocks the vertically polarized light, as expected.
To complete this section, a final example of a more complicated system will be shown to further familiarize the reader with Jones matrices and Jones calculus.

**MALUS’ LAW WITH JONES CALCULUS**

As an example of the simplicity and power of Jones calculus, Malus’ law will be derived with Jones matrices. Malus’ law gives the intensity of light transmitted through two linear polarizers as a function of the relative angle between their transmission axes. To set up the system, we first realize that unpolarized light passed through the first polarizer will simply be polarized at the angle of the first polarizer. To simplify the system, the first polarizer will be set to vertical and the second polarizer will be at an angle $\theta$ with respect to vertical.

The output light from the first polarizer is simply vertically polarized, so this component can be ignored and we begin with the vertically polarized light with amplitude $E_{in}$ entering the second polarizer. Recall that a rotation matrix must be used to rotate the vertically polarized light into the basis of the second polarizer. The resulting Jones system is written as

$$\vec{E}_{out} = [R(-\theta)] [LP] [R(\theta)] \vec{E}_{in}$$

$$= E_{in} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= E_{in} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

producing the final result,

$$E_{out} = E_{in} \begin{pmatrix} -\sin(\theta) \cos(\theta) \\ \cos^2(\theta) \end{pmatrix}$$

The result for $E_{out}$ shows that the resulting electric field will have a horizontal and vertical component where the amplitudes depend on the input angle. If the polarizers are
perpendicular to each other ($\theta = \pi/2$), the output is 0, while the output is maximized when they are parallel ($\theta = 0$ or $\pi$), and leaves only vertically polarized light.

The observed intensity can be found using Equation 11, where the initial intensity (after passing through the first polarizer) is given by $I_0 = E_{in}^2$

\[
I = E_x E_x^* + E_y E_y^* \\
= I_0 \left[ \sin^2(\theta) \cos^2(\theta) + \cos^4(\theta) \right] \\
= I_0 \cos^2(\theta) \left[ \sin^2(\theta) + \cos^2(\theta) \right] \\

I = I_0 \cos^2(\theta)
\]

which is Malus’ law. Again, the intensity is maximized when the polarizers have parallel transmission axes, decreases quickly as the angle is increased, and is 0 when they are perpendicular.

**Summary**

This chapter began by discussing Maxwell’s equations and introducing the wave equation, from which the equations describing the electric and magnetic fields were found. The fields are transverse waves with $x$ and $y$ components that oscillate perpendicular to each other as the wave propagates.

The origins of the index of refraction were also shown, relating the index of refraction to the permittivity and permeability of different materials, which ultimately reduces the phase velocity of electromagnetic radiation by a factor dependent on $n$. Birefringent materials were then discussed, in which the index of refraction is dependent on the axis of propagation of the light through the material. The index ellipsoid illustrates how the index of refraction for the $x$ and $y$ components of the electric field change depending on the angle at which the field propagates through the ellipsoid.

The polarization states of light were shown, including linear and elliptical polarization, and the special cases of left- and right-circularly polarized light. The creation of elliptically polarized light requires a phase difference between the components of the electric field, which can be achieved with the use of birefringent materials.

Finally, Jones calculus was described, where the electric field is written as a vector made of the electric field components and optical components are represented by $2 \times 2$
matrices. The methods for using Jones calculus show that the output electric field of an optical system can be determined, leading to the determination of the resulting polarization and intensity.

The next chapter will combine the concepts of birefringence and Jones calculus to discuss the main component of the experiments in this thesis, the LCD.
CHAPTER 3

THE LIQUID CRYSTAL DISPLAY - SPATIAL LIGHT MODULATOR

The liquid crystal display-spatial light modulator (LCD-SLM) is a valuable tool in our experimental setups. The LCD-SLM is a programmable display that can be programmed with any number of complex diffractive optical elements. Pre-manufactured optical elements such as diffraction gratings are available, but they cannot be modified, and may become too expensive or difficult to manufacture as the optical elements become more complex. The greatest advantage of the LCD is the ease and speed in which the display can be changed and modified, allowing for nearly instantaneous experimental results and unlimited possibilities as far as creating optical elements. In this chapter we will discuss the operation of the SLM and the calibration procedure, which are key to a thorough understanding of the experiments in this work.

LIQUID CRYSTALS

Birefringence and birefringent materials were introduced in Chapter 2, where a difference in the index of refraction along different axes of a material allows for phase shifts in the components of an electric field propagating through the material. A liquid crystal is another material that exhibits birefringence, where a single crystal can be considered as an elongated ellipsoid as seen in Figure 5. The crystals are uniaxial, meaning two of the axes have the same index, referred to as the ordinary index of refraction, $n_O$, while the elongated axis has a different, extraordinary index of refraction, $n_E$.

What makes the liquid crystal more valuable is its ability to align to an applied voltage, as seen in Figure 6. When placed in an electric field, the crystal acts as a dipole and rotates to align to the field. The stronger the field, the greater the rotation. This provides a means of control for the orientation of the crystals, and therefore a means to control the angle at which incident light propagates through the crystal. From the discussion on the index ellipsoid in the previous chapter, the index of refraction encountered by the electric field components depends
Figure 5. A single liquid crystal ellipsoid viewed from the side (left) and top (right). The short axes have the ordinary index of refraction while the long axis has the extraordinary index.

Figure 6. The extraordinary and ordinary index of refraction of a liquid crystal as it is rotated.
on the axis through which the wave propagates in the crystal. This means the relative phase shift of the components can be changed as the crystal is rotated. Note that the ordinary index of refraction stays the same no matter what the rotation of the crystal is, and when the crystal is completely rotated, the extraordinary index is equal to the ordinary index.

Recall that for waveplates, the electric field components of a wave all experience the same relative phase shift through the crystal. With the LCD-SLM, a display can be created where the phase shift at any point of the display can be programmed to any value, allowing for the creation of diffractive optical elements.

**Liquid Crystal Display**

We use an Epson liquid crystal display (LCD), which is constructed of an array of 640×480 pixels with a spacing of 42µm. Each pixel contains a number of liquid crystal molecules which are all parallel aligned due to grooves in the plates that contain the crystals. A grid of wiring allows a different voltage to be applied to each pixel, thereby changing the rotation and therefore the extraordinary index of refraction for any given pixel. The phase shift of the y component to the x component, \( \phi \), is then a function of voltage, display thickness, \( d \), and wavelength of laser light, \( \lambda \), written as

\[
\phi(V) = \frac{2\pi d}{\lambda} [n_E(V) - n_O]
\]

(16)

For our experimental setup, 514.5nm light is used from an air-cooled Argon laser source.

With no applied voltage, the crystals are vertical, and with a maximum applied voltage, the crystals will lay flat. Notice that with no applied voltage, the extraordinary index is at its largest, while at the maximum applied voltage, \( n_E = n_O \). This means the phase shift between the x and y components is greatest with no applied voltage, and is 0 at the max voltage.

Because the liquid crystals in all pixels are vertically parallel aligned, the index of refraction is constant across the horizontal axis of the LCD. As a result of the constant ordinary index of refraction, it is important to note that regardless of the applied voltage at each pixel, the phase shift of the horizontal component of the electric field will be constant for every pixel across the display, regardless of the applied voltage. This means that any diffractive optical element programmed on the LCD only affects the vertical component of the incident light, and the horizontal component will pass through undiffracted.
A computer program is used to program the LCD, where the desired phase shift for each pixel is displayed as a gray level from 0 to 255. Each gray level corresponds to a different applied voltage, so the birefringence and phase shift can also be considered as a function of gray level. In order to apply the desired phase shift at each pixel, the LCD must be calibrated to understand the phase shift as a function of gray level.

**CALIBRATING THE LCD**

The LCD must first be calibrated to gain an understanding of the variation of the phase shift for each pixel as a function of the gray level on the screen. Knowing this relationship will allow for the creation of a calibration curve providing the exact phase shift versus gray level.

To calibrate the LCD, it is first pointed out that the screen may be treated as an adjustable waveplate. Before putting complex patterns on the screen, a uniform gray level can be displayed across the LCD where the gray level can be changed, allowing for determination of the phase shift.

The experimental setup for the calibration is as follows. First, a laser beam is spatially filtered, expanded, and collimated to illuminate the LCD. The collimated beam is passed through a linear polarizer oriented at 45° with respect to vertical, then passed through the LCD. Following the LCD, another linear polarizer is placed, and its orientation is switched between ±45° with respect to vertical. The last component is a photodiode detector from which beam intensity is recorded, as seen in Figure 7.

**Figure 7. Setup for calibration of the LCD phase shift vs. gray level.**
Jones calculus can be used to analyze this setup and determine how the intensity at the detector is expected to vary as the phase shift of the LCD is changed. First, since the LCD will act as an adjustable waveplate in this setup, the Jones matrix for the LCD is represented by

\[
M_{\text{LCD}} = e^{i\phi_0} \begin{pmatrix}
 e^{i\Delta \phi(V)} & 0 \\
 0 & 1
\end{pmatrix}.
\]  

(17)

Here, the exponential term outside the matrix represents the phase shift that is common to both the $x$ and $y$ components. Since we are looking at the intensity in this case, this term will fall out of the final calculation as it is multiplied by its complex conjugate. The exponential term inside the matrix represents the difference in phase shift between the $x$ and $y$ components of the electric field, and it is a function of voltage, and therefore gray level.

The linear polarizer and rotation matrix can be used to write the system as a series of Jones matrices. The resulting electric field is then (for the crossed polarizers)

\[
\vec{E}_{\text{out},\perp}(\phi) = e^{i\phi_0} \frac{1}{4} \begin{pmatrix}
 e^{i\Delta \phi} & -1 \\
 -1 & e^{i\Delta \phi}
\end{pmatrix}
\]  

(19)

The intensity is what is measured at the output plane, which is found by calculating the magnitude of the complex vector. Using some trigonometric identities, the intensity is

\[
I_{\perp}(\phi) = \frac{I_0}{2} \sin^2 \left( \frac{\Delta \phi}{2} \right).
\]  

(20)

The matrix multiplication and intensity are then repeated for the case of parallel polarizers, resulting in a similar expression.
$I_{||}(\phi) = \frac{I_0}{2} \cos^2 \left( \frac{\Delta \phi}{2} \right).$  \hspace{1cm} (21)

As the gray level of the LCD is changed from 0 to 255 for each case of perpendicular or parallel polarizers, we can expect to see an intensity relationship according to Equations 20 and 21. When $\Delta \phi = \pi$ in the perpendicular case, the intensity will be maximized because at this value the LCD will act as a half-wave plate and rotate the polarization from $+45^\circ$ to $-45^\circ$, which will be completely passed by the 2nd polarizer. Likewise, when the phase shift is 0 or $2\pi$, the LCD will induce no phase shift on the light, so the $+45^\circ$ polarized light will be completely blocked by the perpendicular polarizer.

So far it has been assumed that the maximum phase shift will occur with no applied voltage (gray = 0), and no phase shift will occur at a maximum applied voltage (gray = 255). In reality, there could be an offset of phase shift in the LCD, and the range of 0 to 255 gray level could actually not induce a full $2\pi$ phase shift, or it could induce a phase shift greater than $2\pi$ over that range. The calibration will reveal these effects and will also provide a reliable understanding of phase shift versus gray level.

**CALIBRATION RESULTS**

The setup was constructed according to Figure 7, passing a collimated laser through a polarizer at $+45^\circ$ with respect to vertical. The beam then passed through the LCD, and finally through another polarizer. Data was taken for two setups, one with the 2nd polarizer angle at $+45^\circ$ with respect to vertical, and the other at $-45^\circ$. Intensity readings were recorded as the gray level on the LCD was varied over the range of 0 to 255 in increments of 4.

The results of Figure 8 show the resemblance of $\sin^2$ and $\cos^2$ curves. Distinct phase shift points can easily be spotted, like at $g = 84$ where the perpendicular curve is at a minimum while the parallel curve is maximized, meaning this is a $\pi$ phase shift. The points where the two curves are equal ($g = 50$ and $g = 118$) represent phase shifts of $\pi/4$.

Some differences can be noted between the curves. First, we see that the maximum intensity achieved by the parallel polarizer setup is lower than the perpendicular polarizers. Also of note, the curves seem to taper off at a slower rate at higher gray values than at the lower gray values, indicating that the phase shift is not linear with respect to gray. Finally, we see that the LCD is achieving a phase shift greater than $2\pi$, as each curve completes a full cycle to the beginning intensity value before $g = 255$. 
Figure 8. Intensity through perpendicular and parallel polarizers as a function of LCD gray level used for calibration of the LCD.

Previous work has shown that the differences in intensity are primarily due to thin-film interference effects due to the construction of the LCD,\textsuperscript{15} leading to intensity variations as the phase shift is changed. To determine the relationship between phase shift and gray level more accurately, the intensity values of the perpendicular and parallel curves are added. Mathematically, this results in an equation where the phase as a function of gray level can be expressed in terms of the intensities as a function of gray level for the perpendicular and parallel polarizer setup.

\[ \Delta \phi(g) = 2 \arcsin \left( \sqrt{\frac{I(g)_{\perp}}{I(g)_{\perp} + I(g)_{\parallel}}} \right) \]  

(22)

Equation 22 was used to find the phase shift for each gray level by using the appropriate intensity values at each gray level. The cyclical nature of the arcsin function requires some additional phase shifts to be added to certain areas to obtain a smooth curve.

Figure 9 shows the final calibration curve, which has been adjusted to show the true phase shift versus gray level, such that a $2\pi$ phase shift occurs around $g = 84$ and a $\pi$ phase
shift occurs around \( g = 172 \). The curve does show that the LCD spans a phase shift greater than \( 2\pi \), so a portion at either end is removed to obtain a curve that only spans \( 2\pi \). In this case, the portion at the lower gray levels is most flat, meaning that a change in gray level has little change in phase shift in this region, so some of this area is disregarded. Therefore, the final calibration curve actually only contains gray levels from 34 to 255 rather than 0 to 255.

It is important to note that this curve only applies to 514nm wavelength light, and if other wavelengths are to be used, a different calibration curve needs to be obtained. We also note that this calibration was performed on a newer Macintosh computer with a higher resolution and different display characteristics than the typical older computer in the lab. At one point during the experiments, the older computer was used, where it was clear that the calibration curve obtained from the new computer was no longer ideal. Therefore, the display characteristics of the computer also have an effect on the LCD calibration.

With this curve, a phase shift for an incident electric field can be precisely applied at any pixel in the display. Now rather than using the LCD as an adjustable waveplate as in this section, more complicated phase functions can be applied to the display.
PROGRAMMING THE LCD-SLM

To program the LCD, the program Coherent Optics is utilized, created by Dr. Don Cottrell. This program allows for the creation of any number of complicated phase and amplitude functions that may be encoded onto the LCD. The operation and capabilities of this program will not specifically be explained in this thesis, but the concept of the program’s application will be briefly explained for the purpose of explaining the experimental setup.

Once a desired optical element is created on the Coherent Optics program, the resulting optical element is displayed as a grayscale image. This is the image that will be displayed on the LCD, and the gray levels represent the gray levels and phase shifts determined in the calibration procedure. The Coherent Optics program assumes a linear relationship, where a gray level of 0 represents 0 phase shift, and a gray level of 255 represents a $2\pi$ phase shift. As seen in the previous section, the relationship of phase and gray level is not linear and spans a phase shift greater than $2\pi$. A lookup table is created from the calibration curve such that the total span is $2\pi$, and the lookup table relates the assumed shift for a gray level from the Coherent Optics program to the actual gray level that produces this shift.

Understanding exactly how the LCD operates to produce these phase shifts is key to achieving the desired outcome for the optical elements in this thesis. We move on to discuss diffraction theory for gratings, and examine some simpler gratings and the resulting far-field diffraction patterns.
CHAPTER 4

GRATING THEORY

This chapter begins with a brief introduction to diffraction theory, which is used to discuss some common diffraction gratings. The theory and concepts of near and far-field diffraction are covered in detail in the Appendix and other sources, but some key points will be brought up to show how diffractive optical elements can be treated, as well as provide a description of the optical Fourier transform and imaging system used in experiments.

DIFFRACTION

A diffraction grating is a periodic structure that acts as a dispersive medium for incident electric fields. The field passes through the grating and spreads out at any given point. After propagating some distance from the aperture, the superposition of electric fields at the observation plane produces a diffraction pattern. As discussed in the Appendix, Fresnel diffraction describes the near-field pattern, while Fraunhofer diffraction describes the far-field pattern. It turns out that the Fraunhofer diffraction pattern can be found as the Fourier transform of the transmission function of the diffractive optical element.

A useful tool then becomes the common converging lens, which takes the optical Fourier transform of the transmission function and allows for the Fraunhofer diffraction pattern to be obtained over a much shorter distance. A 2-f optical Fourier transform system is created when a lens is placed one focal length from the aperture, and the diffraction pattern is then viewed one focal length past the lens, as seen in Figure 10.

At the observation plane in Figure 10, the diffraction pattern represents the spatial frequencies that make up the transmission function, as will be discussed in more detail shortly. A second 2-f Fourier can be placed after the first, and the observation plane in Figure 10 is then referred to as the Fourier plane.

A second 2-f system can be used to take the inverse Fourier transform of the Fourier plane. This essentially images the transmission function, and the image will appear upside-down and backwards at the new observation plane. Initially this seems redundant, but
Figure 10. The $2f$ system, also called an optical Fourier system, utilizes a lens to achieve the far-field diffraction pattern over a distance of 2 focal lengths.

Note that the spatial frequencies of the transmission function occur at different locations at the Fourier plane. This means the frequencies can be physically blocked or passed at the Fourier plane, altering how the aperture is reconstructed and imaged at the observation plane. This is a type of image processing, and the system seen in Figure 11 will be used in experiments to image one spatial frequency at a time.

Figure 11. A $4f$ system takes two optical Fourier transforms and images the aperture at the observation plane. The first Fourier transform allows for spatial filtering at the Fourier plane.

Next, Fourier analysis of a grating is used to write a transmission function as an infinite sum of frequencies to better understand the diffraction pattern at the Fourier plane.

**Fourier Analysis of Gratings**

Many parameters of a transmission function have an effect on the resulting diffraction pattern, particularly the characteristics of the grating itself. The period of the grating and
transmission function of the grating are the greatest factors, however the wavelength of the incident field is also very important. Different wavelengths are diffracted to different points at the observation plane, which describes one of the functions of a grating as a tool for separating an incident field into its frequency components.

A diffraction grating modifies the electric field that passes through it. In experiments, a collimated laser beam is used to illuminate any particular diffraction grating, which means the incident electric field is monochromatic and coherent. Therefore, as the beam passes through a grating, different parts of the beam acquire phase and amplitude shifts depending on the portion of the grating it passes through. The grating can then be considered as the transmission function at the aperture planes discussed in the previous section.

As with any periodic function, the transmission function may be rewritten as a Fourier series. For an infinitely long grating, this is

\[ t(x) = \sum_{n=\infty}^{\infty} c_n e^{-in\gamma x} \] (23)

In this equation, \( \gamma \) is the spatial frequency \( (2\pi/d) \), \( d \) is the grating period, \( x \) is a point along the grating period, and the \( c_n \) values represent the Fourier coefficients. For a Fourier series, the Fourier coefficients are found using

\[ c_n = \frac{1}{d} \int_{0}^{d} t(x)e^{in\gamma x}dx \] (24)

Equations 23 and 24 provide the spatial frequency components of the grating. The optical Fourier system will simply take the Fourier transform of each of the terms of the transmission function. Equation 23 shows that each component has an exponential term, \( e^{-in\gamma x} \), and the Fourier transform of this exponential function is given by

\[ \mathcal{F}(e^{-in\gamma x}) = \delta(p - n\gamma). \]

This shows that each component is represented by a delta function in the Fourier plane, meaning a series of spots at the locations of \( p = n\gamma \). Here, \( p \) is the spatial frequency, which from diffraction theory is also given as \( p = (2\pi/\lambda) \sin\theta \), where \( \lambda \) is the wavelength of the light.

The Fourier coefficients represent the electric field amplitude for each order, meaning the intensity may be found by multiplying each Fourier coefficient by its complex conjugate.
This leads to the definition of the *diffraction efficiency*. If the incident beam has an intensity $I_0$, we can consider it to be normalized to $I_0 = 1$. Now, as the intensity of each order is calculated, it will represent a fraction of the initial intensity. For example, if for a particular grating $c_1c_1^*$ is calculated to be 0.5, it means that the 1st diffracted order carries half of the initial beam intensity. For an ideal grating, the sum of the intensities of the diffracted orders equals the initial intensity. The diffraction efficiency is a measure of the intensity each order possesses at the Fourier plane for a given grating, given as a percentage of the incident beam.

The above theory shows that any grating can be written in terms of its Fourier series, and from this series, the pattern at the Fourier plane can be calculated. Many Fourier transform relations are known, such as finite aperture functions or rectangular aperture functions, and they all apply in optical Fourier systems. This technique will now be used to examine some simpler, common gratings to provide a basis for some of the experiments and theory in this thesis.

**Amplitude Grating**

One of the simplest gratings is an amplitude grating, which can be considered to be a slide with a periodic ruling of black lines painted on it. Light is completely blocked by the painted lines, while between the lines the light is completely transmitted. The period of the grating, $d$, consists of a painted section of width $w$ and the remainder is unpainted, as seen in Figure 12. This results in a transmission function of the form

$$t(x) = \begin{cases} 
1, & 0 < x < w \\
0, & w < x < d 
\end{cases}$$

To begin, the Fourier coefficients can be determined using Equation 24. In this case, the integral is split into two parts and the respective $t(x)$ is substituted for each section of the period.

$$c_n = \frac{1}{d} \left( \int_0^w (1)e^{in\gamma x}dx + \int_w^d (0)e^{in\gamma x}dx \right)$$

$$= \frac{1}{in\gamma d} [e^{in\gamma x}]_0^w$$

$$= \frac{1}{i2\pi n} \left[ e^{in2\pi \frac{w}{d}} - 1 \right]$$
Figure 12. Transmission function of an amplitude grating where the incident electric field is passed from 0 to \( w \) and blocked for the remainder of the period \( d \).

The term \( n = 0 \) must be calculated separately, or else \( c_0 \) is undefined due to the \( n = 0 \) in the denominator. This simply makes the exponential term \( e^0 = 1 \), so the integral is over \( dx \) from 0 to \( w \). The result is \( c_0 = w/d \). Therefore, the first term can be substituted into the Fourier series expansion of \( t(x) \), giving

\[
t(x) = \frac{w}{d} e^{-i(0)\gamma x} + \sum_{n=\pm1}^{\pm\infty} c_n e^{-i n\gamma x} = \frac{w}{d} + \sum_{n=\pm1}^{\pm\infty} c_n e^{-i n\gamma x}
\]

(25)

The Fourier transform of this term puts the 0th order, also called the DC order, at \( \delta(p - 0) \), which means it is undiffracted and occurs at the origin in the Fourier plane. The intensity is simply \( w^2/d^2 \), which means the intensity will depend only on the ratio of \( w \) to \( d \). The intensity of the other terms comes from multiplying \( c_n \) by its complex conjugate, resulting in

\[
c_n c_n^* = \frac{1}{4\pi^2 n^2} (e^{i\pi w/d} - 1)(e^{-i\pi w/d} - 1) = \frac{1}{2\pi^2 n^2} (1 - \cos(2\pi n w/d))
\]

which shows that the intensity of each successive order will quickly decrease and even be zero for specific orders. The resulting intensity profile of the orders at the Fourier plane, for \( w/d = 1/2 \), would appear as in Figure 13.
Figure 13 shows that for the special case where \( \frac{w}{d} = 1/2 \), the even orders are non-existent, while the odd orders are symmetric about the origin and decrease in intensity. Examining the intensities, it is notable that the sum of the order intensities will approach \( I_{\text{sum}} = 0.5I_0 \), where \( I_0 \) is the incident beam intensity. This makes sense because by the nature of this grating, half of the incident light is blocked by the painted lines. We can now examine a grating that modulates by changing the phase of the electric field rather than amplitude modulation.

### Phase Grating

As its name implies, a phase grating alters an incident electric field by creating a different phase shift at different points of the grating. The changes in the phase then alter the superposition of the electric fields at the observation plane, resulting in a diffraction pattern. To begin, consider another simple periodic grating with a period of \( d \), as shown in Figure 14.

In contrast to the amplitude grating, this grating transmits all of the incident electric field. The difference is that for the width \( w \) of the period, the electric field is unaltered, while for the remainder of the period the electric field experiences a \( \pi \) phase shift. Fourier analysis is used once again to determine the Fraunhofer diffraction pattern. The transmission function is now written as
Figure 14. Transmission function of a phase grating with a period \( d \).
The phase of the incident field is unaltered from 0 to \( w \), and experiences a \( \pi \) phase shift for the remainder of the period.

\[
 t(x) = \begin{cases} 
 1, & 0 < x < w \\
 -1, & w < x < d 
\end{cases}
\]

The Fourier coefficients \( c_n \) are again found using the integral

\[
 c_n = \frac{1}{d} \left( \int_0^w (1)e^{i\gamma x} dx + \int_w^d (-1)e^{i\gamma x} dx \right)
\]

Where again \( c_0 \) is evaluated first because otherwise the answer is undefined. Entering \( n = 0 \) results in a very simple integral of \( dx \) over the limits. The result is

\[
 c_0 = \frac{2w}{d} - 1,
\]

showing that the DC order depends on the ratio of \( w \) to \( d \), similar to the amplitude grating.

The integral for the remaining Fourier coefficients is then found.

\[
 c_n = \frac{1}{d} \left( \int_0^w e^{i\gamma x} dx - \int_w^d e^{i\gamma x} dx \right)
\]

\[
 = \frac{1}{i\gamma d} \left( \left[e^{i\gamma x}\right]^w_0 - \left[e^{i\gamma x}\right]^d_w \right)
\]

\[
 = \frac{1}{i2\pi n} \left[2e^{i2\pi \frac{w}{d}} - e^{i2\pi} - 1\right]
\]
The transmission function is a series of delta functions, which resembles the amplitude grating in Equation 25, but with different Fourier coefficients. The intensity of each delta function is again found by multiplying the coefficients by their complex conjugate,

\[ c_0 c_0^* = \left[ \frac{4w^2}{d^2} - \frac{4w}{d} + 1 \right] \]

and

\[ c_n c_n^* = \frac{1}{4\pi^2 n^2} \left( 2e^{in2\pi \frac{w}{d}} - e^{i2\pi} - 1 \right) \left( 2e^{-i2\pi \frac{w}{d}} - e^{-i2\pi} - 1 \right) \]
\[ = \frac{1}{4\pi^2 n^2} \left( 6 + e^{i2\pi} + e^{-i2\pi} - 2e^{i2\pi \frac{w}{d}} - 2e^{-i2\pi \frac{w}{d}} - 2e^{i2\pi \left( \frac{w}{d} - 1 \right)} - 2e^{-i2\pi \left( \frac{w}{d} - 1 \right)} \right) \]
\[ = \frac{1}{4\pi^2 n^2} \left[ 6 + 2\cos(2\pi n) - 4\cos \left( 2\pi n \frac{w}{d} \right) - 4\cos \left( 2\pi n \left( \frac{w}{d} - 1 \right) \right) \right] \]

The intensity profile can again be determined by plotting these for each order \( n \) from 0 to \( \pm n \), shown in Figure 15.

![Intensity profile of the diffracted orders at the Fourier plane for a phase grating with a w/d ratio of 0.5.](image)

Figure 15. Intensity profile of the diffracted orders at the Fourier plane for a phase grating with a w/d ratio of 0.5.

The amplitude and phase grating patterns can briefly be compared for the case where the w/d ratio is 1/2. In the phase grating, the DC order is non-existent, while it is most prominent in the amplitude grating. Meanwhile, the \( \pm 1 \) orders hold the majority of the initial intensity in the phase grating. We notice that both the amplitude grating and phase grating are
missing the even orders. It is also obvious that while the summation of the intensity of the amplitude grating approached $0.5I_0$, in the case of the phase grating the intensities quickly approach $I_0$, meaning all of the initial intensity is passed.

So far, both the amplitude and phase grating have produced symmetric diffraction patterns. This is mainly due to the symmetry of the gratings themselves. In the next section, an asymmetric grating is considered to examine the resulting diffraction pattern.

**Blazed Grating**

In this section we consider a very special type of grating called a blazed grating. This is a phase grating, but rather than a step-type grating, the phase increases linearly from $-\pi$ to $\pi$ over the period $d$. The result is a sawtooth shape, as seen in Figure 16.

![Figure 16. Transmission function of a blazed grating, where the phase increases linearly from $-\pi$ to $\pi$ over the period $d$.]

For this grating, the phase can be written as a function of $x$ such that $\phi = \frac{2\pi}{d}x$. This leads to the expression of the transmission function, $t(x)$, as

$$t(x) = e^{\frac{2\pi i x}{d}} \quad (27)$$

Following the same method for the Fourier series, we can determine the Fourier coefficients by the integral

$$c_n = \frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} e^{i2\pi \frac{x}{d}} e^{-in2\pi \frac{x}{d}} dx$$

This simplifies to the fairly simple integral
\[ c_n = \frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} e^{i2\pi \frac{x}{d}(1-n)} dx \]
\[ = \frac{1}{2\pi i (1-n)} \left[ e^{i2\pi \frac{x}{d}(1-n)} \right]_{-\frac{d}{2}}^{\frac{d}{2}} \]
\[ = \frac{1}{2\pi i (1-n)} \left[ e^{i\pi(1-n)} - e^{-i\pi(1-n)} \right] \]
\[ = \frac{1}{2i\pi (1-n)} \left[ 2i \sin(\pi(1-n)) \right] \]  

(28)

The solution is undefined when \( n = 1 \), so the integral will be re-evaluated for this point. Interestingly, we see that for all other values of \( n \), the solution is the sine of an integer value of \( \pi \), which is always 0. This leads to the conclusion that all \( c_n = 0 \) for \( n \neq 1 \). For \( n = 1 \), the Fourier coefficient is found to be

\[ c_1 = \frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} e^0 dx \]
\[ = \frac{1}{d} \left[ \frac{d}{2} + \frac{d}{2} \right] \]
\[ = 1 \]  

(29)

Therefore, the resulting Fraunhofer diffraction pattern at the observation plane is simply a delta function at the +1 order, carrying all of the initial intensity. We can see that if the grating were changed to decrease linearly from \( \pi \) to \(-\pi\), the intensity would instead all go to the -1 order. This is a very unique grating and is very important to the work in this thesis. The following chapters will further explore this grating to discover how it can be utilized for polarization control.
CHAPTER 5

POLARIZATION CONTROL OF THE DC ORDER

In this chapter, we will examine how the polarization of the DC diffracted order can be manipulated with great precision with a single pass through the LCD-SLM. A mixture of Jones calculus and Fourier analysis will be used to show how changing the phase depth and adding a phase shift to a $-\pi$ to $\pi$ blazed grating gives amplitude and phase control\textsuperscript{17} of the polarization state of the undiffracted order. Experimental results will confirm that the polarization can be changed from linear to elliptical, and the ellipticity and angle of linear polarization can be controlled as well.

MODIFIED BLAZED GRATINGS

In the previous chapter, a $-\pi$ to $\pi$ blazed grating was examined, where it was found that nearly all of the light incident on the grating is diffracted into the $+1$ order. Here, the grating is explored further by examining the same grating with an adjustable phase depth and incorporated phase shift. The phase depth refers to the overall height of the grating, and is denoted by a constant, $M$, multiplied by the original $2\pi$ grating. Therefore, $M$ can be any value between 0 and 1. The additional phase shift, $\Phi$, is added to the grating, changing the DC offset of the grating. Figure 17 illustrates the effects of $M$ and $\Phi$ on the blazed grating.

With these additional parameters, the transmission function of the grating can be rewritten as

$$t(x) = e^{i(Mx^2\pi + \Phi)}.$$  

Using Fourier analysis, the transmission function can again be rewritten as the sum of the Fourier coefficients multiplied by $e^{\frac{2\pi n x}{d}}$, where the Fourier coefficients are found using

$$c_n = \frac{1}{d} \int_{-d/2}^{d/2} t(x) e^{-i n x \frac{2\pi}{d}} dx.$$  

(30)
Figure 17. (a) A $-\pi$ to $+\pi$ blazed grating modified by (b) changing the phase depth as $M/2\pi$, and (c) adding a phase shift $\Phi$.

Notice that with the transmission function inserted in Equation 30, the term $e^{i\Phi}$ can be factored outside the integral since it does not depend on $x$. This leads to the realization that the phase shift will be added to all diffracted orders equally, and will have no effect on intensity because the intensity is simply the coefficient multiplied by its complex conjugate.

Working out the remainder of the integral, the Fourier coefficients are found to be

\[ c_0 = e^{i\Phi} \text{sinc}(M\pi) \quad \text{and} \quad c_1 = e^{i\Phi} \text{sinc} \left[ \pi(M - 1) \right], \quad (31) \]

where $\text{sinc}(x) = \sin(x)/x$. The transmission function is then rewritten as a Fourier series with two terms as

\[ t(x) = e^{i\Phi} \text{sinc}(M\pi) + e^{i\Phi} e^{ix\frac{2\pi}{d}} \text{sinc} \left[ \pi(M - 1) \right]. \]

Again, we see that the diffracted intensity is kept between the DC and +1 orders. These coefficients will now be used to determine how $M$ and $\Phi$ affect the intensity and polarization of these orders.
CHANGING THE PHASE DEPTH, $M$

As noted in the previous section, the phase shift $\Phi$ has no effect on the intensity for any diffracted orders in this grating. We notice however that the intensity of either order will vary as $M$ is varied. To determine the intensity, the coefficients are multiplied by their complex conjugates, where the leading exponential terms drop out for each order. This produces a $\text{sinc}^2$ function for each, given by

$$I_0 = \text{sinc}^2(M\pi) \quad \text{and} \quad I_1 = \text{sinc}^2(\pi(M-1))$$

(32)

These functions are undefined at $M = 0$ and $M = 1$ for the DC and 1st orders respectively, but L'Hopital's rule shows that the normalized intensity is 1 for each instance. Notice that the orders are directly opposed, where the DC order is a maximum when $M = 0$ and 0 at $M = 1$, while the diffracted order is exactly the opposite.

To verify these functions of intensity, a programmable blazed grating was encoded on the SLM, where the parameter $M$ could be changed. The phase shift $\Phi$ was ignored because, as mentioned earlier, it has no impact on intensity. In the experimental setup shown in Figure 18, collimated, vertically polarized light was passed through the LCD, at which point a 2-f optical Fourier system was used to examine the diffraction pattern on a linear diode array photodetector. The value of $M$ was changed in steps of 0.1 from 0 to 1, and the intensities of the 0 and +1 orders were measured.

Figure 18. The setup for measuring the intensities of the DC and +1 diffracted order as the phase depth, $M$, is changed on a blazed grating.
Figure 19 clearly shows that as $M$ is increased from 0 to 1, the DC order slowly loses intensity while the +1 order gains intensity. Though some of the initial intensity is lost, the transfer of intensity between the 0 and +1 orders is seen between images $a$ and $f$, where $M = 0$ and 1 respectively. When $M = 0.5$, the intensities of the two orders are nearly equal.

![Image](attachment:figure19.png)

**Figure 19.** The intensities of the 0 and +1 orders for $M =$ (a) 0, (b) 0.2, (c) 0.3, (d) 0.5, (e) 0.7, and (f) 1.0.

The normalized measured intensities of the 0 and +1 orders were then plotted versus $M$, along with the theoretical curve as seen in Figure 20. It should be noted that the obtained experimental intensities of the +1 order fall below the theoretical curve for nearly every point. This is a consequence of the pixelation of the transmission function due to the LCD. The theoretical transmission function is a smooth, increasing linear transition from $-M\pi$ to $+M\pi$. In the programming of the LCD however, the period of the transmission function is divided into a number of pixels, which is 32 in this case. This means the smooth, linearly increasing period in Figure 17 is now represented by a series of 32 steps. Obviously, a better representation would occur if the transmission function were programmed onto a larger number of pixels. The drawback of this, however, is that as the period of the grating becomes
Figure 20. Theoretical curves and experimental data points for the normalized intensity of the 0 and +1 orders as a function of $M$.

larger, the spacing between the diffracted orders decreases. A balance must be found where the grating period is large enough to encode a decent representation of the transmission function, but small enough that the diffracted orders may be resolved in the Fourier plane. From the data, a 32 pixel period appears to meet these requirements.

With the Fourier coefficients from Equation 31 and the experimental data in Figure 20, the amount of intensity divided between the 0 and +1 orders in a $M\pi$ grating can accurately be predicted as a function of $M$. Our attention can now be shifted to the other modification of the blazed grating, which is the phase bias, $\Phi$.

**Changing the Phase Bias, $\Phi$**

In the last section, it was noted that adding a phase bias to a blazed grating will add the phase to all of the orders of the diffraction pattern, but will have no effect on the intensity. Referring back to the description of the SLM, recall that only the vertically polarized component of the incident beam is affected by the phase difference of the liquid crystals, while any horizontally polarized light will experience no phase shift and will go straight to the DC order. In previous experiments, the input beam has been vertically polarized, so all of the incident light is subject to diffraction. Normally this is the preferred setup, however we will
examine the result if the incident light is polarized at an angle $\theta$ with respect to the vertical axis.

Figure 21 shows a representation of this, and shows that the electric field, $E_0$, can be written in terms of its horizontal and vertical components, $E_{0x}$ and $E_{0y}$, respectively.

\[ E_{0x} = E_0 \sin(\theta_{in}) \quad \text{and} \quad E_{0y} = E_0 \cos(\theta_{in}). \]

The components are easily found as

The incident electric field can be rewritten with Jones matrices as

\[ \vec{E}_0 = \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} = E_0 \begin{pmatrix} \sin(\theta_{in}) \\ \cos(\theta_{in}) \end{pmatrix}. \] (33)

The transmission function of the blazed grating with an adjustable phase depth, $M$, only has Fourier coefficients $c_0$ and $c_1$. Using the expression for these from Equation 31, the transmission function is written in Jones matrix form as

\[ t(x) = \begin{pmatrix} 1 & 0 \\ 0 & c_0 + c_1 e^{ix \frac{2\pi}{\lambda}} \end{pmatrix}. \]

The input electric field is multiplied by the transmission function of the SLM, resulting in
\[ t'(x) = t(x) \vec{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & c_0 + c_1 e^{i \frac{2\pi}{d}} \end{pmatrix} \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} = E_0 \begin{pmatrix} \sin(\theta_{in}) \\ c_0 \cos(\theta_{in}) + c_1 e^{i \frac{2\pi}{d}} \cos(\theta_{in}) \end{pmatrix} \] (34)

The matrix in Equation 34 can now be split into the DC and +1 order components, noting that the \(x\) component is undiffracted by the SLM, so it is directly carried to the DC order. The resulting transmission function is written as

\[ t'(x) = E_0 \begin{pmatrix} \sin(\theta_{in}) \\ c_0 \cos(\theta_{in}) \end{pmatrix} + E_0 \begin{pmatrix} 0 \\ c_1 e^{i \frac{2\pi}{d}} \cos(\theta_{in}) \end{pmatrix} . \]

For a normalized intensity, the \(E_0\) term can be ignored, and substituting \(c_0\) and \(c_1\) leaves

\[ t'(x) = \begin{pmatrix} \sin(\theta_{in}) \\ e^{i \Phi} \sin(M\pi) \cos(\theta_{in}) \end{pmatrix} + \begin{pmatrix} 0 \\ e^{i \Phi} e^{i \frac{2\pi}{d}} \sin(\pi(M-1)) \cos(\theta_{in}) \end{pmatrix} . \] (35)

Taking the Fourier transform of Equation 35 provides the electric field at the Fourier plane when using a 2-f Fourier transform lens system. This results in a similar looking equation, where each term is now multiplied by a delta function. In Equation 36, we see that the first matrix has a delta function at \(p = 0\), meaning it represents the DC term, and the second term occurs at \(p = \gamma = 2\pi/d\), representing the +1 order.

\[ T'(p) = \begin{pmatrix} \sin(\theta_{in}) \\ e^{i \Phi} \cos(\theta_{in}) \sin(M\pi) \end{pmatrix} \delta(p) + \begin{pmatrix} 0 \\ e^{i \Phi} \cos(\theta_{in}) \sin(\pi(M-1)) \end{pmatrix} \delta(p - \gamma) \] (36)

This result verifies that the +1 diffracted order will always be vertically polarized, regardless of the value of \(M\) and \(\Phi\). The DC term on the other hand is much more interesting. The existence of an \(x\) and \(y\) component of the electric field tells us that the polarization orientation can be controlled as a function of the parameters \(\theta_{in}, M,\) and \(\Phi\). Also, with the phase term \(e^{i \Phi}\) in the \(y\) component, the phase of the vertical component of the electric field can be shifted with respect to the horizontal component. This provides control over the state...
of the polarization. Each of these elements of polarization control over the DC order will be explored in the following sections.

**Changing the Polarization State of the DC Order**

Equation 36 provides the resulting $x$ and $y$ components of the DC and +1 diffracted order for a modified blazed grating. Because the +1 diffracted order is always vertically polarized, it is not of interest because all that can be controlled is intensity. For the DC order then, the output electric field as a function of phase depth, $M$, phase bias, $\Phi$, and input angle, $\theta_{\text{in}}$ is

$$E_{\text{DC}} = \begin{pmatrix} \sin(\theta_{\text{in}}) \\ e^{i\Phi} \cos(\theta_{\text{in}}) \text{sinc}(M\pi) \end{pmatrix} \quad (37)$$

Due to the $e^{i\Phi}$ term in the vertical component of the electric field of the DC order, the relative phase of the vertical component of the electric field can be varied with respect to the horizontal component. This provides the ability to change the polarization state from linear to elliptical or elliptical to linear, as seen in the *Polarization States of Electromagnetic Radiation* section of Chapter 2. For a special case where the $x$ and $y$ components of the electric field are equal, circularly polarized light can be created.

To experimentally demonstrate the effects of changing the phase bias $\Phi$, the input beam is linearly polarized at an angle $\theta_{\text{in}} = 22.5^\circ$ with respect to the vertical. The LCD is programmed with a blazed diffraction grating, where the grating is split into quadrants, as seen in Figure 22. The grating is programmed such that the top left quadrant has no additional phase bias, the upper right has a phase bias of $\pi$, and the lower sections have phase biases of $\pi/2$ and $3\pi/2$. The phase depth, $M$, is set to a constant value for all quadrants. The creation of circularly polarized light requires that the amplitude of the $x$ component of the electric field be equal to the amplitude of the $y$ component. Referring to Equation 37, this requires

$$\sin(\theta_{\text{in}}) = \cos(\theta_{\text{in}}) \frac{\sin(M\pi)}{M\pi}$$

This is satisfied when $M \approx 0.65$, and therefore the phase of the grating programmed on the LCD will run from $-1.3\pi$ to $+1.3\pi$. 


Figure 22. Grating encoded on the SLM where the quadrants have an added phase bias of (clock-wise from upper-left) 0, $\pi$, $3\pi/2$, and $\pi/2$.

We expect the DC order of the output to have two linearly polarized areas and two circularly polarized areas. Because the $x$ and $y$ components of the DC order electric field will have equal amplitude after passing through the SLM, the linearly polarized sections will have polarizations of $\pm 45^\circ$.

Figure 23 shows the setup for observing the polarization of the DC order for the different applied phase bias, $\Phi$.

![Figure 23. Setup for imaging the DC order to observe the effects of a phase bias on the polarization.](image)

In the experimental setup, laser light is linearly polarized at 22.5°, and after passing through the SLM, the optical Fourier transform of the diffracted light is taken. At the Fourier
plane in the experimental setup, the diffracted orders are blocked, while the DC order is passed. Another lens takes the inverse Fourier transform of the order, which allows for imaging on a CCD detector. Before the detector, another polarizer is used as an analyzer to determine the polarization of the output beam.

Figure 24 shows the experimental images obtained for the different analyzers, proving that each quadrant does hold a distinct polarization.

<table>
<thead>
<tr>
<th>Analyzer</th>
<th>Theoretical</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Analyzer</td>
<td><img src="image" alt="No Analyzer" /></td>
<td><img src="image" alt="Experimental" /></td>
</tr>
<tr>
<td>(a) LP @ +45°</td>
<td><img src="image" alt="Theoretical" /></td>
<td><img src="image" alt="Experimental" /></td>
</tr>
<tr>
<td>(b) LP @ -45°</td>
<td><img src="image" alt="Theoretical" /></td>
<td><img src="image" alt="Experimental" /></td>
</tr>
<tr>
<td>(c) LC-Analyzer</td>
<td><img src="image" alt="Theoretical" /></td>
<td><img src="image" alt="Experimental" /></td>
</tr>
<tr>
<td>(d) RC-Analyzer</td>
<td><img src="image" alt="Theoretical" /></td>
<td><img src="image" alt="Experimental" /></td>
</tr>
</tbody>
</table>

**Figure 24.** Experimental results showing the image of the DC order when viewed through (a) no analyzer, (b) a linear analyzer at +45°, (c) a linear analyzer at -45°, (d) a left-circular analyzer, and (e) a right-circular analyzer. The black sections in the theoretical column represent the areas expected to be blocked by the analyzer.

In Figure 24, the center column shows the expected blocked sections at each analyzer angle, with the exception of the “No Analyzer” row, where the programmed polarization states of each section are shown. The theoretical column is compared to the experimentally
observed results in the far right column. We notice that with no analyzers, the intensity of the beam is fairly uniform, confirming that the added phase biases have no impact on the beam intensity. Also, the polarized sections that match the analyzer used have the greatest intensity, as expected.

We point out that we expect two sections to have equal intensity for each analyzer. For example, with the linear analyzer at $+45^\circ$, we expect the intensity of each circularly polarized section to be reduced by an equal amount. The results consistently show that one section has slightly less intensity than the other, which may indicate that the phase bias or phase shift for each section may need some adjustment. If these parameters are not precise, the polarizations may be slightly elliptical in all cases, which would cause some intensity variation with different analyzers.

This experiment shows that for a known phase depth $2\pi M$, the state of the polarization can be well controlled to produce linearly polarized or elliptically polarized light. We showed that with calculations, a value of $M$ can be found that produces equal electric field component amplitudes for any angle of linearly polarized input light. In the next section the linear polarization control capabilities with the use of the same parameters, $M$, $\theta_{in}$, and $\Phi$ will be explored.

**Linear Polarization Control of the DC-Order**

Once again, we look back at the electric field of the DC order for a modified blazed grating.

$$E_{DC} = \binom{\sin(\theta_{in})}{e^{i\Phi} \cos(\theta_{in}) \text{sinc}(M\pi)}$$ (38)

In the past section it was shown that for light of any input polarization angle incident on a modified blazed grating, the output polarization of the DC order will be linear if the phase bias of the grating is $\Phi = 0$ or $\Phi = \pi$. At these values, the term $e^{i\Phi}$ will be 1 or -1, respectively. The horizontal component of the input electric field passes through the SLM undisturbed and becomes the horizontal component of the DC order. The magnitude of the vertical component of the DC order is now primarily dependent on the phase depth value $M$. When $M = 0$, all of the vertical component will go the DC, meaning the DC will have the
same polarization as the input light. On the other hand, when \( M = 1 \), all of the vertically polarized component will be diffracted to the +1 order. This will leave the DC with only a horizontal electric field component, and therefore horizontally polarized.

This shows that the linear polarization can be changed to any value between \( 90^\circ \) and \( \theta_{in} \) and by adding a phase bias of \( \pi \), we can achieve polarizations between \(-90^\circ\) and \(-\theta_{in}\).

From Equation 37, we found the horizontal and vertical electric field components of the DC order were found as a function of the polarization angle of the input beam \( \theta_{in} \), the phase depth multiplier \( M \), and the phase bias \( \Phi \). Again, with a \( \Phi \) of 0 or \( \pi \), the phase bias term is simply either \( \pm 1 \). We can now write the output polarization angle as the arctangent of the \( x \) component of the electric field divided by the \( y \) component. The result is

\[
\theta_{out} = \arctan \left( \frac{M \pi \sin(\theta_{in})}{\cos(\theta_{in}) \sin(M \pi)} \right)
\]  

(39)

Before performing experiments, we can see how the output polarization angle behaves for different input values. The theoretical plots in Figure 25 were created from Equation 39, and show \( \theta_{out} \) plotted from \( M = 0 \) to \( M = 1 \) for \( \theta_{in} \) values of \( 45^\circ \), \( 22.5^\circ \), and \( 5^\circ \).

![Figure 25. Theoretical output polarization angles versus \( M \) value for \( \theta_{in} \) of \( 45^\circ \), \( 22.5^\circ \), and \( 5^\circ \) with respect to vertical.](image)

Figure 25 shows one of the drawbacks of creating polarized light using a modified blazed grating. Decreasing the value of \( \theta_{in} \) gives a larger range of possible output polarization
angles, which is beneficial. However, as the input polarization angle decreases, the range of control with the value of $M$ is constricted to a much smaller range. For example, with the 5° input polarization, $M$ is essentially non-effective from 0 to 0.5, at which point the output polarization angle becomes increasingly sensitive, and even a very small change in $M$ results in a large change in the output polarization angle.

To confirm the theoretical control of the DC linear polarization angle given by Equation 39, the setup in Figure 26 is utilized. The modified grating is programmed across the entire LCD rather than splitting the grating into sections, and the detector is simply placed at the Fourier plane in the 2-f system. An analyzer is placed between the LCD and the Fourier plane, which may be rotated to determine the polarization angle of the DC order. With the light for the orders in a concentrated area as spots, it is much easier to determine when the intensity is at a minimum as the analyzer is rotated.

The input beam was linearly polarized at 10° with respect to the vertical. We began with the phase bias at 0 for the blazed grating, and the value of $M$ was changed in 0.1 increments from 0 to 1. A linear polarizer in a rotation mount was placed between the Fourier lens and the detector, and for each value of $M$, the analyzer was rotated until the intensity of the DC order was at a minimum as observed on the detector. The angle of the analyzer was recorded and the polarization angle of the output beam is simply at 90° to the analyzer. After recording the polarization angle for all values of $M$ with a phase bias of 0, a phase bias of $\pi$
was added to the grating, and the angles were again recorded. The experimental results are shown in Figure 27, plotted against the theoretical curves determined by Equation 39.

Figure 27. Experimental results of linear polarization rotation of the DC order by changing $M$ for $10^\circ$ linearly polarized input light. The theoretical curves are plotted along with experimental data points.

The experimental results in Figure 27 show very close agreement to the theoretical output polarization angles as $M$ is changed. We see that any polarization angle between $\theta_{in}$ and $(360^\circ - \theta_{in})$ can be achieved by adding a phase bias of $\pi$ to the modified grating. After seeing how well the polarization of the DC order can be controlled, we can now construct more complicated gratings to create a beam with a much more complicated polarization profile.

**Radially Polarized Beam**

Beam polarization has become very important in many laser applications, but some require more complex polarizations. Some examples include radially and azimuthally polarized beams, which have applications in laser particle accelerators and particle trapping. In a radially polarized beam, the beam has multiple linear polarizations along the
radius vector of the beam. An azimuthally polarized beam has multiple polarizations that are perpendicular to the radius vector of the beam. These are illustrated in Figure 28.

![Figure 28. (a) Radially and (b) azimuthally polarized beam cross-sections. At each section the beam is linearly polarized to point either along or perpendicular, respectively, to the radius vector.](image)

From the previous experimental results, it was shown that the angle of linear polarization on the DC order can be controlled by changing the phase depth, $M$, with $\Phi = \pm \pi$ in the equation

$$E_{DC} = \begin{pmatrix} \sin(\theta_{in}) \\ e^{i\phi} \cos(\theta_{in}) \text{sinc}(M\pi) \end{pmatrix}$$

With this capability, a grating can be split into sections where each section has a different phase depth, and therefore a different polarization. For this experiment, a grating was created that split a circle into 8 sections. Each section can be considered to have a radius vector of $\pm 22.5^\circ$ and $\pm 67.5^\circ$ with respect to the vertical, as seen in Figure 29.

In Figure 29, the grating is programmed for $22.5^\circ$ input light, for reasons which will be explained shortly. To create sections of $22.5^\circ$ and $67.5^\circ$ polarization, the $M$ value of the gratings are found to be 0 and 0.85, respectively. Since the $M$ value is 0 for the $22.5^\circ$ sections, the grating depth is 0 so there is no grating, just a uniform gray level for phase bias. To create positive or negative polarizations, we take advantage of the phase bias value, $\Phi$, where $\Phi = 0$ creates positive polarization and $\Phi = \pi$ creates negative polarization.
One important factor to note about Figure 29 is that it is not a “pure” radial polarizer. Note how half of the sections point radially inward while the other half point radially outward. This is due to the instantaneous polarization of the incident beam. Referring to Figure 2 in Chapter 2, the resultant electric field of the incident beam oscillates, meaning the instantaneous polarization angle also oscillates. For example, a +45° linearly polarized beam spends half of the time at +45°, and another half of the time at 180° from this, or 225°. In a pure radial polarizer, all sections would point either radially inward or outward simultaneously. Because the single pass system only provides the capability to apply a phase shift to the \( y \) component and not both components, a pure radial polarizer cannot be created with a single pass.

To test this grating, the 4-\( f \) Fourier transform system in Figure 23 was again used to image the DC order on the detector. The analyzer was set to angles of \( \pm 22.5° \) and \( \pm 67.5° \) and images were taken at each setting. The results are shown in Figure 30, where the top row shows results with no analyzer. The theoretical drawing for the “No Analyzer” image shows the polarization angles and relative magnitudes for each section. Some image saturation was necessary to capture weaker portions of the beam, which will be explained shortly.

From these results we can see that the polarization of each section is indeed along the radius vector. For each analyzer setting there are two sections that are polarized perpendicular to the analyzer transmission axis, and therefore have no intensity. As the analyzer is rotated, the blocked sections rotate with it.
Figure 30. Experimental results of the imaged, radially polarized DC order as viewed through (a) no analyzer, and an analyzer at (b) -22.5°, (c) +22.5°, (d) +67.5°, and (e) -67.5° with respect to vertical. The black sections in the theoretical column represent the sections that are expected to be blocked by the analyzer.

Figure 30 shows another drawback of using the single pass method for polarization control. In subset a, with no analyzer in front of the detector, it is clear that the ±22.5° sections have more intensity than the ±67.5° sections. Recall that the polarization angle is changed by changing the phase depth of the blazed grating, which adjusts the amount of the vertical component of the electric field that is sent to the diffracted orders. Because the 67.5° sections have a polarization closer to horizontal, they have a weaker vertical component of the electric field, which results in a lower intensity. Figure 31 illustrates this effect by plotting the theoretical normalized intensity of the output beam versus the programmed output polarization angle for a 22.5° linearly polarized input beam.
Figure 31. A plot showing the theoretical normalized intensity of the output beam as the angle of polarization is changed for a 22.5° input beam.

Figure 31 shows that the intensity drops quickly as the output polarization angle is programmed closer to horizontal. For a smaller $\theta_{in}$ like 10°, the horizontal component is even weaker, and we would see the intensity drop faster, and reach a lower minimum as well. This is the reason 22.5° input light was used. This maximizes the strength of the horizontal component, which allows for the weaker sections to be more visible. If the input is only 10°, the same polarizations can be created, but the horizontal component would be much weaker and therefore much more difficult to see on the images.

**Rotating Polarization in a Sectioned Beam**

We can continue to play with the sectioned DC order of the diffracted beam and create a more complex polarized beam. Here, the beam will be split into a total of 18 sections, and rather than having polarizations directed along the radius vector as in the previous section, we will take a more complicated approach and rotate the linear polarization to complete a nearly 180° rotation over 5 sections in increments of 31.5°. This is more clearly illustrated in Figure 32.

In Figure 32 the limitations of the single pass are again realized as the nature of the setup inhibits the ability to have complete control of the rotation and achieve a complete 360° rotation. Each section of the left half of the beam has a $\pi$ phase shift in the vertical component in comparison to the corresponding mirrored section on the right half, for the sake of making
Figure 32. An 18-section beam in which each section is linearly polarized in a manner where the angle of polarization rotates 31.5° between sections.

the optical element more interesting. Again, the polarization angles will either be as shown or exactly 180° opposites depending on the instantaneous polarization angle of the incident 22.5° linearly polarized beam. Excel Solver was used to determine the $M$ values required to create the desired polarization angles for each section. Figure 33 shows the optical element displayed on the LCD.

Figure 33. The 18-section grating displayed on the LCD to create the polarization profile illustrated in Figure 32.
The optical element was created and displayed on the LCD, and we used the same setup shown in Figure 23. Once again, images were taken at different analyzer angles to verify the angle of polarization of each section. Figure 34 first shows the imaged DC order with no analyzer, and with an analyzer where the transmission axis is at 90° with respect to vertical. For no analyzer, the theoretical drawing shows the expected intensity variation as a result of the decrease of the magnitude of the vertical electric field component, as shown in the plot in Figure 31. The uniform intensity with the horizontal analyzer shows that all sections carry a horizontal component of equal magnitude.

![Table and images](image)

**Figure 34. DC-Order of the 18-Section grating viewed through (a) no analyzer, and (b) analyzer at 90° with respect to vertical.**

The results for the remaining images taken with different analyzer angles follow in Figure 35. It is important to note that the intensity is adjusted between images for these results. Contrary to the results in Figure 30 where the weaker intensity at polarization angles close to horizontal was emphasized, in these results we emphasize the actual polarization angle of each section. Some image saturation was necessary to make the weaker sections more visible, which makes it easier to discern when they are blocked at points where the analyzer is perpendicular to that section’s polarization.

As the analyzer is rotated in Figure 35, the blocked sections can be seen to transition from the left side to the right side of the circle. The center theoretical column shows the expected blocked sections at each analyzer angle in black. Some of the sections appear blocked due to the fact that their polarization angle is so close to the target polarization angle...
Figure 35. DC-Order of the 18-Section grating viewed through an analyzer at angles of (a) -67.5°, (b) -36°, (c) -4.5°, (d) +27°, (e) +58.5°, (f) -58.5°, (g) -27°, (h) +4.5°, (i) +36°, and (j) +67.5° with respect to vertical. Black sections represent the area expected to be completely blocked by the analyzer and gray sections represent areas that are expected to be mostly blocked due to a programmed polarization angle that is within 10° of the black sections.

for that image. These sections are shaded gray in the theoretical column to help distinguish them from the targeted black sections.

**SUMMARY**

In this chapter, the blazed grating was explored in greater detail to reveal that two parameters, the phase bias and grating depth, can be incorporated into the blazed grating. With these parameters, it was shown that changing the grating depth parameter determines how the incident beam intensity is diffracted and divided into the DC and +1 diffracted orders.
The phase bias parameter allows for control of the overall phase shift imparted on the vertical component of the electric field, with no impact on the intensity.

Jones calculus was used to find the Jones matrix for the electric fields of the DC and +1 order as a function of these two parameters and the angle of the linearly polarized input beam with respect to vertical. Using this matrix, it was determined that these parameters provide a great amount of control over the polarization of the DC order. It was experimentally determined that the polarization state of the DC order could be made to be linear, elliptical, or circular. The linear polarization angle can also be adjusted to any angle between horizontally polarized and the polarization angle of the input beam.

With these results, two complex gratings were created where the beam was split into 8 and 18 sections. These experiments verified that not only could a DC order be created which is divided into sections of different polarizations, but that the polarization of the sections can be very well controlled to create beams with complex polarizations.

The single pass system showed the limitations of the polarization control as well. Obviously, only the DC order has good control while the diffracted 1st order is only vertically polarized. The DC order linear polarization is limited by the angle of the input beam’s linear polarization, and due to the instantaneous state of the input beam, absolute 360° control of the DC order linear polarization cannot be attained. Finally, the closer the DC order polarization is to horizontally polarized, the weaker the intensity, which creates large intensity variations for the sectioned beams.
CHAPTER 6

CONCLUSION

In conclusion, the experimental evidence shows the polarization control capabilities of the DC order for a single pass through a LCD-SLM. First, it was shown that the amplitude of the vertical electric field component of the DC order can be controlled with a modified blazed grating by changing the phase depth, $M$. Changing the phase bias, $\Phi$, provided control over the phase of the vertical component, and we showed that the polarization state of the DC order could be changed to be linearly or elliptically polarized. With the combination of these two parameters, the polarization angle of the DC order was precisely controlled. Finally, the sectioned LCD was used to successfully create a quasi-radially polarized beam, as well as a complex 18-section beam where the polarization angle rotated nearly $180^\circ$ over 5 sections.

We began with the wave equation, where the origins of the index of refraction and polarization were introduced. The various polarization states were covered, and the concepts of birefringent materials showed how the polarization can be altered due to a phase shift between the electric field components. A treatment of Jones matrices provided a useful tool for calculating the resulting electric field, polarization, and intensity of a beam passing through any complex optical system.

We described the LCD-SLM to provide an understanding of how the applied voltage to a pixel in the display rotates the birefringent liquid crystals, providing a means to change the phase shift imparted on an incident electric field. The LCD was calibrated to produce a curve that correctly described the phase shift as a function of gray level on the LCD.

Diffraction theory was used to describe the optical Fourier transform and imaging systems used in the experiments, where a 2-$f$ system allowed for observation of the Fraunhofer diffraction pattern, and a 4-$f$ system allowed for spatial filtering at the Fourier plane and imaging of the LCD transmission function. We used this setup to image the DC order alone and analyze the polarizations for various experiments. Fourier analysis showed
how the amplitude and phase of the vertical component of the diffracted electric field could be determined.

A theoretical treatment of the modified blazed grating was used to carefully determine the electric field components of the DC and first diffracted orders for a modified blazed grating. We found that the first diffracted order is always vertically polarized, while the DC order polarization can be changed by varying the parameters of $\theta$, $\Phi$, and $M$. Experimentally, we showed that the results match the expectations, and the linear polarization angle and polarization state of the DC order can be controlled.

For our last experiments, we combined our previous results to create a quasi-radially polarized beam and a very complex beam with a rotating polarization behavior. The predicted sections were shown to have the desired polarizations, which displayed the amount of control, programmability, and adaptability the LCD-SLM has with just a one way pass.

Analysis of the system shows that the single pass polarization control using a diffractive optical element has several drawbacks. These include a large variation of intensity based on the output linear polarization angle, less than 360° control over the polarization angle, and the lack of ability to produce pure radially polarized beams. Despite these drawbacks, we can see that the control is quite remarkable.

In a two pass system, it could be possible to control the polarization of not just the DC order, but the diffracted orders as well. The dual pass could also allow for intensity control, and a pure radially polarized beam could also be created. Other diffraction gratings could be encoded and treated like the blazed grating to produce multiple diffracted orders where perhaps each order could have a different polarization. Ultimately, there is plenty more research to be done based on these experiments, and the results could lead to a highly adaptable system with precise polarization control of multiple orders, including the more complex beams like pure radially and azimuthally polarized beams.
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APPENDIX

DIFFRACTION THEORY
DIFFRACTION THEORY

Diffraction can be examined either in the near-field (Fresnel Diffraction) or the far-field (Fraunhofer Diffraction). This chapter will recite key concepts and principles for the experiments of this thesis. The theory behind diffraction will show that the diffraction pattern of a given grating may be determined by taking the Fourier transform of the transmission function of the grating.

SCALAR DIFFRACTION

To begin, consider a planar electromagnetic wave propagating along $z$ and incident on a diffracting aperture at a plane in space, as seen in Figure 36. The aperture consists of a distribution of amplitude and phase, $\vec{E}_i(r_1)$, which will alter the amplitude and phase of the incident wave at each point $r_1$ in the plane. The amplitude and phase of the resulting electric field, $\vec{E}(r_2)$, can then be considered at each point $r_2$ in an observation plane further along the propagation path $z$.

![Figure 36. A representation of a diffraction pattern at the observation plane as a result of a plane wave $\vec{E}_i$ passing through a diffracting aperture.](image)

The amplitude and phase from each point at the aperture plane is different at each point in the observation plane. Therefore, the resulting electric field can be determined by individually adding the phase and amplitude from each aperture point at every point in the
observation plane. The waves leaving the aperture are spherically diverging (represented by $e^{-ik\vec{r}}$) with a wavelength $\lambda$, where $k = 2\pi/\lambda$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$. This is written as the integral

$$\vec{E}(x_2, y_2) = C \int \frac{\vec{E}_i(x_1, y_1)e^{-ik\vec{r}}}{\vec{r}} d\sigma,$$  \hspace{1cm} (40)

where $C$ is a constant and $d\sigma$ is an infinitesimal area in the aperture plane. The aperture can be redefined as a transmission function of the form

$$t(x_1, y_1) = |t(x_1, y_1)|e^{i\phi(x_1, y_1)}$$ \hspace{1cm} (41)

In Equation 41, the portion in the vertical brackets modifies the amplitude of $E_i(x_1, y_1)$ while the exponential modifies the phase at that point. The transmission is defined as 0 outside of the aperture, which allows for the insertion of the transmission function in Eq.40 and for the integration limits to be taken from $-\infty$ to $+\infty$, producing

$$E(x_2, y_2) = C \int_{-\infty}^{\infty} \frac{E_i(x_1, y_1)t(x_1, y_1)e^{-ik\vec{r}}}{r} d\sigma.$$ \hspace{1cm} (42)

**Fresnel Diffraction**

With Eq.42 we can now investigate the electric field at the observation plane. In Cartesian coordinates, the vectors $\vec{r}_1$ and $\vec{r}_2$ can be written as $\vec{r}_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $\vec{r}_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$. Therefore, the vector $\vec{r} = \vec{r}_2 - \vec{r}_1$ can be written as

$$\vec{r} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2}.$$  \hspace{1cm} (43)

Note that $z$ is the same for all points between the aperture and observation planes, and can be factored from inside the square root, leaving

$$\vec{r} = z\sqrt{1 + \frac{(x_2 - x_1)^2}{z^2} + \frac{(y_2 - y_1)^2}{z^2}}.$$ \hspace{1cm} (43)

The fractions inside the square root of Eq.43 can be momentarily represented by the symbol $\delta$, and the if the square root is expanded using a binomial approximation, we find

$$\vec{r} = z \left[ 1 + \frac{1}{2}\delta - \frac{1}{4 \cdot 2!}\delta^2 + \frac{3}{8 \cdot 3!}\delta^3 + \ldots \right].$$ \hspace{1cm} (44)
The vector $\vec{r}$ can be approximated by neglecting terms higher than the second-order. This means the third term in Eq.44 should be considerably less than one wavelength ($\lambda$) so as not to drastically contribute to the phase shift in the exponential term of Eq.42. Therefore,

$$\frac{z}{4 \cdot 2!} \left[ \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{z^2} \right]^2 \ll \lambda$$

and after simplifying,

$$\left[ (\Delta x)^2 + (\Delta y)^2 \right]^2 \ll 8\lambda z^3. \quad (45)$$

Equation 45 is known as the Fresnel limit. Given the physical parameters of the aperture and the wavelength of light passing through the aperture, the minimum distance $z$ can be determined, at which point the second-order and higher terms can be neglected. If the observation plane is placed at a distance less than this calculated $z$, the Fresnel approximation is no longer valid. The approximation is not necessary for the $\vec{r}$ term in the denominator of Equation 42 because the higher order terms are only on the order of wavelengths, which is a small contribution compared to the large distance of $z$ between the aperture and observation planes. In the Fresnel limit, the first two terms of Equation 44 can now be substituted into Equation 42 to find the approximation of the electric field at the observation plane.

$$E(x_2, y_2) = C \int_{-\infty}^{\infty} E_i t(x_1, y_1) e^{-ik\left(\frac{1+(x_1-x_2)^2+(y_1-y_2)^2}{2z}\right)} dx_1 dy_1 \quad (46)$$

Equation 46 describes the amplitude and phase of the electric field at an observation plane placed after the Fresnel limit, and is referred to as the Fresnel diffraction equation. While greatly simplified from the form in Equation 42, the equation is still difficult to solve. In the next section, further steps will be taken to find the resulting electric field if the distance $z$ from the aperture to the observation plane becomes even larger, entering what is known as the Fraunhofer limit.
**Fraunhofer Diffraction**

In order to further simplify the Fresnel diffraction equation, we examine what happens as the distance between the aperture plane and observation is increased. First, we look at the two exponential terms in Equation 46 with a $2z$ in the denominator. As $z$ is increased, the denominator will eventually become much larger than the numerator. The limit of these two exponential terms as $z$ approaches infinity is 1. We can now direct our attention to the final exponential term inside the integral,

$$e^{i(k \frac{x_1^2 + y_1^2}{z})}$$

The quantities $x_2/z$ and $y_2/z$ in the exponential are the components of the angles with respect to the $z$ axis at which the electric field is diffracted from the aperture plane. This is seen geometrically in Figure 37.

![Figure 37. The tangent of angles $\theta_x$ and $\theta_y$ are represented by $x_2/z$ and $y_2/z$.](image)

Recalling that $k = 2\pi/\lambda$, the spatial frequencies $p$ and $q$ can be defined as

$$p = k \frac{x_2}{z} = \frac{2\pi}{\lambda} \tan \theta_x$$
$$q = k \frac{y_2}{z} = \frac{2\pi}{\lambda} \tan \theta_y$$

Spatial frequencies $p$ and $q$ can now be substituted into Equation 46. The two exponential terms are dropped due to the limit as $z$ approaches infinity as discussed earlier, and the constants can all be combined to create $C_1$. The resulting equation, now depending on $p$ and $q$, is
\[ E(p, q) = C_1 \int_{-\infty}^{\infty} t(x_1, y_1) e^{i(px_1 + qy_1)} dx_1 dy_1. \] (48)

Examining Equation 48 more closely, the resulting electric field at the observation plane in the Fraunhofer limit is actually the two-dimensional Fourier transform of the transmission function at the aperture plane. This concept is used extensively in this thesis to examine different transmission functions and determine the resulting electric fields in the Fraunhofer limit.

With the near-field (Fresnel) and far-field (Fraunhofer) limit diffraction equations, the distances required to make the approximations of the Fraunhofer diffraction equation valid can be examined through the condition

\[ \frac{x^2 + y^2}{2z} \ll \lambda. \]

Compared to the Fresnel limit requirement in Equation 45, the Fraunhofer limit requires a considerably longer propagation along \( z \) for the approximations to be valid. In the next section, we will examine methods to reach the Fraunhofer region over a much shorter propagation distance with the use of a Fourier transform lens system.

**Fourier Transform Lens Systems**

As stated in the previous section, the Fraunhofer diffraction pattern of an electric field through a specific transmission function can be easily calculated as the Fourier transform of the transmission function. The disadvantage seen is that long propagation distances may be required to achieve Fraunhofer diffraction, especially as the size of the transmission function aperture increases. For experimental purposes, it would be convenient if the far-field pattern could be observed over a shorter distance.

Referring back to the terms in Equation 46, the terms

\[ e^{-ik\left(\frac{x_2^2 + y_2^2}{2z}\right)} \quad \text{and} \quad e^{-ik\left(\frac{x_1^2 + y_1^2}{2z}\right)} \]

can be recognized as spherically diverging wavefronts. In the previous section, these terms were eliminated by using a large distance \( z \). Another way to eliminate these terms would be to multiply them by exponentials with opposite signs,

\[ e^{ik\left(\frac{x_2^2 + y_2^2}{2z}\right)} \quad \text{and} \quad e^{ik\left(\frac{x_1^2 + y_1^2}{2z}\right)}. \] (49)
The terms in Equation 49 are converging spherical wavefronts, which are created by converging spherical lenses. Therefore, if lenses with a focal length of $z$ are placed at the aperture plane and observation plane, the diverging terms can also be eliminated. Furthermore, the far-field diffraction pattern is now achieved in a distance that is equal to the focal length of the lenses used, saving a considerable amount of space as seen in Figure 38. This is called a 1-$f$ Fourier transform lens system.

![Figure 38. The far-field Fraunhofer diffraction pattern can be observed over a much shorter distance with the use of a 1-$f$ optical Fourier transform system. Lenses of focal length $f$ are placed at the aperture and observation plane, separated by a distance $f$.](image)

The same effect can be achieved through the use of a 2-$f$ system. In this case, one lens is used instead of two and the lens is placed at a distance $f$ from both the aperture plane and observation plane, as seen in Figure 39. The analysis of this system is described in brief. The Fresnel diffraction pattern is found at the first surface of the lens, which is then multiplied by the converging lens exponential term to give the electric field at the other lens surface. Finally, the Fresnel diffraction pattern is again determined from the second lens surface to the observation plane. This system takes more space than a 1-$f$ system, but is used for most of the experiments in this thesis because it uses only one lens and is less clumsy as the lens does not have to be placed as close as possible to the aperture or observation plane.
Figure 39. A 2-f is used in the lab, which utilizes only one lens to achieve the far-field diffraction pattern over a distance of two focal lengths.

These systems have been discussed as displaying the Fraunhofer diffraction pattern by effectively taking the optical Fourier transform of the electric field at the aperture plane. At other times, two successive optical Fourier transforms may be taken, which is discussed next.

4-f Fourier Transform Lens System

In the 4-f optical Fourier system, two 2-f Fourier systems are placed in sequence, as seen in Figure 40. This takes the Fourier transform of a Fourier transform, which is simply found to be the original function, or in this case, the electric field at the aperture plane. While this seems redundant at first, the system has important uses.

Figure 40. A 4-f system takes two optical Fourier transforms and images the aperture at the observation plane. The first Fourier transform allows for spatial filtering at the Fourier plane.

In this system, the appearance of the “Fourier plane” is seen, which has taken the place of the observation plane for the first optical Fourier transform. Just as with the 1-f and 2-f
systems, the Fourier transform of the electric field at the aperture plane occurs at this point. This means the transmission function is broken down to its spatial frequency components at this point, and each component occurs at a different physical location in the Fourier plane. This provides for the opportunity to spatially filter the components of the transmission function and re-image the aperture.

If all spatial frequencies are re-collected by the second Fourier system, the transmission function at the aperture plane will simply be re-imaged. However, the Fourier plane provides the opportunity to block or pass any of the desired spatial frequencies. Now the second Fourier system will collect a frequency set that lacks some of the initial information of the original aperture. This is referred to as spatial filtering, which results in a form of image processing. Spatial filtering can also be used to image one spatial frequency component at a time, which is a technique that is used in many of our experiments.