Deriving Pseduocodewords from Bethe permanents

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Deriving Pseduocodewords from Bethe permanents

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DEDICATION

To my dog Mishka, woof woof
If we hit that bullseye, the rest of the dominoes should fall like a house of cards. Checkmate.

– Captain Zapp Brannigan
In this thesis, we classify all degree $p$ Bethe permanent vectors and Bethe permanent vectors for size $2 \times n$ binary matrices with no zero rows or columns and show that they are pseudocodewords. In doing so, we were able to find a formula for the permanent of the matrix $\text{perm} \left[ I + P \right]$ that is dependent on the size and number of cycles in $P$, a permutation matrix. We subsequently utilized this result coupled with our novel way to partition $S_n$, the symmetric group of $n$ elements, to calculate the Bethe permanent of all size $2 \times 2$ binary matrices.
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CHAPTER 1

INTRODUCTION

As stated in the title, this thesis will focus on calculating the Bethe permanent of size $2p \times 2p$ binary matrices. The Bethe permanent was discovered by Vontobel in [5] while researching the error correcting capabilities of low density parity check codes (LDPC). He found that the inherent iterative message-passing algorithm fails when there exist pseudocodewords of low weight. Thus, as a way to calculate pseudocodewords for large LDPC codes, he created the Bethe permanent. In this introduction, we will begin with a brief history of coding theory and discuss the current research of finding pseudocodewords using the Bethe permanent.

Since the advent of communication devices, physical phenomena have degraded the signal quality of transmitted information, causing a loss or misinterpretation of the data. The problem of loss or misinterpreted information had no solution in sight until the introduction of the 1948 article entitled *A Mathematical Theory of Communication* by Claude Shannon. In this groundbreaking article, Dr. Shannon identified these errors as "noise" and proposed a solution through the use of redundant information. He concluded, however, that adding redundant information increases the delivery time of the message and ultimately realized that these errors could only be matching redundancy to the error probability. In addition, Dr. Shannon also proved that codes achieving nearly perfect error correction exist, however he was unable to construct such codes.

Since Dr. Shannons proof of limit approaching codes, researchers have been trying to produce them. One such researcher was Robert G. Gallagher. In 1960, Robert G. Gallagher introduced a new family of codes in his doctoral thesis named Gallagher codes or Low Density Parity Check Codes. These codes are typically defined with a sparse binary matrix, a matrix mostly populated with zero’s. Due the complexity of his proposed decoding scheme, his research was largely ignored.

In 1993, Turbo codes were found which approached the Shannon limit and researchers once again attempted to construct Shannon Limit approaching codes. Gallaghers work was then rediscovered by David MacKay and Radford M. Neal in 1996. Their paper entitled *Near Shannon Limit Performance of Low Density Parity CheckCodes* was able to prove the effectiveness of Dr. Gallaghers discovery and show how LDPC codes could perform better than the standard convolutional and concatenated codes. In their paper, they first defined four types of parity check matrices and then compared the decoding time and cost of these
matrices with other coding schemes of the same length and dimension. They were only able to test four possible constructions, however a later work [4] by David MacKay and Matthew Davey investigated other types of constructions. The current research of LDPC codes uses the connection between a low density parity-check matrix and a bipartite graph. The importance of researching LDPC codes is due to the more recent decoding methods, min sum and sum product [6].

The min sum and sum product decoding schemes are called message passing iterative decoding schemes as they utilize the bipartite graph of a code to find the influence of each node in our graph and correct errors. A problem which can occur with these decoding methods is the lack of convergence to a solution. Vontobel, in [5], showed a connection between the sum product decoding algorithm and the Bethe free energy function. ”It has been recently observed that the permanent of a non-negative square matrix, i.e., of a square matrix containing only non-negative real entries, can be approximated by solving a certain Bethe free energy function minimization problem with the help of the sum-product algorithm. We call the resulting approximation of the permanent the Bethe permanent”. In [1] and [5] Kelley and Vontobel extended the work done by MacKay and Davey by defining a new term, psuedocodewords. These pseudocodewords satisfy a set of inequalities that are defined as the fundamental cone $K(H)$. They are used to identify when the decoding method is unable to find a solution and thus unable to rectify an error. Kelley and Vontobel claim that the failure of the decoding scheme is due to non-codeword pseudocodewords. They found that when low weight non-codeword pseudocodewords exist, the decoding schemes fail to converge. By identifying codes which have low weight non-codeword pseudocodewords, Kelley was able to select better performing codes. Finding such pseudocodewords is a difficult task.

Via the utilization of Kelley and Vontobels results, Roxana Smarandache showed that the Bethe Permanent based vector of a $m \times n$ with $m > n$ binary matrix with no zero rows or columns is a pseudocodeword. She was also able to classify the Bethe permanent based vector for all binary parity check matrices of size $2 \times 3$ with no zero rows or columns. The main goal of this thesis is to explore in more detail the results in the unpublished work of Smarandache in [2].

In this thesis, we begin Chapter 2 by giving the reader a brief introduction to coding theory and LDPC codes. Next, we discuss an alternative way to define LDPC codes in terms of bipartite graphs. We conclude the second chapter by defining a covering graph.

In Chapter 3, we are introduced to pseudocodewords and permanents. We first define a pseudocodeword and illustrate the difficulty of finding pseudocodewords using the linear programming method. Next, we define the permanent of a matrix and the permanent based vector of a $m \times n$ with $m > n$ matrix. We conclude the third chapter by proving that the
permanent based vectors of a \( m \times n \) with \( m > n \) binary matrix with no zero rows or columns is a pseudocodeword.

In the fourth chapter, we define the lift of a matrix by way of permutation matrices and show that the lift is always a covering graph. Next, we define the Bethe permanent of a matrix and identify the difficulty in calculating the Bethe permanent of a matrix by way of example. The remainder of this chapter is dedicated to calculating the Bethe permanent of the all ones \( 2 \times 2 \) matrix. We conclude chapter four by defining the Bethe Permanent vector and prove that the Bethe Permanent vector produces a pseudocodeword for all \( 2 \times n \) binary matrices with no zero rows or columns.
CHAPTER 2
LINEAR CODES

This chapter will start with a brief introduction to binary linear codes followed by defining a good code by its capability of error correction, specifically through the use of an algorithm. We will then introduce the notion of a covering graph and briefly discuss its use in regards to the error correcting algorithms.

**Definition 2.1.** A $[n, k]$ binary linear code $C$ of length $n$ is a subspace over $\mathbb{F}_2$ of dimension $k$ of the vector space $\mathbb{F}_2^n$. Elements of $C$ are called codewords. A $k \times n$ matrix $G$ whose rows form a basis for a $[n, k]$ binary linear code $C$ is called a generator matrix of the code $C$. An $r \times n$ matrix $H$ for which the left nullspace of $H^T$ is $C$ is called a parity check matrix for $C$. Note that the rank of $H$ must be $n - k$.

**Example 2.2.** Let $C$ be the $[3, 1]$ binary linear code with the $1 \times 3$ generator matrix $G$ and the $2 \times 3$ parity check matrix $H$ are defined as:

$$G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then, $(1, 1, 1)$ is a codeword of $C$ and satisfies Definition 2.1 since,

$$(1, 1, 1)^T H^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly $G$ is the generator matrix since $G$ spans the nullspace of $H^T$. Also notice that the rank-nullity theorem is satisfied.

**Definition 2.3.** The Hamming distance between two codewords $c$ and $c'$ denoted $H(c, c')$ is equal to the number of positions in which the codewords differ.

In general, creating a good code requires an interplay between the dimension, length, and minimum distance. A larger minimum distance ensures better decoding, since there is a larger distance between possible codewords which allows more room for errors to be corrected. Increasing the dimension increases the amount of codewords and decreases the minimum distance. Once a code is selected we would like a quick and effective algorithmic decoding
A method to correct and detect errors. One family of codes which can have many of our desired traits including quick and effective decoding schemes is called a low-density parity-check code, LDPC for short. As stated in [6] LDPC codes are ideal because there exists computationally simple decoding algorithms. Many of the algorithmic decoding approaches are based on linear programming and belief propagation. Linear programming decoding is based on solving a system of equations. Belief propagation decoding is based on the relationship of graph nodes in bipartite graphs. A LDPC code is constructed by a sparse parity check matrix. It is useful to use bipartite graphs as an alternative way to define the code $C$.

**Definition 2.4.** A bipartite graph is a graph $G = (V, E)$ where $V$ is a set of vertices and $E$ is a set of edges such that:

1. $V$ is partitioned into two sets $A$ and $B$,
2. Each edge $e \in E$ is incident to a vertex in $A$ and a vertex in $B$.

**Definition 2.5.** Let $G$ be a bipartite graph $G = (V, E)$. We define the neighborhood of a vertex $v \in V$, $\text{nbd}(v)$, as the set of vertices that have an edge $e \in E$ which is incident to $v$. Since $G$ is a bipartite graph we have that for a vertex $v \in A$, its neighborhood $\text{nbd}(v) \subseteq B$.

**Definition 2.6.** Let $H$ be an $r \times n$ parity check matrix. The bipartite graph of $H$ is obtained by the following: For each of the $n$ columns of $H$ define a vertex $a_1 \cdots a_n \in A$ and for each of the $r$ rows of $H$ define a vertex $b_1 \cdots b_r \in B$. Then for all $i = \{1, \cdots, r\}, j = \{1, \cdots, n\}$ such that $h_{i,j} = 1 \in H$ define an edge between $a_j \in A$ and $b_i \in B$.

**Example 2.7.** Let $C$ be the code defined by the parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The bipartite graph defined by $H$ may be seen in figure 2.1.

## 2.1 Covering Graphs

As stated in [1], to ensure the convergence of the min-sum decoding algorithm we need to investigate the covering graph of our bipartite graph derived from our LDPC code.

**Definition 2.8.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, and let $f : V_2 \to V_1$ be a surjection. Then $f$ is a covering map from $G_2$ to $G_1$ if for each $v \in V_2$, the restriction of $f$ to the neighborhood of $v$ is a bijection onto the neighborhood of $f(v) \in V_1$ in $G_1$. If there exists a covering map from $G_2$ to $G_1$, then $G_2$ is also called a lift of $G_1$. A $p$-lift is a lift such that the covering map $f$ has the property that for every vertex $v$ of $G_1$, its pre-image $f^{-1}(v)$ has exactly $p$ elements.
Later in this paper we will see lifting graphs are obtained by replacing the entries of our original matrix $H$ with permutation matrices of size $p \times p$. For the sake of the reader, we will start with a trivial example of a covering graph obtained by using two copies of our original matrix $H$ and show it is a 2-lift. Then through matrix manipulation we will obtain the standard lifting matrix, our matrix $H$ with its entries replaced by permutation matrices.

**Definition 2.9.** Let $G = (V, E)$ be a bipartite graph. Then $G$ is a connected graph if for any vertex $v_1, v_2 \in G$ there exists a collection of edges that join $v_1$ to $v_2$.

The first 2-lift, which we will label $\tilde{H}$, will be an example of a disconnected graph and our second 2-lift labeled $\tilde{J}$ will be an example of a connected graph.

**Example 2.10.** Continuing with our example, assume our code is defined by the following parity check matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

We wish to create a degree 2-lift of the bipartite graph obtained from the parity check matrix $H$. We may start with a simple 2-lift created by replacing any non zero entries in $H$ with the $2 \times 2$ identity matrix and any zero entries by the $2 \times 2$ zero matrix. This 2-lift is not very useful however it clearly illustrates the properties we described in Definition 2.8.

$$\tilde{H} = \begin{bmatrix} I & I & I \\ I & 0 & I \end{bmatrix}$$

The bipartite graph described by our matrix $\tilde{H}$ may be seen in Figure 2.2. We will now prove that the bipartite graph described by $\tilde{H}$ is a 2-lift of the bipartite graph described by our original matrix $H$. By Definition 2.8 we have the following sets:

$V_1 = \{a_1, a_2, a_3, b_1, b_2\}$ and $V_2 = \{\tilde{a}_1, \ldots, \tilde{a}_6, \tilde{b}_1, \ldots, \tilde{b}_4\}$
We define the function \( f : V_2 \to V_1 \) as follows:

\[
f(n) = \begin{cases} 
  a_i \mod 3 & \text{if } n \text{ is } \bar{a} \\
  b_i \mod 2 & \text{if } n \text{ is } \bar{b}
\end{cases}
\]  

(2.1)

(Note we use \( n \) to denote \( 0 \mod n \))

Select the neighborhood around \( \bar{b}_3 \) consisting of the vertices \( \text{nbd}(\bar{b}_3) = \{\bar{a}_1, \bar{a}_5\} \). Similarly select the neighborhood around \( f(\bar{b}_3) = b_1 \), \( \text{nbd}(f(\bar{b}_3)) = \{a_1, a_5\} \). Computing the neighborhood around the image of the neighborhood around \( \bar{b}_3 \), we obtain:

\( f(\text{nbd}(\bar{b}_3)) = \{a_1, a_5\} \).

Thus, \( f \) restricted to the neighborhood of \( \bar{b}_3 \in V_2 \) is a bijection onto the neighborhood of \( f(\bar{b}_3) \in V_1 \). By repeating this process for each vertex \( \bar{v} \in V_2 \), and since \( f \) is a surjection, we conclude that \( \tilde{H} \) is a covering map of \( H \). In fact, for any \( v \in V_1 \) we have \( |f^{-1}(v)| = 2 \), thus \( \tilde{H} \) is a 2-lift. In order to visualize the lift we have created, we may untwist the graph with the help of permutation matrices that will manipulate the rows and columns of \( \tilde{H} \).

**Figure 2.2. Trivial degree 2- lift of Bipartite graph for Binary Parity Check Matrix**

---

\[ \tilde{A} \]

\[ \tilde{B} \]
Define the $6 \times 6$ permutation matrix $T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$

Define the $4 \times 4$ permutation matrix $R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$

Next we may apply the permutations to our matrix $\tilde{H}$.

$$R \tilde{H} T = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
H & 0 \\
0 & H \\
\end{bmatrix}$$

Our untangled bipartite graph as seen in Figure 2.3 is disconnected. Notice as described in Definition 2.9 we can select the vertices $\tilde{a}_6$ and $\tilde{b}_1$, clearly there is no collection of edges that join $\tilde{a}_6$ and $\tilde{b}_1$. In general, we wish to create a non-trivial lift such that the covers are interleaved between one another. By using the permutation matrix $P = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}$ in place of one of the identity matrix in $\tilde{H}$ we will obtain our non-trivial example. Let $\tilde{J} = \begin{bmatrix}
I & I & I \\
P & 0 & I \\
\end{bmatrix}$. The associate bipartite graph can be seen in Figure 2.4. By reordering the rows and columns of $\tilde{J}$ we obtain Figure 2.5. The following path connects every vertex of Figure 2.5:

$$(\tilde{a}_3 \sim \tilde{b}_1 \sim \tilde{a}_1 \sim \tilde{b}_4 \sim \tilde{a}_6 \sim \tilde{b}_2 \sim \tilde{a}_2 \sim \tilde{b}_3 \sim \tilde{a}_5 \sim \tilde{b}_1 \sim \tilde{a}_1 \sim \tilde{b}_4 \sim \tilde{a}_6 \sim \tilde{b}_2 \sim \tilde{a}_4).$$

Also notice that by Definition 2.9 the graph described by matrix $\tilde{J}$ is connected. Once again we may prove this graph is in fact a lift employing the same methods used for Figure 2.2.

Select the neighborhood around $\tilde{b}_3$ consisting of the vertices $\text{nbd}(\tilde{b}_3) = \{\tilde{a}_5, \tilde{a}_2\}$. Similarly select the neighborhood around $f(\tilde{b}_3)$ $\text{nbd}(f(\tilde{b}_3)) = \{a_2, a_5\}$. Computing the neighborhood around the image of the neighborhood around $\tilde{b}_3$, we obtain: $f(\text{nbd}(\tilde{b}_3)) = \{a_2, a_5\}$. Thus, $f$ restricted to the neighborhood of $\tilde{b}_3 \in V_2$ is a bijection onto the neighborhood of $f(\tilde{b}_3) \in V_1$. By repeating this process for each vertex $\tilde{v} \in V_2$, and since $f$ is a surjection, we conclude that $\tilde{H}$ is a covering map of $H$. In fact, for any $v \in V_1$ we have $|f^{-1}(v)| = 2$, thus $\tilde{H}$ is a 2-lift.
Figure 2.3. Degree 2- lift of Bipartite graph for Binary Parity Check Matrix

In the next chapter, we will begin our study of pseudocodewords and permanents. Note that we will return to the examples used in this chapter when we revisit the $p$- lift of a matrix in Chapter 4.
Figure 2.4. Non-trivial 2-lift of Bipartite graph for Binary Parity Check Matrix

Figure 2.5. Non-trivial 2-lift of Bipartite graph for Binary Parity Check Matrix
CHAPTER 3
PSEUDOCODEWORDS AND PERMANENTS

We begin this chapter by introducing pseudocodewords of a binary matrix $H$. We then give an example, using the linear programming method, to obtain pseudocodewords. Next, we define the permanent, and construct a permanent based vector for a binary matrix $H$ with no zero rows or columns. We conclude this chapter by showing that the permanent based vector is a pseudocodeword.

3.1 PSEUDOCODEWORDS

We will now proceed to define pseudocodewords and discuss the methods used to obtain them.

Definition 3.1. Let $H = (h_{i,j})$ be a $m \times n$ binary matrix. Denote the set $[n] = \{1, \cdots, n\}$. The fundamental cone $K(H)$ of $H$ is the set of all vectors $w = (w_j) \in \mathbb{R}^n$ that satisfy

1. $w_j \geq 0, \forall j \in [n]$
2. $w_j \leq \sum_{j' \in \text{supp}(R_i) \setminus j} w_{j'}, \forall i \in [m], j \in \text{supp}(R_i)$

where $R_i$ is the $i^{th}$ row vector of $H$ and $\text{supp}(R_i)$ is its support (the positions where the vector is non-zero). A vector $w \in K(H)$ is called a pseudocodeword. Two pseudocodewords $w, w' \in K(H)$ are said to be in the same equivalence class if there exists an $\alpha > 0$ such that $w = \alpha w'$. In this case we write $w \propto w'$. Note that if $w$ is a pseudocodeword, so is $\alpha w$ for $\alpha > 0$. Since $(1, 1, \cdots, 1)$ is always a pseudocodeword we will have an infinite number of pseudocodewords of the form $(\alpha, \alpha, \cdots, \alpha)$ for $\alpha > 0$. To each codeword $c \in C$ there is a pseudocodeword $w \in K(H)$ which is $1 \in \mathbb{R}$ in the position where $c$ is non zero and 0 elsewhere. A pseudocodeword which is not equivalent to a codeword will be called a nc-pseudocodeword.

We will now give an example to illustrate the fundamental cone $K(H)$ as well as how the above definition can be used to find a nc-pseudocodeword.

Example 3.2. Let $H$ be the binary check matrix defined below:

$$
H = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
$$
Let $x$ be a pseudocodeword. By applying Definition 3.1 (1) we obtain:

$x_i \geq 0$ for $i = 1, 2, 3, 4$. Applying (2) of Definition 3.1 to the first row of $H$ we derive the following: $x_1 \leq x_4$ and $x_4 \geq x_1$.

These together imply $x_1 = x_4$.

Applying part (2) of Definition 3.1 to the second row of $H$ we derive the following 3 equations:

$x_1 \leq x_3 + x_4$, $x_3 \leq x_1 + x_4$ and $x_4 \leq x_1 + x_3$

Substituting $x_1 = x_4$ we obtain

$x_3 \leq 2x_1$

Applying part (2) of Definition 3.1 to row 3 of $H$ we obtain the following 4 equations:

$x_1 \leq x_2 + x_3 + x_4$, $x_2 \leq x_1 + x_3 + x_4$, $x_3 \leq x_1 + x_2 + x_4$, and $x_4 \leq x_1 + x_2 + x_3$

Substituting $x_1 = x_4$ we obtain

$x_2 \leq 2x_1 + x_3$

Therefore we are left with only 2 inequalities to satisfy:

$x_3 \leq 2x_1$

$x_2 \leq 2x_1 + x_3$

Graphing these equations we obtain a subset of $\mathbb{R}^3$ that contains all possible solutions to this system of equations. If we assume our solution lies on the plane $x_3 = 2x_1$ and apply this to $x_2 \leq 2x_1 + x_3$, we obtain:

$x_2 \leq 4x_1$

This leads to an infinite number of solutions of the form:

$(1, a, 2, 1)$, where $0 \leq a \leq 4$

Choosing the integer values of $a$ we have the following nc-pseudocodewords:

$(1, 0, 2, 1) \quad (1, 1, 2, 1) \quad (1, 2, 2, 1) \quad (1, 3, 2, 1) \quad (1, 4, 2, 1)$

As seen in Example 3.2 we may use Definition 3.1 to obtain pseudocodewords, however this method requires solving a system of inequalities. Another possible method which does not require solving a system of inequalities is obtained through the use of the permanent. We will define the permanent and a permanent based vector, then prove that this vector is indeed a pseudocodeword. We conclude this chapter by investigating the permanent based pseudocodeword of a covering graph.
3.2 PERMANENTS

Definition 3.3. Let $S_n$ be the symmetric group on $n$ elements. The permanent of a $n \times n$ matrix $H$ is:

$$\text{perm}(H) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} h_{i,\sigma(i)}$$

Now we will apply the above definition to an example:

Example 3.4. Let $H$ be the $2 \times 2$ matrix with entries $h_{i,j}$ defined as follows:

$$H = \begin{bmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{bmatrix}$$

Then,

$$\text{perm}(H) = \sum_{\sigma \in S_2} \prod_{i=1}^{2} h_{i,\sigma(i)} = h_{1,1}h_{2,2} + h_{1,2}h_{2,1}$$

Definition 3.5. Let $S \subseteq [n]$. By $H_S$ we denote the submatrix of $H$ consisting of the columns indexed by the elements of $S$.

We now have the tools to define the permanent based vector as follows:

Definition 3.6. Let $n > m$ and let $H$ be a binary matrix that has no zero columns or rows. For a size $(m + 1)$ subset $\beta$ of $[n]$ we define the perm-vector based on $\beta$ to be the vector $w^{\beta} \in \mathbb{Z}^n$ with components
In the case that $H$ is a $m \times (m+1)$ matrix, we will denote our perm based vector as $w_i$, with components $w_i = \text{perm}(H_{\beta \setminus i})$, where $H_{\beta \setminus i}$ is the $m \times m$ matrix $H$ with its $i^{th}$ column removed.

When $H$ is a size $m \times (m+1)$ matrix we only obtain one permanent based vector, however when $H$ is a size $m \times n$ matrix where $n > m + 1$, we may obtain $\binom{n}{m+1}$ possible permanent based vectors. We will now proceed with an example of a matrix $H$ that will illustrate how we obtain multiple permanent based vectors.

**Example 3.7.** Let $H$ be the matrix defined as follows:

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Then each subset $\beta$ is of size 3. We have $\binom{4}{3}$ choices for $\beta$, which in turn will gives us 4 perm-vectors.

$$\beta_1 = \{2, 3, 4\} \quad \beta_2 = \{1, 3, 4\} \quad \beta_3 = \{1, 2, 4\} \quad \beta_4 = \{1, 2, 3\}$$

Consider $\beta_1$. The matrix associated to it is:

$$H_{\beta_1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The perm-vector for $H_{\beta_1}$ is obtained as follows:

$$w_{\beta_1}^i := \begin{cases} \text{perm}(H_{\beta_1 \setminus i}) & \text{if } i \in \beta_1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$w_{1}^{\beta_1} = 0 \quad w_{2}^{\beta_1} = \text{perm}(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) \quad w_{3}^{\beta_1} = \text{perm}(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) \quad w_{4}^{\beta_1} = \text{perm}(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$$

Thus $w^{\beta_1} = (0, 1, 1, 2)$. Similarly for $\beta_2$,

$$H_{\beta_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The perm-vector $w$ for $\beta_2$ is

$$w_{1}^{\beta_2} = \text{perm}(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) \quad w_{2}^{\beta_2} = 0 \quad w_{3}^{\beta_2} = \text{perm}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \quad w_{4}^{\beta_2} = \text{perm}(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix})$$
Thus $w^{\beta_2} = (1, 0, 1, 1)$. Similarly for $\beta_3$,

$$H_{\beta_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The perm-vector $w$ for $\beta_3$ is

$$w^\beta_1 = \text{perm}(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) \quad w^\beta_2 = \text{perm}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \quad w^\beta_3 = 0 \quad w^\beta_4 = \text{perm}(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix})$$

Thus $w^{\beta_3} = (1, 1, 0, 1)$. Similarly for $\beta_4$,

$$H_{\beta_4} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The perm-vector $w$ for $\beta_4$ is

$$w^\beta_1 = \text{perm}(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) \quad w^\beta_2 = \text{perm}(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \quad w^\beta_3 = \text{perm}(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \quad w^\beta_4 = 0$$

Thus $w^{\beta_4} = (2, 1, 1, 0)$. Notice that $w^{\beta_1}, w^{\beta_2}, w^{\beta_3}$ and $w^{\beta_4}$ are in $K(H)$ since they satisfy Definition 3.1.

We now wish to prove that these permanent based vectors are always pseudocodewords. We will first give an example that proves the permanent based vectors of any matrix $H$ of size $2 \times 4$ that has no zero columns or rows are pseudocodewords. Then by utilizing the same process in the example, we will prove Proposition 3.9, which states that the permanent based vector of any matrix $H$ of size $m \times (m + 1)$ that has no zero columns or rows is a pseudocodeword. Finally, Corollary 3.10 concludes that all permanent based vectors obtained from a matrix $H$ of size $m \times n$, where $n > m$ and $H$ contains no zero rows or columns are pseudocodewords.

**Example 3.8.** Let $H$ be the following matrix with entries $h_{i,j} \in \{0, 1\}$

$$H = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} \\ h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} \end{bmatrix}$$

By Definition 3.6 we must select $\beta$ to be a 3 element subset of $\{1, 2, 3, 4\}$. Without loss of generality let $\beta = \{1, 2, 3\}$. Since $\{4\} \notin \beta$, $w^\beta_4 = 0$ and clearly $w^\beta_3 = 0 \leq w^\beta_1 + w^\beta_2 + w^\beta_3$. Now select an element of $\beta$, say $j = 1$. In order to prove $w^\beta_1 \leq w^\beta_2 + w^\beta_3 + w^\beta_4$ we do the following:

First we find $w^\beta_2$ and $w^\beta_3$, notice both are calculated with the following matrices.
\[ H' = \begin{bmatrix} h_{1,1} & h_{1,3} \\ h_{2,1} & h_{2,3} \end{bmatrix} \]

\[ H'' = \begin{bmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{bmatrix} \]

The first column in \( H' \) and \( H'' \) are contributed from column 1 of matrix \( H \). Since each column is non-zero, we may assume \( h_{1,1} \neq 0 \) (the situation for \( h_{2,1} \neq 0 \) is similar). Expanding along the first column we obtain:

\[ w_2^\beta + w_3^\beta = h_{1,1}h_{2,3} + h_{1,3}h_{2,1} + h_{1,1}h_{2,2} + h_{1,2}h_{2,1} \]

\[ \geq h_{1,1}h_{2,3} + h_{1,1}h_{2,2} \]

Since \( h_{1,1} = 1 \)

\[ = h_{2,3} + h_{2,2} \]

\[ \geq h_{1,2}h_{2,3} + h_{1,3}h_{2,2} = w_1 \]

Since \( j = 1 \) was arbitrary we can reuse this proof for \( j = 2, 3 \).

**Proposition 3.9.** Let \( H \) be a \( m \times (m + 1) \) binary matrix with no zero rows or columns. The perm-vector \( w \) of \( H \) is a pseudocodeword of \( H \) and satisfies

1. \( w_j \geq 0, \forall j \in [m + 1] \)

2. \( w_j \leq \sum_{k=1}^{m+1} w_k \)

**Proof.** We will show for an arbitrary \( j \) that \( w_j \) satisfies Definition 3.1.

First note that item (1) is immediate, \( w_j \geq 0, \forall j \in [m + 1] \) from the definition of the perm vector since the entries of \( H \) are positive.

For item (2), fix \( j_o \) such that \( h_{i,j_o} = 1 \). Then,

\[ w_j = \sum_{\substack{j'=1 \\ j' \neq j}}^{m+1} h_{i,j'} \text{ perm } H_{\forall,j'}^{\forall,j} \geq h_{i,j_o} \text{ perm } H_{\forall,j_o}^{\forall,j}, \text{ since all terms are } \geq 0. \]

So,

\[ \sum_{\substack{j=1 \\ j \neq j_o}}^{m+1} w_j \geq \sum_{\substack{j' \neq j_o}} h_{i,j_o} \text{ perm } H_{\forall,j_o}^{\forall,j} = w_{j_o} \]
Corollary 3.10. Let $H$ be a $m \times n$ binary matrix with no zero rows or columns. For all $\binom{n}{m+1}$ size $m + 1$ subsets of $[n]$, the perm-vector $w$ based on each subset is a pseudocodeword of $H$.

Proof. For each subset $\beta$ of $[n]$ we form a $m \times (m + 1)$ binary matrix with no zero rows or columns. Then by Proposition 3.9 each perm-vector $w^\beta$ of $H$ is a pseudocodeword. \hfill \Box

We will now revisit the matrix from Example 3.2 and show that the permanent based vector is indeed a pseudocodeword as stated by Proposition 3.9.

Example 3.11. Recall our matrix $H$ from example 3.2 is

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Since $H$ is of size $m \times (m + 1)$ our perm-vector $w = (\text{perm}(H_{\backslash 1}), \text{perm}(H_{\backslash 2}), \text{perm}(H_{\backslash 3}), \text{perm}(H_{\backslash 4}))$ where,

- $H_{\backslash 1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $H_{\backslash 2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $H_{\backslash 3} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $H_{\backslash 4} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

By direct calculation, $w = (1, 3, 2, 1)$. We will now show that $w$ satisfies the inequalities obtained from Example 3.2 and is a pseudocodeword as stated in Lemma 3.9:

1. Clearly $w_j \geq 0$ for $j = \{1, 2, 3, 4\}$
2. $x_1 = x_4$ is satisfied since $1 = 1$
3. $x_3 \leq 2x_1$ is satisfied since $2 \leq 2 \times 1 = 2$
4. $x_2 \leq 2x_1 + x_3$ is satisfied since $3 \leq 2 \times 1 + 2 = 4$

Thus, $w$ is a pseudocodeword, $w \in K(H)$.

In the next chapter, by taking a $p$-lift of the graph obtained from our parity check matrix $H$ we hope to find nc-pseudocodewords. We will see that the simplistic permanent based vector of a lifting graph is not always a pseudocodeword. In order to get a pseudocodeword we will need to define a new permanent based vector.
CHAPTER 4

PSEUDOCODEWORDS DERIVED FROM LIFTING GRAPHS

As noted, our current permanent based vector described by Definition 3.1 does not always produce a pseudocodeword for lifting graphs. In [2] the points of convergence of the min-sum algorithm is realized as the solution of the Bethe Free Energy minimization problem. The result is a definition for a permanent based vector for lifting graphs. In this chapter we will first define the permanent based vector for lifting matrices. We will continue by proving some important properties of the permanent that will allow us to quickly determine the permanent of $2 \times 2$ block matrices. Our final goal is to classify all of the permanent based vectors derived from matrices of size $2 \times n$, where $n \geq 3$, and prove that they are pseudocodewords. We conclude the chapter with an open ended question of determining how to extend the definition of pseudocodeword and permanent based vectors to $q$-ary linear codes of size $2 \times n$.

4.1 PERMANENTS AND LIFTS

Let $\mathcal{R}_p$ be the set of $p \times p$ permutation matrices. We first show that the matrix formed by replacing nonzero entries of $H$ by permutation matrices is indeed a $p$-lift, where $p$ is the size of the permutation matrices. Then, we will show that for a positive integer $p$ and a binary matrix $H$ we may obtain a $p$-lift of the graph described through $H$ by replacing the non zero entries of $H$ with size $p \times p$ permutation matrices and the 0 entries by a $p \times p$ matrix of 0’s.

Example 4.1. For $p = 2$, $\mathcal{R}_p$ contains two matrices, $I$ and $P$:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which correspond to (1)(2) and (12) respectively.

Recall that in Example 2.10 we were able to construct the matrices $\tilde{H}$ and $\tilde{J}$, with both being 2-lifts of our matrix $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ by replacing the non zero entries of $H$ with size...
2 \times 2 permutation matrices and our zero entries with the size 2 \times 2 zero matrix. Our result was the matrix 
\tilde{H} = \begin{bmatrix} I & I & I \\ I & 0 & I \end{bmatrix} \quad \text{and} \quad \tilde{J} = \begin{bmatrix} I & I & I \\ P & 0 & I \end{bmatrix}

For the rest of the paper we will assume that the p-lift of a graph is obtained through the use of size p \times p permutation matrices applied to the graph’s corresponding matrix.

**Definition 4.2.** Let \( \psi_{m,n,p} \) be the set of \( m \times n \) matrices whose entries are \( p \times p \) permutation matrices, \( \psi_{m,n,p} = \mathcal{R}_{m \times n}^{p^m \times n} \). Thus if \( Q \in \psi_{m,n,p} \), then \( Q_{i,j} \in \mathcal{R}_p \) for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Let \( H \) be a non-negative (with non-negative real entries) \( m \times n \) matrix with \( n > m \). If \( m = n \) we will use the notation \( \psi_{m,p} \). For \( H \) a binary \( m \times n \) matrix and \( Q \in \psi_{m,n,p} \) we define \( H^{\uparrow Q} \), the \( mp \times np \) binary matrix as:

\[
H^{\uparrow Q} = \begin{bmatrix}
h_{11}Q_{11} & \cdots & h_{1n}Q_{1n} \\
\vdots & \ddots & \vdots \\
h_{m1}Q_{m1} & \cdots & h_{mn}Q_{mn}
\end{bmatrix}
\]

**Proposition 4.3.** Let \( H \) be a \( m \times n \) binary parity check matrix such that \( n \geq m \). Let \( G = (V,E) \) represent the associated bipartite graph. For any positive integer \( p \) and \( Q \in \psi_{m,n,p} \), the graph \( G_1 = (V_1,E_1) \) representing the matrix \( H^{\uparrow Q} \) is a p-lift of our graph \( G = (V,E) \) described by \( H \).

We will accomplish this proof by showing that there exists a surjective function from the graph \( G_1 \) described by \( H^{\uparrow Q} \) to the graph \( G \) described by \( H \). Then we will show that when restricted to the neighborhood around a single vertex in \( G_1 \) there is a bijection from the neighborhood of \( v \in G_1 \) to the neighborhood \( f(v) \in G \) and finish the proof by showing there exist \( p \) such unique neighborhoods. This will establish \( G_1 \) as a p-lift of \( G \).

**Proof.** Let \( H \) be a binary matrix of size \( m \times n \), where \( n \geq m \). Then we compute the rowsum for each row and label them \( r_1, \ldots, r_m \) as well as compute the colsum for each column and label them \( c_1, \ldots, c_n \).

\[
\begin{array}{c|ccccc}
\text{vertex} & a_1 & \cdots & a_n & \text{rowsum} \\
\hline
b_1 & \begin{pmatrix} h_{1,1} & \cdots & h_{1,n} \end{pmatrix} & r_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_m & \begin{pmatrix} h_{m,1} & \cdots & h_{m,n} \end{pmatrix} & r_m \\
\text{colsum} & c_1 & \cdots & c_n \\
\end{array}
\]

The bipartite graph \( G \) derived from our matrix \( H \) is comprised of the two sets of vertices \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_m\} \) which comprise the set of vertices \( V_1 \). Notice that for any vertex \( a_i \in A \) the size of its neighborhood is equal to \( c_i \), the sum of column \( i \).
Similarly for any vertex $b_j \in B$ the size of its neighborhood is equal to $r_j$ the sum of row $j$.

Let $p$ be given, then the matrix $H^\uparrow Q$ has the graph $\tilde{G}$ associated to it. Where $\tilde{G}$ is the bipartite graph defined by the two sets $\tilde{A} = \{\tilde{a}_{1,1}, \ldots, \tilde{a}_{1,k}, \ldots, \tilde{a}_{n,1}, \ldots, \tilde{a}_{n,k}\}$ and $\tilde{B} = \{\tilde{b}_{1,1}, \ldots, \tilde{b}_{1,k}, \ldots, \tilde{b}_{m,1}, \ldots, \tilde{b}_{m,k}\}$ which comprise the set of vertices $V_2$. In order to show that $H^\uparrow Q$ is a $p$-lift we need to verify that Definition 2.8 is satisfied by defining a surjective function $f$ from $V_2 \to V_1$ such that for any $v \in V_2$ $f$ restricted to the neighborhood of $v$ is a bijection to the neighborhood of $f(v)$, $\text{nbd}(v) \to \text{nbd}(f(v))$. For any $\tilde{h}_{r,s}$ for $r = 1, 2, \ldots, mp$ and for $s = 1, 2, \ldots, np$ we have $f(\tilde{h}_{r,s}) = h_{\left\lceil \frac{r}{p} \right\rceil, \left\lfloor \frac{s}{p} \right\rfloor}$. First we must show that this map is onto. Select any $h_{i,j} = 1 \in H$ such that $i = 1, \ldots, m$ and any $j = 1, \ldots, n$. Then by construction there exist $p$ non zero entries $\tilde{h}_{u,v}$ for some $u = (i-1)p + 1, \ldots, up$ and $v = (j-1)p + 1, \ldots, vp$, any of which will map to $h_{i,j}$. Thus $f$ is surjective. Without loss of generality select the vertex $\tilde{b}_k \in V_2$ corresponding to row $k$ in $H^\uparrow Q$. Then, by construction $\tilde{b}_k$ has $r = r\left\lceil \frac{k}{p} \right\rceil$ neighbors. (Note we can use $a_j \in V_2$ with $r = r\left\lceil \frac{j}{p} \right\rceil$ neighbors). We will show that each of the $r$ neighbors is mapped by $f$ to a different vertex in $G$ by contradiction.

Assume $r > 0$, (if $r = 0$ we clearly have a bijection). Assume $f$ maps two of $\tilde{b}_k$’s neighbors to the same vertex in $G$, say $\tilde{a}_j$ and $\tilde{a}_j'$, where $j \neq j'$. These vertices correspond to $h_{k,j}$ and $h_{k,j'}$ in $H^\uparrow Q$. If they both map to the same vertex in $G$ it implies that $\left\lceil \frac{j}{p} \right\rceil = \left\lceil \frac{j'}{p} \right\rceil$. However since $j \neq j'$ there are two non zero entries in the same row of the permutation matrix $P\left[\begin{array}{c}j \\ p \end{array}\right]$. This contradicts the definition of a permutation matrix. So all of the $r$ neighbors of $\tilde{b}_k$ must map to different vertices in $V_1$. Since $\text{nbd}(f(b_k))$ also has exactly $r$ neighbors and the $r$ neighbors of $\text{nbd}(b_k)$ map to different vertices we have that $f$ is a bijection from the neighborhood around $b_k$ to the neighborhood around $f(b_k)$. Since $b_k$ was arbitrary we have $f$ is a surjection and for any vertex $v \in V_2$, $f$ is a bijection from $\text{nbd}(v) \to \text{nbd}(f(v))$. Thus $H^\uparrow Q$ is a covering graph. Now select any vertex $v \in G$. Let $b_j$ be an arbitrary vertex in $G$ for $j = 1, \ldots, m$. The $f^{-1}(b_j)$ has $p$ possible choices. By construction we may select any of the $p$ non zero rows $p(j - 1) + 1, \ldots, pj$ of the $H^\uparrow Q$ and each row will describe the vertex $b_j$. (We may also do this for $a_k$ by selecting the rows). Then since $b_j$ was arbitrary we have for any vertex $v \in G$, $f^{-1}(v)$ has exactly $p$ elements. Thus $H^\uparrow Q$ is a $p$-lift.

We will now define how to obtain a permanent based vector for a matrix $H$, from a covering graph $H^\uparrow Q$.

**Definition 4.4.** Let $H$ be a matrix of size $m \times n$ and let $p$ be a positive integer. For a matrix $Q \in \Psi_{m,n,p}$ and a subset $S = \{s_1, s_2, \ldots, s_k\}$ of $[n]$, where $k \leq n$, define $H^\uparrow Q_S$ as the
following matrix:

\[ H^{Q_S} = \begin{bmatrix}
    h_{1,s_1} Q_{1,s_1} & \cdots & h_{1,s_k} Q_{1,s_k} \\
    \vdots & \ddots & \vdots \\
    h_{m,s_1} Q_{m,s_1} & \cdots & h_{m,s_k} Q_{m,s_k}
\end{bmatrix} \]

**Definition 4.5.** Let \( H \) be a matrix of size \( m \times n \) and let \( p \) be a positive integer. For a matrix \( Q \in \psi_{m,n,p} \) and a size \((m + 1)\) subset \( \beta \) of \([n]\), we define the perm-vector based on \( \beta \) to be the vector \( w^\beta \in \mathbb{R}^n \) with components:

\[ w_i^\beta := \begin{cases} 
    \text{perm} H^{Q_{\beta \setminus i}} & \text{if } i \in \beta \\
    0 & \text{otherwise}
\end{cases} \quad (4.1) \]

**Example 4.6.** Revisiting the matrix from Example 2.10, we have \( H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \). A 2-lift of our matrix can be formed by selecting the matrix \( Q \in \psi_{2,3,2} \).

\[ Q = \begin{bmatrix} I & I & P \\ I & I & P \end{bmatrix} \]

Applying our \( p \)-lift we obtain \( H^{\uparrow Q} = \begin{bmatrix} I & I & P \\ I & 0 & P \end{bmatrix} \). Now we may compute the perm based vector \( \tilde{w} \) of \( H^{\uparrow Q} \) as described in Equation (4.15). Since \( H \) is of size \( m \times (m + 1) \) for \( m = 2 \), \( \beta = \{1, 2, 3\} \)

\[ \tilde{w}_1^\beta = \text{perm} \begin{bmatrix} h_{1,2} Q_{1,1} & h_{1,3} Q_{1,2} \\ h_{2,2} Q_{2,1} & h_{2,3} Q_{2,2} \end{bmatrix} = \text{perm} \begin{bmatrix} I & P \\ 0 & P \end{bmatrix} \]

\[ \tilde{w}_2^\beta = \text{perm} \begin{bmatrix} h_{1,1} Q_{1,1} & h_{1,3} Q_{1,2} \\ h_{2,1} Q_{2,1} & h_{2,3} Q_{2,2} \end{bmatrix} = \text{perm} \begin{bmatrix} I & P \\ I & P \end{bmatrix} \]

\[ \tilde{w}_3^\beta = \text{perm} \begin{bmatrix} h_{1,1} Q_{1,1} & h_{1,2} Q_{1,2} \\ h_{2,1} Q_{2,1} & h_{2,2} Q_{2,2} \end{bmatrix} = \text{perm} \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \]

By direct calculation we obtain \( \tilde{w}^\beta = (1, 4, 1) \)
As we can see this method does not produce a pseudocodeword. We will utilize the method introduced by Vontobel in [5], by defining a new permanent based vector for a binary matrix $H$.

**Definition 4.7.** Let $H$ be a binary parity check matrix of size $m \times m$ and let $p$ be a positive integer. Define the degree-$p$ Bethe permanent of $H$ as:

$$B_{\text{perm}}_p(H) \triangleq \sqrt[1/p]{\langle \text{perm}(H^\top Q) \rangle}_{Q \in \psi_{m,p}}$$

where the angular brackets $\langle \rangle$ represent the arithmetic average of $\text{perm}(H^\top Q)$ over all $Q \in \psi_{m,p}$. Then the Bethe permanent of $H$ is defined as

$$B_{\text{perm}}(H) = \lim_{p \to \infty} \sup B_{\text{perm}}_p(H)$$

We will now illustrate Definition 4.7 by way of example. We will calculate $B_{\text{perm}}_1(H)$ the Bethe permanent for $p = 1$ for all size $2 \times 2$ matrices with no zero rows or columns. Then we will illustrate the difficulty in calculating the permanent for larger $p$’s by calculating $B_{\text{perm}}_2(H)$ the Bethe permanent for $p = 2$ for a single size $2 \times 2$ matrix with no zero rows or columns. We shall see for $p = 2$, the number of computations required increases drastically. We will then prove useful properties that will allow us to calculate the permanent for larger matrices. We finish the chapter by computing $B_{\text{perm}}_k(H)$, for $k \in \mathbb{N}$ and for

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

**Example 4.8.** We will calculate the $B_{\text{perm}}_1(H)$ for each possible $H$ of size $2 \times 2$. Notice that for $p = 1$, there is only one permutation matrix, namely the identity matrix. Thus, the only $p$-lift of $H$ is $H$ itself and we have $\text{perm}_1(H^\top Q) = \text{perm}_1(H)$. We will now list the 8 possible matrices for $H$ and calculate their Bethe permanents. For $H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have the Bethe permanent as:

$$B_{\text{perm}}_1(H) = (\text{perm} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})^{1/1} = 2$$

Similarly for all the following matrices the Bethe permanent is the same, $B_{\text{perm}}(H) = 1$.\[ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \]

All other $2 \times 2$ matrices have a zero row or a zero column.

Now assume $p = 2$ and let $H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. When $p = 2$, there are 2 possible $2 \times 2$ permutation matrices, namely the identity matrix $I$ and the matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In order to
find $B_{\text{perm}_2}(H)$ we need to calculate the permanent of all possible 2-lifts. Notice there are exactly $2^4 = 16$ possible 2-lifts of our matrix $H$, i.e. $|\psi_{2,2}| = 16$. Below we have $H^\uparrow Q_i$ for $i = 1, \cdots, 16$, the 16 possible lifts of $H$:

\[
\begin{align*}
H^\uparrow Q_1 &= \begin{bmatrix} I & I \\ I & I \end{bmatrix}, &
H^\uparrow Q_2 &= \begin{bmatrix} P & I \\ I & P \end{bmatrix}, &
H^\uparrow Q_3 &= \begin{bmatrix} P & P \\ I & I \end{bmatrix}, &
H^\uparrow Q_4 &= \begin{bmatrix} P & I \\ P & I \end{bmatrix}, \\
H^\uparrow Q_5 &= \begin{bmatrix} I & P \\ I & P \end{bmatrix}, &
H^\uparrow Q_6 &= \begin{bmatrix} I & P \\ P & I \end{bmatrix}, &
H^\uparrow Q_7 &= \begin{bmatrix} I & I \\ P & P \end{bmatrix}, &
H^\uparrow Q_8 &= \begin{bmatrix} P & P \\ P & P \end{bmatrix}, \\
H^\uparrow Q_9 &= \begin{bmatrix} I & I \\ I & P \end{bmatrix}, &
H^\uparrow Q_{10} &= \begin{bmatrix} I & P \\ I & I \end{bmatrix}, &
H^\uparrow Q_{11} &= \begin{bmatrix} I & I \\ P & I \end{bmatrix}, &
H^\uparrow Q_{12} &= \begin{bmatrix} P & P \\ I & I \end{bmatrix}, \\
H^\uparrow Q_{13} &= \begin{bmatrix} P & P \\ I & P \end{bmatrix}, &
H^\uparrow Q_{14} &= \begin{bmatrix} P & P \\ P & I \end{bmatrix}, &
H^\uparrow Q_{15} &= \begin{bmatrix} I & P \\ P & P \end{bmatrix}, &
H^\uparrow Q_{16} &= \begin{bmatrix} P & P \\ P & P \end{bmatrix}
\end{align*}
\]

By direct calculation we have

$$\text{perm}(H^\uparrow Q_i) = \begin{cases} 4 & \text{if } i = 1, \cdots, 8 \\ 2 & \text{if } i = 9, \cdots, 16 \end{cases}$$

Then, $B_{\text{perm}_2}(H) = \sqrt{\frac{8 \times 4 + 8 \times 2}{16}} = \sqrt{3}$

We would like to compute all degree-$p$ Bethe permanents and all $2 \times 2$ matrices of size $m = 2$, i.e.

$$B_{\text{perm}_p}(H) \triangleq \sqrt[p]{\langle \text{perm}(H^\uparrow Q) \rangle_{Q \in \psi_{2,p}}}$$

The difficulty in calculating comes from having to compute the permanent for large sized matrices, this task is emphasized since we are required to find the average permanent. In general, for a matrix $H$ of size $m \times m$ and for $p$, a positive integer, we are required to take the permanent of $(p!)^m$ size $mp \times mp$ matrices. The fastest known exact formula for calculating the permanent is known as Rysers Formula. For a size $n$ matrix the number of arithmetic operations in Rysers algorithm is bounded by $2^n n^2$. In the next section we will compute a formula for the permanent with the goal of increasing computation speed of the Bethe permanent.

### 4.2 Block Permanents

This section begins by investigating the relationship of the permanent of a matrix $H$ and the permanent of $H$ under row and column manipulation by way of multiplying by permutation matrices. Then we will derive and prove a formula for the permanent of a more
generalized block matrix, that was discussed in [2]. The end of this section will consist of an example illustrating the derived formula for a block permanent.

We begin by proving that the permanent of a matrix $A$ is unchanged by the permutation of rows and columns.

**Definition 4.9.** Let $\sigma \in S_n$. The permutation matrix associated to $\sigma$ is $P_\sigma$ where,

\[
(P_\sigma)_{i,j} = \begin{cases} 
1 & \text{if } \sigma(i) = j \\
0 & \text{otherwise} 
\end{cases}
\]  

(4.2)

**Lemma 4.10.** Let $A$ be an $n \times n$ matrix. Then,

\[
\text{perm}(A) = \text{perm}(AP) = \text{perm}(PA) \text{ for } P \in \mathcal{R}_n
\]

**Proof.** Recall the definition of the permanent of a $n \times n$ matrix $H$ is:

\[
\text{perm}(H) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} h_{i,\sigma(i)}.
\]

Let $P$ be a permutation matrix and let $\sigma_P$ denote the permutation which defines $P$.

Then $(AP)_{i,j} = \sum_{j=1}^{n} a_{i,k}P_{k,j} = a_{i,\sigma_P^{-1}(j)}P_{\sigma_P^{-1}(j),j} = a_{i,\sigma_P^{-1}(j)}$

\[
\text{perm}(AP) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (AP)_{i,\sigma(i)}
\]

\[
= \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_P^{-1}(\sigma(i))}
\]

\[
= \sum_{\sigma' \in S_n} \prod_{i=1}^{n} a_{i,\sigma_P^{-1}(\sigma_P\sigma'(i))}, \text{ where } \sigma = \sigma_P\sigma'
\]

(4.3)

\[
= \sum_{\sigma' \in S_n} \prod_{i=1}^{n} a_{i,\sigma'(i)}
\]

\[
= \text{perm}(A)
\]
Similarly, \((PA)_{i,j} = a_{\sigma P(i),j}\)

\[
\text{perm}(PA) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (PA)_{i,\sigma(i)}
\]

\[
= \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma P(i),\sigma(i)}
\]

\[
= \sum_{\sigma' \in S_n} \prod_{i=1}^{n} a_{j,\sigma'(j)}
\]

\[
= \text{perm}(A)
\]

Thus \(\text{perm}(PA) = \text{perm}(AP) = \text{perm}(A)\)

Now we consider the permanent of a special block matrix.

**Corollary 4.11.** Let \(P, Q, R\) be \(n \times n\) permutation matrices. Let \(S\) be a \(n \times n\) matrix with positive entries. Then the following holds:

\[
\text{perm} \left[ \begin{array}{cc} P & Q \\ R & S \end{array} \right] = \text{perm} \left[ \begin{array}{cc} I & I \\ I & T \end{array} \right], \text{ where } T = PR^{-1}SQ^{-1}
\]

**Proof.** Let \(P, Q, R\) be \(n \times n\) permutation matrices, and \(S\) a \(n \times n\) matrix with positive entries. Let

\[
B = \left[ \begin{array}{cc} P & Q \\ R & S \end{array} \right], \quad A = \left[ \begin{array}{cc} I & 0 \\ 0 & PR^{-1} \end{array} \right], \quad C = \left[ \begin{array}{cc} P^{-1} & 0 \\ 0 & Q^{-1} \end{array} \right]
\]

Notice that \(A, C \in \mathcal{R}_{2n}\)

By Lemma 4.10, we have:

\[
\text{perm}(B) = \text{perm}(ABC)
\]

where,

\[
ABC = \left[ \begin{array}{cc} I & I \\ I & T \end{array} \right], \text{ and } T = PR^{-1}SQ^{-1}
\]

\(\square\)
Although the matrix obtained from the product $ABC$ is presented in a simpler form, the permanent still requires the same amount of sums and products as the permanent of our original matrix $B$. We proceed by proving a solution to reducing the number of operations for the block permanent and illustrating the solution through an example. The following subsets of $S_{2n}$ will be used in the proof of Proposition 4.15.

**Definition 4.12.** For $n \in \mathbb{N}$. For each $\alpha \subseteq [n]$ let $\bar{\alpha} = \{1, \ldots, n\} - \alpha$. Then define $T_\alpha \subseteq S_{2n}$ as the following:

$$T_\alpha = \{ \sigma \in S_{2n} \text{ such that } \sigma(i) = i \quad \text{if } i \in \alpha \} \quad \text{and} \quad \sigma(i) = n + i \quad \text{if } i \in \bar{\alpha}$$

Notice there are no restrictions on $\sigma(i)$ for $i \in \{n + 1, \ldots, 2n\}$.

We will now illustrate how to calculate $T_\alpha$ in the following two examples.

**Example 4.13.** Consider $n = 1$. Recall the set $S_2$ consists of the two permutations $(1)(2)$ and $(12)$. There are two choices for $T_\alpha$ since we have two choices for $\alpha$, $\alpha = \{1\}$ with $\bar{\alpha} = \emptyset$ or $\alpha = \emptyset$ with $\bar{\alpha} = \{1\}$. Lets begin with $\alpha = \{1\}$. Since $\bar{\alpha}$ is empty, the $\sigma \in T_\alpha$ are dependent upon $i \in \alpha$:

$$T_\alpha = \{ \sigma \in S_2 \text{ such that } \sigma(1) = 1 \} = \{(1)(2)\}$$

Similarly if $\alpha = \emptyset$ the $\sigma \in T_\alpha$ are dependent upon $i \in \bar{\alpha}$.

$$T_\alpha = \{ \sigma \in S_2 \text{ such that } \sigma(1) = 2 \} = \{(12)\}$$

**Example 4.14.** Consider $n = 2$. Recall the set $S_4$ consists of 24 permutations. There are four choices for $T_\alpha$ since we have four choices for $\alpha$, $\alpha = \{1\}$ with $\bar{\alpha} = \{2\}$, $\alpha = \{2\}$ with $\bar{\alpha} = \{1\}$, $\alpha = \{1, 2\}$ with $\bar{\alpha} = \emptyset$ or $\alpha = \emptyset$ with $\bar{\alpha} = \{1, 2\}$. Lets begin with $\alpha = \{1\}$ and $\bar{\alpha} = \{2\}$:

$$T_\alpha = \{ \sigma \in S_4 \text{ such that } \sigma(1) = 1 \text{ and } \sigma(2) = 4 \} = \{(1)(24)(3), (1)(243)\}$$

If $\alpha = \{2\}$ and $\bar{\alpha} = \{1\}$.

$$T_\alpha = \{ \sigma \in S_4 \text{ such that } \sigma(2) = 2 \text{ and } \sigma(1) = 3 \} = \{(2)(13)(4), (2)(134)\}$$

If $\alpha = \{1, 2\}$ and $\bar{\alpha} = \emptyset$.

$$T_\alpha = \{ \sigma \in S_4 \text{ such that } \sigma(1) = 1 \text{ and } \sigma(2) = 2 \} = \{(1)(2)(3)(4), (1)(2)(34)\}$$

Lastly, if $\alpha = \emptyset$ and $\bar{\alpha} = \{1, 2\}$.

$$T_\alpha = \{ \sigma \in S_4 \text{ such that } \sigma(1) = 3 \text{ and } \sigma(2) = 4 \} = \{(13)(24), (1324)\}$$
Proposition 4.15. Let \( P \) be a permutation matrix of size \( n \times n \), then we have the following:

\[
\text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix} = \text{perm}(I + P)
\]

Proof. Let

\[
M = \begin{bmatrix} I & I \\ I & P \end{bmatrix}
\]

then by definition,

\[
\text{perm}(M) = \sum_{\sigma \in S_{2n}} \prod_{i=1}^{2n} m_{i,\sigma(i)} = \sum_{\sigma \in S_{2n}} (\prod_{i=1}^{n} m_{i,\sigma(i)}) (\prod_{i=n+1}^{2n} m_{i,\sigma(i)})
\]

We only need to consider the \( \sigma \) such that \( \sigma(i) = i \) or \( \sigma(i) = n + i \) for \( i \in \{1, \cdots, n\} \).

\[
\text{perm}(M) = \sum_{\sigma \in T_{\alpha}} (\prod_{i=1}^{n} m_{i,\sigma(i)}) (\prod_{i=n+1}^{2n} m_{i,\sigma(i)}) = \sum_{\alpha \subseteq [n]} \sum_{\sigma \in T_{\alpha}} \prod_{i=n+1}^{2n} m_{i,\sigma(i)}.
\]

The last equality holds since for each \( \sigma \in T_{\alpha} \), \( \prod_{i=1}^{n} m_{i,\sigma(i)} = 1 \).

Let \( \beta = \alpha \cup (\bar{\alpha} + n) \), where \( \bar{\alpha} + n \) is the set containing \( i + n \) for all \( i \in \bar{\alpha} \). For each \( \alpha \subseteq [n] \) we have:

\[
\sum_{\sigma \in T_{\alpha}} \prod_{i=n+1}^{2n} m_{i,\sigma(i)} = \text{perm}(M_{[2n]\backslash[n],[2n]\backslash\beta}),
\]

where \( M_{[2n]\backslash[n],[2n]\backslash\beta} \) is the matrix \( M \) consisting of the rows indexed in the set \([2n]\backslash[n]\) and the columns indexed in the set \([2n]\backslash\beta\). For each \( \alpha \subseteq [n] \), \( M_{[2n]\backslash[n],[2n]\backslash\beta} \) is the matrix formed by selecting the \( n \) columns not indexed by \( \beta \) of \( I \ P \). The columns indexed by \( [2n]\backslash\beta \) are the columns not chosen from the first \( 1 \) to \( n \) columns of \( M \) by \( \alpha \) and the \( n + 1 \) to \( 2n \) columns of \( M \) not chosen by \( \bar{\alpha} \). However, the columns not selected from \( 1 \) to \( n \) by \( \alpha \) are the columns of \( 1 \) to \( n \) indexed by \( [n] - \alpha = \bar{\alpha} \) and similarly the columns not selected from \( n + 1 \) to \( 2n \) by \( \bar{\alpha} \) are the columns of \( 1 \) to \( n \) indexed by \( [n] - \bar{\alpha} = \alpha \). Thus for each \( \alpha \subseteq [n] \),

\[
M_{[2n]\backslash[n],[2n]\backslash\beta} = \begin{bmatrix} I_{\bar{\alpha}} & P_{\alpha} \end{bmatrix}
\]

We have shown

\[
\text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix} = \sum_{\alpha \subseteq [n]} \text{perm} \begin{bmatrix} I_{\bar{\alpha}} & P_{\alpha} \end{bmatrix}.
\]

The following Lemma will allows us to conclude that

\[
\text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix} = \text{perm}(I + P)
\]
Lemma 4.16. Let $A$ and $B$ be square matrices of size $n$ with non negative real entries then

$$
\sum_{\alpha \subseteq [n]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right] = \text{perm}(A + B)
$$

Proof. We will proceed by way of induction on $n$, the size of our matrices. Let $n = 1$, then $A = \begin{bmatrix} a_{1,1} \end{bmatrix}$ and $B = \begin{bmatrix} b_{1,1} \end{bmatrix}$. Then, we have two choices for $\alpha$ and $\bar{\alpha}$ as described in Example 4.13. By direct computation:

$$
\sum_{\alpha \subseteq [n]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right] = \text{perm}(a_{1,1}) + \text{perm}(b_{1,1}) = a_{1,1} + b_{1,1}.
$$

Similarly by direct computation:

$$
\text{perm}(A + B) = \text{perm}\left(\begin{bmatrix} a_{1,1} + b_{1,1} \end{bmatrix}\right) = a_{1,1} + b_{1,1}.
$$

Assume $\sum_{\alpha \subseteq [n]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right] = \text{perm}(A + B)$ for $n = k$. We will now prove this holds for $n = k + 1$.

Let $n = k + 1$ and let $A$ and $B$ be be square matrices of size $(k + 1)$ with non negative real entries $a_{i,j}$ and $b_{i,j}$ respectively, then we have the following:

$$
\sum_{\alpha \subseteq [k+1]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right] =
$$

$$
= \sum_{1 \not\in \alpha \subseteq [k+1]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right] + \sum_{1 \in \alpha \subseteq [k+1]} \text{perm} \left[ \begin{array}{cc} A_{\bar{\alpha}} & B_{\alpha} \end{array} \right].
$$

In the first the summation, our restriction of $1 \not\in \alpha$ is equivalent to always selecting the first column of the matrix $A$, while simultaneously never selecting the first column of the matrix $B$. Let $a_1$ be the first column of $A$, $a_1 = \begin{bmatrix} a_{1,1} & \cdots & a_{k+1,1} \end{bmatrix}^T$. Since we have fixed $1 \not\in \alpha \subseteq [k+1]$, we may only select $k$ more columns from the matrices $A^{(-1)}$ and $B^{(-1)}$ to use in the matrix $\begin{bmatrix} A_{\bar{\alpha}} & B_{\alpha} \end{bmatrix}$, where $M^{(-j)}$ is the matrix $M$ without its $j^{th}$ column. Similarly for our second summation, we always select the first column of the matrix $B$. Let $b_1$ be the first column of $B$, $b_1 = \begin{bmatrix} b_{1,1} & \cdots & b_{k+1,1} \end{bmatrix}^T$. The we may rewrite our summation as follows:

$$
= \sum_{\alpha_1 \subseteq [k+1] \setminus \{1\}} \text{perm} \left[ \begin{array}{cc} a_{1} & A_{\alpha_1} \ B_{\alpha_1} \end{array} \right] + \sum_{\alpha_1 \subseteq [k+1] \setminus \{1\}} \text{perm} \left[ \begin{array}{cc} A^{(-1)}_{\alpha_1} & b_{1} \ B^{(-1)}_{\alpha_1} \end{array} \right],
$$

where the compliment of the set $\alpha_1$ is $\bar{\alpha}_1 = [k+1] \setminus \{1\} \setminus \alpha_1$. In our first summation we may do a cofactor expansion along column $a_1$ and similarly in our second summation we may do a cofactor expasion along column $b_1$.

$$
= \sum_{i=1}^{k+1} a_{i,1} \sum_{\alpha_1 \subseteq [k+1] \setminus \{1\}} \text{perm} \left[ \begin{array}{cc} A^{(i,-)}_{\alpha_1} & B^{(i,-)}_{\alpha_1} \end{array} \right] + \sum_{i=1}^{k+1} b_{i,1} \sum_{\alpha_1 \subseteq [k+1] \setminus \{1\}} \text{perm} \left[ \begin{array}{cc} A^{(i,-)}_{\alpha_1} & B^{(i,-)}_{\alpha_1} \end{array} \right],
$$
For each \( i \), we now select our columns from the matrices \( A^{(i,1)} \) and \( B^{(i,1)} \) to use in the matrix 
\[
\begin{bmatrix}
A^{(i,-)}_{\alpha_1} & B^{(i,-)}_{\alpha_1}
\end{bmatrix},
\]
where the \( i^{th} \) row was removed by the cofactor expansion along row \( i \).

For each \( i = 1, \ldots, k + 1 \), let \( R^i = A^{(i,1)} \) and \( S^i = B^{(i,1)} \), then
\[
\sum_{\alpha_1 \subseteq [k+1] \setminus \{1\}} \text{perm}
\begin{bmatrix}
A^{(i,-)}_{\alpha_1} & B^{(i,-)}_{\alpha_1}
\end{bmatrix} = \sum_{\alpha_2 \subseteq [k]} \text{perm}
\begin{bmatrix}
R^i_{\alpha_2} & S^i_{\alpha_2}
\end{bmatrix}.
\]
Rewriting our equation we obtain:
\[
= \sum_{i=1}^{k+1} a_{i,1} \sum_{\alpha_2 \subseteq [k]} \text{perm}
\begin{bmatrix}
R^i_{\alpha_2} & S^i_{\alpha_2}
\end{bmatrix} + \sum_{i=1}^{k+1} b_{i,1} \sum_{\alpha_2 \subseteq [k]} \text{perm}
\begin{bmatrix}
R^i_{\alpha_2} & S^i_{\alpha_2}
\end{bmatrix},
\]
Notice that \( R^i \) and \( S^i \) are square matrices of size \( k \) with non negative real entries, thus we may apply our induction hypothesis and obtain:
\[
= \sum_{i=1}^{k+1} a_{i,1} \text{perm}(R^i + S^i) + \sum_{i=1}^{k+1} b_{i,1} \text{perm}(R^i + S^i)
\]
Combining like terms we obtain:
\[
= \sum_{i=1}^{k+1} a_{i,1} + b_{i,1} \text{perm}(R^i + S^i)
\]
Which is just the cofactor expansion along the first column of the matrix \( A + B \)
\[
= \text{perm}(A + B)
\]
\[\square\]
Since \( I \) and \( P \) are square matrices with non negative real entries, we may set \( A = I \) and \( B = P \) to obtain our desired result in Proposition 4.15. Thus,
\[
\text{perm}
\begin{bmatrix}
I & I \\
I & P
\end{bmatrix} = \text{perm}(I + P)
\]
\[\square\]
Now we shall give an example to illustrate Proposition 4.15, Equation (4.5) and Equation (4.7).

**Example 4.17.** Consider the 2-lift \( \tilde{H} \) of \( H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)
\[
\tilde{H} = \begin{bmatrix} I & I \\ I & P \end{bmatrix}, \text{ where } P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
By direct computation \( \text{perm}(\tilde{H}) = \text{perm} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = 2 \). Computing the permanent of \( \tilde{H} \) by applying Lemma 4.15 we obtain:
\[
\text{perm}(\tilde{H}) = \text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix} = \text{perm}(I + P) = \text{perm} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2
\]

We may also compute the permanent of \( \tilde{H} \) utilizing Equation (4.5).
\[
\text{perm}(\tilde{H}) = \sum_{\alpha \subseteq [n]} \sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)}
\]

We have computed the values from Equation (4.5) for 4 possible sets of \( \alpha \) and \( \bar{\alpha} \) with their corresponding \( T_{\alpha} \)’s as discussed in Example 4.14:

\[ \alpha = \emptyset, \bar{\alpha} = \{1, 2\} \text{ then, } T_{\alpha} = \{(13)(24), (1324)\} \]
\[
\sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)} = \tilde{h}_{3,3} \tilde{h}_{4,2} + \tilde{h}_{3,2} \tilde{h}_{4,1} = 1 + 0
\]

\[ \alpha = \{1\}, \bar{\alpha} = \{2\} \text{ then, } T_{\alpha} = \{(1)(3)(24), (1)(243)\} \]
\[
\sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)} = \tilde{h}_{3,3} \tilde{h}_{4,2} + \tilde{h}_{3,2} \tilde{h}_{4,1} = 0 + 0
\]

\[ \alpha = \{2\}, \bar{\alpha} = \{1\} \text{ then, } T_{\alpha} = \{(2)(13)(4), (2)(134)\} \]
\[
\sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)} = \tilde{h}_{3,3} \tilde{h}_{4,4} + \tilde{h}_{3,4} \tilde{h}_{4,1} = 0 + 0
\]

\[ \alpha = \{1, 2\}, \bar{\alpha} = \emptyset \text{ then, } T_{\alpha} = \{(1)(2)(3)(4), (1)(2)(34)\} \]
\[
\sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)} = \tilde{h}_{3,3} \tilde{h}_{4,4} + \tilde{h}_{3,4} \tilde{h}_{4,3} = 0 + 1
\]

Summing we obtain:
\[
\sum_{\alpha \subseteq [n]} \sum_{\sigma \in T_{\alpha}} \prod_{i=3}^{4} \tilde{h}_{i,\sigma(i)} = 2
\]

Lastly, we compute the permanent of \( \tilde{H} \) utilizing Equation (4.7).
\[
\text{perm}(\tilde{H}) = \sum_{\alpha \subseteq [2]} \text{perm} \begin{bmatrix} I_{\bar{\alpha}} & P_{\alpha} \end{bmatrix}
\]
We have the following 4 possible sets for \( \alpha \) and \( \bar{\alpha} \) with their corresponding matrices:

\[
\alpha = \emptyset, \bar{\alpha} = \{1, 2\}, \quad [I_{\bar{\alpha}} \, P_{\alpha}] = I
\]

\[
\alpha = \{1\}, \bar{\alpha} = \{2\}, \quad [I_{\bar{\alpha}} \, P_{\alpha}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
\alpha = \{2\}, \bar{\alpha} = \{1\}, \quad [I_{\bar{\alpha}} \, P_{\alpha}] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}
\]

\[
\alpha = \{1, 2\}, \bar{\alpha} = \emptyset, \quad [I_{\bar{\alpha}} \, P_{\alpha}] = P
\]

By direct computation we obtain the following:

\[
\text{perm}(I) + \text{perm}\left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right) + \text{perm}\left( \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right) + \text{perm}(P) = 2
\]

Recall that if \( i \in \alpha \) then \( \text{perm}\left( I_{\bar{\alpha}} \, P_{\alpha} \right) = 1 \) if \( \sigma_P(i) = \sigma(i) \) for \( i \in \bar{\alpha} \) where \( \sigma_P \) is the permutation associated to the matrix \( P \). Clearly \( \sigma_P(1) \neq \sigma(1) \) and \( \sigma_P(2) \neq \sigma(2) \). As stated, Lemma 4.15, Equation (4.5) and Equation (4.7) give us the permanent of \( \tilde{H} \).

We have now simplified our problem of computing the permanent of a block matrix, however we are still left with the problem of finding the average permanent. Once again we are tasked with trying to reduce the number of sums and products required to calculate the permanent.

### 4.3 Cycles and Orbits

In this section we focus our attention on calculating the Bethe permanent \( B_{\text{perm}_p}(H) \). We would like to find a formula for the average permanent of a matrix \( H \) over all possible lifts of size \( p \), \( \langle \text{perm}(H^Q) \rangle_{Q \in \psi_{m,n,p}} \). We begin by partitioning the symmetric group of \( n \) elements in terms of orbits and relate this new definition to our average permanent. Next we classify permutation matrices by the number of non-fixed rows and calculate the permanent for each of the possibilities. Finally we count the number of different ways a row can remain fixed and combine all of these results to find the average permanent of our special block matrix.

First we will define the orbit of an element \( x \) in a group \( G \) then, show how we can apply orbits in order to partition the symmetric group of \( n \) elements.

**Definition 4.18.** For a set \( A \) with \( 1 \in A \subseteq [n] \) define

\[
n_G = \{ \sigma \in S_n : \text{orb}_\sigma(1) = A \}
\]

\[
n_H = \bigcup_{A : |A| = k} n_G
\]
It is clear that the \( nG_A \) partition \( S_n \) because for each \( \sigma \in S_n \), \( \text{orb}_\sigma(1) \) is a uniquely defined subset of \([n]\) containing 1. Note that we could select any \( x = \{2, \ldots, n\} \in A \), instead of 1 to define \( nG_A \).

We will now give two examples of partitioning \( S_n \) utilizing Definition 4.18.

**Example 4.19.** Consider \( S_3 \). Recall that there are 6 permutations in \( S_3 \) namely
\[
S_3 = \{(1)(2)(3), (1)(23), (12)(3), (13)(2), (123), (132)\}
\]
We will now show that the \( 3G_A \) partition \( S_3 \).

For \( k = 1 \) we have one choice for our set \( A \) with it’s associated \( 3G_A \).
\[
A = \{1\}, 3G_A = \{(1)(2)(3), (1)(23)\}
\]
then,
\[
3H_1 = \{(1)(2)(3), (1)(23)\}
\]
For \( k = 2 \) we have the following 2 choices for our set \( A \) with their associated \( 3G_A \)’s.
\[
A = \{1, 2\}, 3G_A = \{(12)(3)\}
\]
\[
A = \{1, 3\}, 3G_A = \{(13)(2)\}
\]
then,
\[
3H_2 = \{(13)(2), (12)(3)\}
\]
For \( k = 3 \) we have one choice for our set \( A \) with it’s associated \( 3G_A \).
\[
A = \{1, 2, 3\}, 3G_A = \{(123), (132)\}
\]
then,
\[
3H_3 = \{(123), (132)\}
\]
We have \( S_3 = 3H_1 \cup 3H_2 \cup 3H_3 \).

**Example 4.20.** Consider \( S_4 \). Recall that there are 24 permutations in \( S_4 \).

For \( k = 1 \) we have one choice for our set \( A \) with it’s associated \( 4G_A \).
\[
\]
then,
\[
\]
For \( k = 2 \) we have the following 3 choices for our set \( A \) with their associated \( 4G_A \)’s.

\[
A = \{1, 2\}, 4G_A = \{(12)(3)(4), (12)(34)\}
\]

\[
A = \{1, 3\}, 4G_A = \{(13)(2)(4), (13)(24)\}
\]

\[
A = \{1, 4\}, 4G_A = \{(14)(2)(3), (14)(23)\}
\]

then,

\[
\]

For \( k = 3 \) we have the following 3 choices for our set \( A \) with their associated \( 4G_A \)’s.

\[
A = \{1, 2, 3\}, 4G_A = \{(123)(4), (132)(4)\}
\]

\[
A = \{1, 2, 4\}, 4G_A = \{(124)(3), (142)(3)\}
\]

\[
A = \{1, 3, 4\}, 4G_A = \{(134)(2), (143)(2)\}
\]

then,

\[
\]

For \( k = 4 \) we have one choice for our set \( A \) with it’s associated \( 4G_A \).

\[
A = \{1, 2, 3, 4\}, 4G_A = \{(1234), (1243), (1324), (1342), (1423), (1432)\}
\]

then,

\[
4H_4 = \{(1234), (1243), (1324), (1342), (1423), (1432)\}
\]

We have \( S_4 = 4H_1 \cup 4H_2 \cup 4H_3 \cup 4H_4 \).

The next lemma tells us the size of \( nG_A \) and \( nH_k \)

**Lemma 4.21.** Let \( n \in \mathbb{N} \). Then,

\[
|nG_A| = (k - 1)!(n - k)!
\]

and

\[
|nH_k| = (n - 1)!
\]

**Proof.** Let \( n \in \mathbb{N} \). For any \( k \in \{1, 2, \cdots, n\} \) and for a fixed \( A \) such that \( 1 \in A \) we have the following: \( |nG_A| \) is the number of permutations \( \sigma \in S_n \) such that \( \text{orb}_\sigma(1) = A \). Notice that since 1 is in the set \( A \) and \( A \) is fixed there exists \( k - 1 \) other elements in the orbit of 1. For each \( A \) these \( k - 1 \) elements are fixed, however there are \((k - 1)!\) ways to rearrange them. Similarly the \( n - k \) element not in \( A \) can be arranged in \((n - k)!\) ways. Thus for a fixed \( A \), \( |nG_A| = (k - 1)!(n - k)! \). To find the size of \( nH_k \) we see that we are actually counting the
number of possible fixed sets $A$ of size $k$ times $|nG_A| = (k - 1)!(n - k)!$ for each $A$. Since 1 is already fixed in $A$ there are only $k - 1$ other elements that may be in $A$. These $k - 1$ elements may be chosen from the remaining $n - 1$ elements left in $[n]$, with the one element already removed being 1. So the number of possible sets $A$ is equal to $\binom{n-1}{k-1}$.

Thus we have

$$|nH_k| = \binom{n-1}{k-1}(k - 1)!(n - k)! = \frac{(n - 1)!}{(k - 1)!(n - k)!}(k - 1)!(n - k)! = (n - 1)!$$

Example 4.22. Revisiting Examples 4.19 and 4.20 it is easy to see that Lemma 4.21 is satisfied.

We still need to find a way to calculate the permanent of our resulting matrix $[I + P]$. We first introduce a solution to finding the permanent of the matrix $[I + P]$, then we find the sum of all permanents as $P$ varies in $\mathcal{R}_n$ and divide by the number of choices to get the average. In order to find the permanent of our special matrix, we will use the fact that row and column permutations do not affect the permanent.

Before proceeding with an example we will need to find the permanent of a special block matrix which will occur in our proof.

**Proposition 4.23.** Let $A$ be an $n \times n$ matrix, let $B$ be a $n \times m$ matrix and let $C$ be a $n \times m$ matrix. Then we have the following:

$$\text{perm}\left( \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \right) = \text{perm}(A) \text{perm}(C)$$

**Proof.** We will prove this using induction on $n$, the size of our matrix $A$. Let $D = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

For $n = 1$ we have the matrix $A = [a_{1,1}]$ and we obtain the following block matrix :

$$\text{perm}\left( \begin{bmatrix} a_{1,1} & B \\ 0 & C \end{bmatrix} \right) = a_{1,1} \times \text{perm}(C) = \text{perm}(A) \text{perm}(C)$$

Assume $\text{perm}\left( \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \right) = \text{perm}(A) \text{perm}(C)$ for $n \leq k$, we will show that it is also true for $n = k + 1$. Let $n = k + 1$, we begin taking the permanent of our block matrix $D$ by selecting the first column as our point of expansion. In general we will have:

$$\sum_{i=1}^{2n} d_{i,1} \text{perm}(D_{i,1})$$

where, $D_{i,1}$ represents the matrix $D$ with its $i^{th}$ row and $1^{st}$ column removed.

(4.8)
By our construction of $D$ we have:

$$d_{i,1} = \begin{cases} a_{i,1} & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases} \quad (4.9)$$

Then Equation (4.8) may be rewritten as the following:

$$\sum_{i=1}^{n} a_{i,1} \operatorname{perm}(D_{i,1}) \quad (4.10)$$

Notice that for every $i = 1, \cdots, n$ the matrix $D_{i,1}$ is actually the matrix

$$\begin{bmatrix} A_{i,1} & B_{i,0} \\ 0_{0,1} & C \end{bmatrix},$$

where $B_{i,0}$ is the matrix $B$ with only its $i^{th}$ row removed and $0_{0,1}$ is the zero matrix with only its $1^{st}$ row removed. Since $A_{i,1}$ is a matrix of size $k \times k$ we may invoke our induction hypothesis on the matrix $D_{i,1}$ as follows: $\operatorname{perm}(D_{i,1}) = \operatorname{perm}(A_{i,1}) \operatorname{perm}(C)$

Applying this to Equation (4.10) we obtain:

$$\sum_{i=1}^{n} a_{i,1} \operatorname{perm}(A_{i,1}) \operatorname{perm}(C)$$

Since $\operatorname{perm}(C)$ is independent of $i$ we may factor it out of our equation and obtain:

$$\operatorname{perm}(C) \times \sum_{i=1}^{n} a_{i,1} \operatorname{perm}(A_{i,1})$$

However $\sum_{i=1}^{n} a_{i,1} \operatorname{perm}(A_{i,1}) = \operatorname{perm}(C) \operatorname{perm}(A)$. Thus we have shown:

$$\operatorname{perm}(A) \operatorname{perm}(C)$$

Lemma 4.24. Let $P$ be a permutation matrix of size $n$ and let $\sigma$ be its associated permutation. Suppose $\sigma$ has a $k$-cycle in its factorization, where $k \leq n$. Let the factorization of $\sigma$ be $\sigma = \sigma_1\sigma_2$. Where $\sigma_1$ is the $k$-cycle and $\sigma_2$ contains the rest of $\sigma$. Let $J = [n] \setminus \text{supp}(\sigma_1)$, where $\text{supp}(\sigma_1)$ is the set of $k$ elements permuted by $\sigma_1$. Let $\pi : [n - k] \to J$ be a bijection. Then:

$$\operatorname{perm}(I + qP_{\sigma}) = (1 + q^k) \operatorname{perm}(I + qP_{\pi^{-1}\sigma\pi})$$

Where, $\pi^{-1}\sigma\pi$ has the same cycle structure as $\sigma_2$. 

\[\square\]
Proof. Let $P$ be a $n \times n$ permutation matrix and $q \in \mathbb{R}$. Let $\sigma \in S_n$ be the permutation associated to $P$ and let $\sigma_1 = (c_1, \cdots, c_k)$ describe one of the non-trivial $k$-cycles in $\sigma$’s factorization and $J = [n] \setminus \text{supp}(\sigma_1)$. By construction
\[ \text{colsum}(I + qP) = \text{rowsum}(I + qP) = 1 + q. \]
Since $\sigma$ contains at least one non trivial $k$-cycle, there exist $k$ rows and columns such that $q + 1$ is not the diagonal entry. In general, we have
\[
(I + qP)_{i,i} = 1 \text{ if } i \in \{c_1, \cdots, c_k\}
\]
Consider $T^{-1}(I + qP)T$, where $T$ is the permutation matrix whose columns are permuted by the $k$ transpositions $(1, c_1), \cdots, (k, c_k)$ and $T^{-1}$ is the permutation matrix whose rows are permuted by the $k$ transpositions $(1, c_1), \cdots, (k, c_k)$. By applying $T$ and $T^{-1}$ to our matrix $I + qP$ we obtain a triangular matrix: $T^{-1}(I + qP)T = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$ where $R$ is the $k \times k$ matrix defined as
\[
R = \begin{bmatrix}
1 & q & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q & 0 & \cdots & 0 & 1
\end{bmatrix}
\]
and $S$ is a $(n - k) \times (n - k)$ matrix. By corollary 4.23 we have that the permanent is equal to $\text{perm}(R) \text{ perm}(S)$. We will proceed to find the permanent by expanding along the first column. Notice that we have two non-zero entries in the first column, thus two possible rows to expand along. We will break the permanent into two cases. First we will expand along the 1 entry in row 1 column 1 and then we will expand along the $q$ entry in row $k$ column 1.

Begin by selecting the 1 entry in position $r_{1,1}$. By construction we have removed two $q$ entries in positions $r_{k,1}$ and $r_{1,2}$. Our new matrix is now of the form
\[
R_{1,1} = \begin{bmatrix}
1 & q & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]
Where $R_{1,1}$ is of size $(k - 1) \times (k - 1)$. We now need to find the permanent of the matrix $R_{1,1}$. We can select any row or column to expand upon. However by construction, the first column only contains one non-zero entry. By selecting this entry in position $(1, 1)$ of the matrix $R_{1,1}$ we remove a $q$ from entry $(1, 2)$ of the matrix $R_{1,1}$. We have removed the first row and column from the matrix $R_{1,1}$. We can rewrite this as the matrix $R_{\{1,2\}\{1,2\}}$ the matrix $R$
with column 1 row 1 and column 2 row 2 removed.

\[
R_{\{1,2\},\{1,2\}} = \begin{bmatrix}
1 & q & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]

We continue this process until we are left with the matrix \(R_{\{1,\ldots,k-1\},\{1,\ldots,k-1\}}\). The matrix \(R\) with the first \(k - 1\) rows and columns removed. This however is just the entry \(r_{k,k}\) which by construction is 1.

\[
R_{\{1,\ldots,k-1\},\{1,\ldots,k-1\}} = [1]
\]

Each of the \(k\) expansions were around a 1 entry in our matrix and by computing their product we obtain:

\[
(1)^k \text{perm}(S)
\]

Similarly we select the \(q\) entry from row \(k\) of column 1, and utilizing the same process above working from the lower right entry \(r_{k-1,k}\) and finishing at entry \(r_{1,2}\) we obtain:

\[
(q)^k \text{perm}(S)
\]

Summing up the two values we obtain \((1 + q^k)\text{perm}(S)\), where \(S\) is the matrix of size \((n - k) \times (n - k)\) of the form \(I + qV\) for some permutation matrix \(V\). Notice that \(V\) is the permutation matrix of size \(n - k\) that is dependent upon the permutation of the \(n - k\) elements in \(J\). Specifically, define a bijective function \(\pi : [n - k] \to J\). Then

\[
\text{perm}(S) = \text{perm}(I + qP_{\pi^{-1}\sigma\pi}).
\]

The last inequality holds since both \(V\) and \(P_{\pi^{-1}\sigma\pi}\) are dependent on the factorization of \(\sigma_2\) and since the permanent is unchanged under row or column permutations as described in Lemma 4.10. Combining these two results we obtain

\[
\text{perm}(I + qP_\sigma) = (1 + q^k)\text{perm}(I + qP_{\pi^{-1}\sigma\pi})
\]

Now we will proceed with an example to illustrate the proof of the Lemma.

**Example 4.25.** Let \(P\) be the \(5 \times 5\) permutation matrix with a 3-cycle. The associated permutation \(\sigma_P = (234)(15)\). Then \(\sigma_1 = (234)\) and \(\sigma_2 = (15)\). Let \(q \in \mathbb{R}\). Consider the matrix formed by the sum of the two matrices \(I\) and \(qP\):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & q \\
0 & 1 & q & 0 & 0 \\
0 & 0 & 1 & q & 0 \\
0 & q & 0 & 1 & 0 \\
q & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Let $T$ be the permutation matrix with the associated permutation $\sigma_T = (1, 2)(2, 3)(3, 4)$:

$$T = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Notice $T^{-1} = T^\top$, since permutation matrices are orthogonal.

Now we will calculate the permanent of the matrix $(I + qP)$. By Lemma 4.10 we know that:

$$\text{perm}(I + qP) = \text{perm}(T^{-1}(I + qP)T),$$

since $T$ is a permutation matrix. Applying $T$ and $T^{-1}$ to our matrix results in the following matrix:

$$T^{-1}(I + qP)T = \begin{bmatrix}
1 & q & 0 & 0 & 0 \\
0 & 1 & q & 0 & 0 \\
q & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & q \\
0 & 0 & 0 & q & 1
\end{bmatrix}$$

Where $S$, the matrix from the proof, is formed by the sum $I + qP_{\pi^{-1}\sigma\pi}$, where $\pi(1) \to 1$ and $\pi(2) \to 5$. By computing the permanent with a cofactor expansion along the first column we obtain:

$$\text{perm}(T^{-1}(I + qP)T) = q^3(q^2 + 1) + (q^2 + 1) = (q^3 + 1)(q^2 + 1) = (1 + q^3) \text{ perm}(I + qP_{\pi^{-1}\sigma\pi})$$

**Corollary 4.26.** If $P$ is a permutation matrix with only one non trivial orbit, $\sigma$ of size $k \leq n$, then:

$$\text{perm}(I + qP) = (1 + q^k)(1 + q)^{n-k}$$

**Proof.** By Lemma 4.24

$$\text{perm}(I + qP) = (1 + q^k) \text{ perm}(I + qP_{\pi^{-1}\sigma\pi})$$

Since we factored out the only non trivial cycle in $\sigma$, $P_{\pi^{-1}\sigma\pi} = I$

$$= (1 + q^k) \text{ perm}(I + qI) = (1 + q^k) \text{ perm}((1 + q)I) = (1 + q^k)(1 + q)^{n-k}$$

□
**Theorem 4.27.** Let $P$ be a $n \times n$ permutation matrix and $q \in \mathbb{R}$. Let $\sigma$ be the permutation associated to $P$ and let $k_1, k_2, \cdots, k_m$ be the length of the nontrivial cycles contained in $\sigma$’s factorization, so $M = \sum_{i=1}^{m} k_i$ and $M \leq n$. Then

$$\text{perm}(I + qP) = (1 + q)^{n-M} \prod_{i=1}^{m} (1 + q^{k_i})$$

Alternatively, let $a_k$ be the number of cycles of size $k$ in the factorization of $\sigma$ into disjoint cycles. Then,

$$\text{perm}(I + qP_\sigma) = \prod_{k=1}^{n} (1 + q^k)^{a_k}$$

**Proof.** Let $P$ be a $n \times n$ permutation matrix and let $q \in \mathbb{R}$. Let $\sigma$ be the permutation associated to $P$ and let $k_1, \cdots, k_m$ such that $M = \sum_{i=1}^{m} k_i$ and $M \leq n$ describe the non-trivial cycles in $\sigma$’s factorization. Select the first cycle of length $k_1$, then there exists a matrix $T_{k_1}$ and a matrix $T_{k_1}^{-1}$ such that:

$$T_{k_1}^{-1}(I + qP)T_{k_1} = \begin{bmatrix} R_1 & 0 \\ 0 & S_1 \end{bmatrix},$$

where $R$ is of size $k_1 \times k_1$ and $S$ is of size $(n - k_1) \times (n - k_1)$. By Lemma 4.24 we have:

$$\text{perm}(\begin{bmatrix} R_1 & 0 \\ 0 & S_1 \end{bmatrix}) = (1 + q^{k_1}) \text{perm}(S_1)$$

We continue this process of selecting cycles of length $k_i$ for $i = 2, \cdots, m - 1$. In each case there exists a $T_{k_i}$ and a $T_{k_i}^{-1}$ such that

$$T_{k_i}^{-1}(I + qP)T_{k_i} = \begin{bmatrix} R_i & 0 \\ 0 & S_i \end{bmatrix}$$

At each step we take the permanent of our matrix and by Lemma 4.24 we obtain:

$$\text{perm}(I + qP) = \prod_{i=1}^{m} (1 + q^{k_i}) \text{perm}(S_{m-1})$$

where $S_{m-1}$ is the matrix of size $(n - M + k_m) \times (n - M + k_m)$ with only one orbit of size $k_m$. By Corollary 4.27

$$\text{perm}(S_{m-1}) = (1 + q^{k_m})(1 + q)^{n-M}$$

Combining all of our terms we have:

$$\text{perm}(I + qP) = \prod_{i=1}^{m} (1 + q^{k_i})(1 + q)^{n-M}$$

We shall illustrate these properties by example.

**Example 4.28.** In this example we will find $\text{perm}(H^Q)_{Q \in \psi_{2,3}}$, the sum of the permanents of all possible 3-lifts for the matrix $\begin{bmatrix} I & I \\ I & qP \end{bmatrix}$, for $Q \in \psi_{2,3}$ and $q \in \mathbb{R}$. Let our matrix $P$ be of size $k \times k$, where $k \leq 3$. We will begin by using $G_{x,A}$ defined in Lemma 4.21.
Consider the case when $k = 1$.

$$\text{perm} \begin{bmatrix} 1 & 1 \\ 1 & q \end{bmatrix} = 1 + q$$

Now consider the case when $k = 2$.

If $|A| = 1$ we have,

$$\text{perm}(I + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}) = (1 + q)^2$$

If $|A| = 2$ we have,

$$\text{perm}(I + \begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix}) = (1 + q^2)$$

Summing the two cases:

$$= \sum_{|A|=1}^2 \left( \sum_{P \in \mathbb{G}_A} \text{perm}(I + qP) \right) = (1 + q^2) + (1 + q)^2 = 2(q^2 + q + 1)$$

Lastly, consider the case when $k = 3$. We may utilize the partition of $S_3$ by the $3H_k$ from Example 4.19.

When the size of $A$ is 1 we have:

$$3H_1 = \{(1)(2)(3), (1)(23)\}$$

By Corollary 4.26

$$\sum_{3G_A \in 3H_1} \sum_{P \in 3G_A} \text{perm}(I + qP) = (1 + q)^3 + (1 + q)(1 + q^2)$$

When the size of $A$ is 2 we have:

$$3H_2 = \{(12)(3), (13)(2)\}$$

By Corollary 4.26

$$\sum_{3G_A \in 3H_2} \sum_{P \in 3G_A} \text{perm}(I + qP) = 2(1 + q^2)(1 + q)$$

When the size of $A$ is 3 we have:

$$3H_3 = \{(123), (132)\}$$
By Corollary 4.26
\[ \sum_{|A|=1} \sum_{P \in \mathcal{A}_3} \text{perm}(I + qP) = 2(1 + q^3) \]

Summing the cases of \( k = 1, 2 \) and 3 we obtain:
\[ \sum_{|A|=1} \left( \sum_{P \in \mathcal{A}_3} \text{perm}(I + qP) \right) = (1 + q)^3 + (1 + q)(1 + q^2) + 2(1 + q^2)(1 + q) + 2(1 + q^3) = 3!(1 + q + q^2 + q^3) \]

We prove one more result that simplifies the calculation of the sum of permanents over a specific \( k \) cycle.

**Lemma 4.29.** Let \( n \in \mathbb{N} \) and suppose \( \sigma_1 \) is a \( k \)-cycle for \( k \leq n \). Define the set 
\[ T = \{ \sigma | \sigma_1 \text{ is contained in the factorization of } \sigma \in S_n \}. \]
Then 
\[ \sum_{\sigma \in T} \text{perm}(I + qP_\sigma) = (1 + q^k) \sum_{\tau \in S_{n-k}} \text{perm}(I + qP_\tau) \]

**Proof.** Suppose \( \sigma_1 \) is a \( k \)-cycle. Then define the set 
\[ T = \{ \sigma | \sigma_1 \text{ is contained in the factorization of } \sigma \} \text{ and } J = [n] \setminus \text{supp}(\sigma_1). \]
Notice that \( T \) may also be thought of as all \((n-k)!\) possible choices for the permutation of our set \( J \). Let 
\( \pi : [n-k] \to J \) be a bijection. We want to calculate \( \sum_{\sigma \in T} \text{perm}(1 + qP_\sigma) \). Then for each 
\( \sigma \in T \) we may apply Lemma 4.24 and obtain:
\[ \sum_{\sigma \in T} \text{perm}(I + qP_\sigma) = \sum_{\sigma \in T} (1 + q^k) \text{perm}(I + qP_{\pi^{-1} \sigma \pi}) \]
Since \((1 + q^k)\) is independent of \( \sigma \) we may factor it out of the sum
\[ = (1 + q^k) \sum_{\sigma \in T} \text{perm}(I + qP_{\pi^{-1} \sigma \pi}) \]
Since \( \pi \) is a bijection and each of the unique \((n-k)!\) \( \sigma \)'s of \( T \) are unchanged we have
\[ = (1 + q^k) \sum_{\sigma \in S_{n-k}} \text{perm}(I + qP_{\pi^{-1} \sigma \pi}) \]
\[ \square \]

**Corollary 4.30.** Let \( n \in \mathbb{N}, k \) be a positive integer such that \( k \leq n \). Then
\[ \sum_{\sigma \in H_k} \text{perm}(I + qP_\sigma) = \binom{n-1}{k-1} (k-1)! (1 + q^k) \sum_{\tau \in S_{n-k}} \text{perm}(I + qP_\tau) \]
Proof. Let \( n \in \mathbb{N} \) and \( k \) be a positive integer such that \( k \leq n \). Then for \( _nH_k \) select \( \sigma_1 = (1, \cdots, k) \) as a representative of all size \( k \) cycles. Notice since 1 has to be in \( \text{supp}(\sigma_1) \), there are \( \binom{n-1}{k-1} \) ways to select the other \( k - 1 \) elements in \( \sigma_1 \) and there are \( (k - 1)! \) ways to reorder the \( k - 1 \) elements in each choice of \( \sigma_1 \). So in total we have \( \binom{n-1}{k-1}(k - 1)! \) total choices for \( \sigma_1 \). Now using our \( \sigma_1 = (1, \cdots, k) \) representative define the set \( T = \{ \sigma | \sigma_1 \text{ is contained in the factorization of } \sigma \} \). We now have the following:

\[
\sum_{\sigma \in _nH_k} \text{perm}(I + qP_{\sigma}) = \binom{n-1}{k-1}(k - 1)! \sum_{\sigma \in T} \text{perm}(I + qP_{\sigma})
\]

(4.11)

By Lemma 4.29 we may rewrite this as:

\[
= \binom{n-1}{k-1}(k - 1)!(1 + q^k) \sum_{\tau \in S_{n-k}} \text{perm}(I + qP_{\tau})
\]

We will now give an example to illustrate Equation 4.11 and Corollary 4.30.

Example 4.31. Let \( n = 4 \) and \( k = 2 \). Let \( \sigma_1 = (12) \). Then the set


By direct calculation we have:

\[
\sum_{\sigma \in _4H_2} \text{perm}(I + qP_{\sigma}) = 3(1 + q^2)(1 + q)^2 + 3(1 + q^2)^2,
\]

where \( 3(1 + q^2)(1 + q)^2 \) is obtained by applying Corollary 4.26 to the 3 permutations \((12)(3)(4), (13)(2)(4) \) and \((14)(2)(3)\) that have only one non trivial orbit of size 2 and \( 3(1 + q^2)^2 \) is obtained by applying Theorem 4.27 to the 3 permutations \((12)(34), (13)(24) \) and \((14)(23)\) that have 2 non trivial orbits of size 2.

\[
= 3(1 + q^2)((1 + q^2) + (1 + q)^2]
= 3(1 + q^2) \sum_{\tau \in S_2} \text{perm}(I + qP_{\tau})
\]

As stated in Corollary 4.30 and where \( \sum_{\tau \in S_2} \text{perm}(I + qP_{\tau}) = (1 + q^2) + (1 + q)^2 \) was derived in Example 4.28
Similarly, using Equation (4.11), when \( \sigma_1 = (12) \) and \( T = \{(12)(3)(4), (12)(34)\} \) we have:

\[
\sum_{\sigma \in \mathcal{H}_2} \text{perm}(I + qP_\sigma) = 3 \sum_{\sigma \in T} \text{perm}(I + qP_\sigma)
\]

By direct calculation we have:

\[
= 3[(1 + q^2)(1 + q)^2 + (1 + q^2)^2],
\]

where \((1 + q^2)(1 + q)^2\) is obtained by applying Corollary 4.26 to the permutation \((12)(3)(4)\) which has only one non trivial orbit of size 2 and \((1 + q^2)^2\) is obtained by applying Theorem 4.27 to the permutation \((12)(34)\) which has 2 non trivial orbits of size 2.

In both cases, Equation (4.11) and Corollary 4.30 obtain the correct permanent.

We now have all of the tools required to find

\[
B_{\text{perm}_p}(H) = \langle \text{perm}(H^{\Delta Q}) \rangle_{Q \in \psi_{2,p}}.
\]

**Theorem 4.32.** Let \( q \in \mathbb{R} \). Then:

\[
\sum_{P \in \mathcal{R}_n} \text{perm} \begin{bmatrix} I & I \\ I & qP \end{bmatrix} = n! \times \sum_{i=0}^{n} q^i
\]

**Proof.** We will prove this by induction on \( k \). The base step of \( k = 1 \) is trivial. Now assume this is true for \( k \leq n \). Let \( k = n + 1 \). By Lemma 4.15,

\[
\sum_{P \in \mathcal{R}_{n+1}} \text{perm} \begin{bmatrix} I & I \\ I & qP \end{bmatrix} = \sum_{P \in \mathcal{R}_{n+1}} \text{perm}(I + qP)
\]

We partition \( \mathcal{R}_{n+1} \) using Definition 4.18. We may rewrite

\[
\sum_{P \in \mathcal{R}_{n+1}} \text{perm}(I + qP) = \sum_{k=1}^{n+1} \sum_{\sigma \in \mathcal{H}_k} \text{perm}(I + qP_\sigma) \quad (4.12)
\]

By Corollary 4.30 we may rewrite Equation (4.12) as the following:

\[
\sum_{k=1}^{n+1} \sum_{\sigma \in \mathcal{H}_k} \text{perm}(I + qP_\sigma) = \sum_{k=1}^{n+1} \binom{n}{k-1} (k-1)! (1 + q^k) \sum_{\tau \in S_{n+1-k}} \text{perm}(I + qP_\tau) \quad (4.13)
\]

Since \( I + qP_\tau \) is of size \((n + 1 - k) \times (n + 1 - k)\) we may apply our induction hypothesis to \( \text{perm}(I + qP_\tau) \) in Equation (4.13)
\[
\sum_{k=1}^{n+1} (k-1)! \binom{n}{k-1} (1+q^k) \sum_{\tau \in S_{n+1-k}} \text{perm}(I+qP_\tau) = \sum_{k=1}^{n+1} (k-1)! \binom{n}{k-1} (1+q^k)(n+1-k)! \sum_{i=0}^{n+1-k} q^i
\]

We may now rewrite our original sum as:

\[
\sum_{k=1}^{n+1} \sum_{\sigma \in S_{n+1} H_k} \text{perm}(I+qP_\sigma) = \left( \sum_{k=1}^{n+1} (1+q) \cdots + q^{n+1-k} \right) \sum_{k=1}^{n+1} (n)!(1+q+\cdots+q^{n+1})
\]

\[
= n! \left[ \sum_{k=1}^{n+1} \sum_{j=0}^{n+1-k} q^j + \sum_{k=1}^{n+1} \sum_{j=0}^{n+1-k} q^{k+j} \right]
\]

For \( k \geq 1, j \geq 0 \), let

\[
a_{k,j} = \begin{cases} 
1 & \text{if } k+j \leq n+1 \\
0 & \text{otherwise}
\end{cases}
\quad (4.14)
\]

By definition, if \( k \leq 0 \) or \( j < 0 \), \( a_{k,j} = 0 \).

\[
\sum_{k=1}^{n+1} \sum_{\sigma \in S_{n+1} H_k} \text{perm}(I+qP_\sigma) = n! \left[ \sum_{k=1}^{n+1} \sum_{j=0}^{n} a_{k,j}q^j + \sum_{k=1}^{n+1} \sum_{j=0}^{n} a_{k,j}q^{k+j} \right]
\]

Substituting \( m = k+j \),

\[
= n! \left[ \sum_{k=1}^{n+1} \sum_{j=0}^{n} a_{k,j}q^j + \sum_{k=1}^{n+1} \sum_{m=1}^{2n+1} a_{k,m-k}q^m \right]
\]

When \( m > n+1 \), \( a_{k,m-k} = 0 \) so our sum only needs to go from \( m = 1 \) to \( n+1 \). Thus we may rewrite our sum as:

\[
= n! \left[ \sum_{j=0}^{n} \sum_{k=1}^{n+1} a_{k,j}q^j + \sum_{m=1}^{n+1} \sum_{k=1}^{n+1} a_{k,m-k}q^m \right]
\]

Factoring out the term \( j = 0 \) and the term \( m = n+1 \) from their respective summations and combining like terms we obtain:

\[
= n! \left[ (n+1) + \sum_{m=1}^{n+1} q^m \sum_{k=1}^{n+1} (a_{k,m} + a_{k,m-k}) + (n+1)q^{n+1} \right]
\]

Notice for each \( m \), \( a_{k,m-k} \neq 0 \) when \( m-k \geq 0 \) or equivalently, when \( m \geq k \). This occurs for the \( m \) values of \( k, k+1, \ldots, m \). Similarly, \( a_{k,m} \geq 0 \) when \( k+m \leq n+1 \) or equivalently,
when \( k \leq n + 1 - m \). This occurs for the \( n + 1 - m \) values of \( k, k = 1, \ldots, n + 1 - m \). Thus for each \( m \), \( a_{k,m-k} + a_{k,m} = n + 1 - m + m = n + 1 \).

\[
= n! \left[ (n + 1) + \sum_{m=1}^{n} q^m(n + 1) + (n + 1)q^{n+1} \right]
= n! \left[ (n + 1) \sum_{m=0}^{n+1} q^m \right]
= (n + 1)! \sum_{m=0}^{n+1} q^m
\]

\[\square\]

**Corollary 4.33.** For \( q = 0 \) and for \( n \in \mathbb{N} \)

\[
\sum_{P \in \mathcal{R}_n} \text{perm} \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} = n!
\]

**Proof.** Applying our results from Theorem 4.32 with \( q = 0 \) we obtain:

\[
\sum_{P \in \mathcal{R}_n} \text{perm} \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} = (n)! \sum_{j=0}^{n} 0^j
\]

Where \( \sum_{j=0}^{n} 0^j = 1 \) since \( 0^0 = 1 \).

\[
= n!
\]

We may also see this directly from Proposition 4.15.

\[
\sum_{P \in \mathcal{R}_n} \text{perm}(I + qP) = \sum_{P \in \mathcal{R}_n} \text{perm}(I) = \sum_{P \in \mathcal{R}_n} (1) = n!
\]

\[\square\]

**Corollary 4.34.** Let \( H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) then,

\[
\mathcal{B}_{\text{perm}}(H) = 1
\]

and

\[
\mathcal{B}_{\text{perm}}(H) = 1.
\]
Proof. By definition,

\[
B_{\text{perm}}_k(H) = \sqrt[k]{\langle \text{perm}(\hat{H}^Q) \rangle_{Q \in \psi_{2,k}}}
\]

\[
= \sqrt[k]{\sum_{P \in \mathcal{R}_k} \text{perm}\left(\begin{bmatrix} I & I \\ I & 0 \end{bmatrix}\right)}
\]

Applying Corollary 4.33 to \(\sum_{P \in \mathcal{R}_k} \text{perm}\left(\begin{bmatrix} I & I \\ I & 0 \end{bmatrix}\right)\) we obtain:

\[
= \sqrt[\frac{k}{k!}]{\frac{(k)!}{k!}} = \sqrt[\frac{k}{k!}]{1} = 1
\]

Then for \(B_{\text{perm}}(H)\) we obtain:

\[
B_{\text{perm}}(H) = \limsup_{k \to \infty} \sqrt[k]{\langle \text{perm}(\hat{H}^Q) \rangle_{Q \in \psi_{2,k}}}
\]

\[
B_{\text{perm}}(H) = \limsup_{k \to \infty} 1
\]

\[
= 1
\]

\[\square\]

Corollary 4.35. For \(q = 1\) and for \(n \in \mathbb{N}\)

\[
\sum_{P \in \mathcal{R}_n} \text{perm}\left(\begin{bmatrix} I & I \\ I & P \end{bmatrix}\right) = (n + 1)!
\]

Proof. Let \(k \in \mathbb{N}\). Applying our results from Theorem 4.32 with \(q = 1\) we obtain:

\[
\sum_{P \in \mathcal{R}_k} \text{perm}\left(\begin{bmatrix} I & I \\ I & P \end{bmatrix}\right) = k! \sum_{i=0}^{k} (1) = k!(k + 1) = (k + 1)!
\]

\[\square\]

Corollary 4.36. Let \(H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\) then,

\[
B_{\text{perm}}_n(H) = \sqrt[n+1]{1}
\]

and

\[
B_{\text{perm}}(H) = 1.
\]
Proof. By definition,

\[
\mathcal{B}_{\text{perm}}(H) = \sqrt[k]{\langle \text{perm}(\tilde{H}^\uparrow Q) \rangle_{Q \in \psi_{2,k}}}
\]

\[
= \sqrt[k]{\sum_{P \in R_k} \text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix}}
\]

Applying Corollary 4.35 to \(\sum_{P \in R_k} \text{perm} \begin{bmatrix} I & I \\ I & P \end{bmatrix}\) we obtain:

\[
= \sqrt[k]{(k+1)!} = \sqrt[k]{k+1}
\]

Then for \(\mathcal{B}_{\text{perm}}(H)\) we obtain:

\[
\mathcal{B}_{\text{perm}}(H) = \lim_{k \to \infty} \sqrt[k]{\langle \text{perm}(\tilde{H}^\uparrow Q) \rangle_{Q \in \psi_{2,k}}}
\]

\[
= \lim_{k \to \infty} \sqrt[k]{k+1} = \sqrt[k]{k+1}
\]

\[
\log(\mathcal{B}_{\text{perm}}(H)) = \lim_{k \to \infty} \log((k+1)^{1/k})
\]

Applying the log power rule and L’Hôpital’s rule

\[
\log(\mathcal{B}_{\text{perm}}(H)) = \lim_{k \to \infty} \frac{\log(k+1)}{k} = 0
\]

Thus,

\[
e^{\log(\mathcal{B}_{\text{perm}}(H))} = e^0 \implies \mathcal{B}_{\text{perm}}(H) = 1
\]

\[
\text{Corollary 4.37.}
\]

\[
\langle \text{perm}(\tilde{H}^\uparrow Q) \rangle_{Q \in \psi_{2,n}} = \frac{(q^{n+1} - 1)}{q - 1} \text{ for } q \neq 1
\]

Proof. Recall that the average Sum of Values

Thus:

\[
\langle \text{perm}(\tilde{H}^\uparrow Q) \rangle_{Q \in \psi_{2,k}} = \frac{\sum_{P \in R_k} \text{perm} \begin{bmatrix} I & I \\ I & qP \end{bmatrix}}{k!}
\]

By Theorem 4.32
\[
\sum_{P \in \mathcal{R}_k} \text{perm} \begin{bmatrix}
I & I \\
I & qP
\end{bmatrix} = \frac{k!(q^{k+1} - 1)}{(q - 1)k!} = \frac{(q^{k+1} - 1)}{q - 1}
\]

We will now investigate the construction of a Bethe permanent vector for $2 \times n$ matrices.

### 4.4 Bethe Permanent Vectors

We begin by defining the Bethe perm vector and then find all of the Bethe perm vectors of size $2 \times 3$.

**Definition 4.38.** Let $H$ be a matrix of size $m \times n$ and let $p$ be a positive integer. For a matrix $Q \in \psi_{m,n,p}$ and a size $(m+1)$ subset $\beta$ of $[n]$, we define the degree $p$ Bethe permanent vector based on $\beta$ to be the vector $w^{\beta}_{B,p} \in \mathbb{R}^n$ with components:

\[
 w^{\beta}_{B,p,i} := \begin{cases} 
 \text{Bperm}_p(H^{\beta \setminus i}_{\beta \setminus i}) & \text{if } i \in \beta \\
 0 & \text{otherwise}
\end{cases}
\]  
(4.15)

As $p$ tends to infinity we obtain the Bethe permanent vector

\[
 w^{\beta}_{B} = \limsup_{p \to \infty} w^{\beta}_{B,p}
\]

To illustrate the definition we will find the Bethe permanent for all possible $2 \times 3$ binary matrices $H$ with no zero columns or rows.

**Proposition 4.39.** For a $2 \times 3$ binary matrix $H$ with no zero rows or columns, the degree $p$ Bethe permanent vector as stated in Definition 4.38 has to be proportional to one of the following vectors:

1. $w^{\beta}_{B,p} \propto (1,1,1)$
2. $w^{\beta}_{B,p} \propto (0,1,1)$
3. $w^{\beta}_{B,p} \propto (1,1,(p+1)^{1/p})$

While the Bethe permanent vector has to be proportional to:

1. $w_{B} \propto (1,1,1)$
2. $w_{B} \propto (0,1,1)$
Proof. Let $H$ be a binary matrix of size $2 \times 3$ with no zero rows or columns. Then, there are only 5 possible choices for $H$. For each possible choice of $H$ we will calculate $w_{B,p}$ and $w_B$.

Let $Q \in \psi_{2,3,p}$.

For $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, we have $H^\top Q = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \end{bmatrix}$.

Then,

$$w_{B,p} = (B_{\text{perm}} p \begin{bmatrix} P_{1,2} & P_{1,3} \\ P_{2,2} & P_{2,3} \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,3} \\ P_{2,1} & P_{2,3} \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,2} \end{bmatrix})$$

Applying Corollary 4.11:

$$w_{B,p} = (B_{\text{perm}} p \begin{bmatrix} I & I \\ I & P \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} I & I \\ I & P \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} I & I \end{bmatrix})$$

Applying Corollary 4.36,

$$w_{B,p} = ((p + 1)^{1\backslash p}, (p + 1)^{1\backslash p}, (p + 1)^{1\backslash p}) \propto (1, 1, 1)$$

As well as:

$$w_B = \lim_{p \to \infty} ((p + 1)^{1\backslash p}, (p + 1)^{1\backslash p}, (p + 1)^{1\backslash p}) = (1, 1, 1)$$

For $H = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have $H^\top Q = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \end{bmatrix}$.

Then,

$$w_{B,p} = (B_{\text{perm}} p \begin{bmatrix} P_{1,2} & P_{1,3} \\ 0 & P_{2,2} \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,3} \\ P_{2,1} & 0 \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,2} \end{bmatrix})$$

Applying Corollary 4.11

$$w_{B,p} = (B_{\text{perm}} p \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} I & I \end{bmatrix})$$

Applying Corollaries 4.34 and 4.36 we obtain:

$$w_{B,p} = (1, 1, (p + 1)^{1\backslash p})$$

As well as:

$$w_B = \lim_{p \to \infty} (1, 1, (p + 1)^{1\backslash p}) = (1, 1, 1).$$

For $H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, we have $H^\top Q = \begin{bmatrix} P_{1,1} & 0 & P_{1,3} \\ P_{2,1} & P_{2,2} & 0 \end{bmatrix}$.

Then,

$$w_{B,p} = (B_{\text{perm}} p \begin{bmatrix} 0 & P_{1,3} \\ P_{2,2} & 0 \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,3} \\ P_{2,1} & 0 \end{bmatrix}, B_{\text{perm}} p \begin{bmatrix} P_{1,1} & 0 \end{bmatrix})$$
Applying Corollary 4.11

$$w_{B,p} = (\mathcal{B}_{\text{perm}} p \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} I & 0 \\ I & I \end{bmatrix})$$

Applying Corollaries 4.34 and 4.36 we obtain:

$$w_{B,p} = (1, 1, 1)$$

As well as:

$$w_B = \lim_{p \to \infty} (1, 1, 1).$$

For $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, we have $H^\triangledown Q = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & 0 & 0 \end{bmatrix}$ Then,

$$w_{B,p} = (\mathcal{B}_{\text{perm}} p \begin{bmatrix} P_{1,2} & P_{1,3} \\ 0 & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,3} \\ P_{2,1} & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & 0 \end{bmatrix})$$

Applying Corollary 4.11

$$w_{B,p} = (\mathcal{B}_{\text{perm}} p \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} I & I \end{bmatrix})$$

Applying Corollaries 4.34 and 4.36 we obtain:

$$w_{B,p} = (0, 1, 1)$$

As well as:

$$w_B = \lim_{p \to \infty} (0, 1, 1) = (0, 1, 1).$$

For $H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, we have $H^\triangledown Q = \begin{bmatrix} 0 & P_{1,2} & P_{1,3} \\ P_{2,1} & 0 & 0 \end{bmatrix}$ Then,

$$w_{B,p} = (\mathcal{B}_{\text{perm}} p \begin{bmatrix} P_{1,2} & P_{1,3} \\ 0 & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} 0 & P_{1,3} \\ P_{2,1} & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} 0 & P_{1,2} \\ P_{2,1} & 0 \end{bmatrix})$$

Applying Corollary 4.11

$$w_{B,p} = (\mathcal{B}_{\text{perm}} p \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \mathcal{B}_{\text{perm}} p \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix})$$

Applying Corollaries 4.34 and 4.36 we obtain:

$$w_{B,p} = (0, 1, 1)$$
As well as:
\[ w_B = \lim_{p \to \infty} (0, 1, 1) = (0, 1, 1). \]

Any 2 × 3 binary matrix \( H \) with no zero rows or columns is permutation equivalent to one of those considered.

**Corollary 4.40.** For a 2 × n binary matrix \( H \) with no zero rows or columns, the degree \( p \) Bethe permanent vector and the Bethe permanent vector as stated in Definition 4.38 are pseudocodewords.

**Proof.** Let \( n \geq 3 \). Let \( H \) be a 2 × n binary matrix with no zero rows or columns. Without loss of generality, let \( \beta = \{1, 2, 3\} \). Then,
\[ w^\beta_{B,p} = (\text{Bperm}_p(H_{\beta \setminus 1}^{\uparrow Q_{\beta \setminus 1}}), \text{Bperm}_p(H_{\beta \setminus 2}^{\uparrow Q_{\beta \setminus 2}}), \text{Bperm}_p(H_{\beta \setminus 3}^{\uparrow Q_{\beta \setminus 3}}), 0, \ldots, 0) \]

Since there are \((n - 3)\) zeros following \( \text{Bperm}_p(H_{\beta \setminus 3}^{\uparrow Q_{\beta \setminus 3}}) \), we only need to show that the terms of \( w_{B,p} \) and \( w_B \) indexed by \( \beta \) satisfy the definition of a pseudocodeword based on the submatrix \( H^\beta \).

Since \( H^\uparrow_{\beta Q_{\beta}} \) is a binary matrix of size 2 × 3 with no zero rows or columns, we may apply Proposition 4.39. Thus, the degree \( p \) Bethe permanent vectors with their corresponding matrices \( H \) must be proportional to one of the following:

1. \( w^\beta_{B,p} \propto (1, 1, 1, 0, \ldots, 0) \) if \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) or \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \).
2. \( w^\beta_{B,p} \propto (0, 1, 1, 0, \ldots, 0) \) if \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) or \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).
3. \( w^\beta_{B,p} \propto (1, 1, (p + 1)^{\uparrow p}, 0, \ldots, 0) \) if \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \)

While the Bethe permanent vectors with their corresponding matrices have to be proportional to:

1. \( w_B \propto (1, 1, 1, 0, \ldots, 0) \) if \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) or \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \) or
\[ H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]
2. \( w_B \propto (0, 1, 1, 0, \ldots, 0) \) if \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) or \( H^\uparrow_{\beta Q_{\beta}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).
By direct calculation, each degree \( p \) Bethe permanent vector and Bethe permanent vector with their corresponding matrices satisfy Definition 3.1. Thus, they are pseudocodewords of their corresponding matrices.
CHAPTER 5

CONCLUSION

As stated in the introduction, the goal of this thesis was to further expand upon the unpublished work of Professor Smarandache in [2] and to clarify a few of her results. In doing so, we were able to include many examples and prove some useful properties about permanents. The most interesting concepts appear in Chapter 4 and have been highlighted below.

Since LDPC decoding requires a covering graph, one of our first results was proving that the \( p \)-lift of a permutation matrix results in a covering graph as stated here: Let \( H \) be a \( m \times n \) binary parity check matrix such that \( n \geq m \). Let \( G = (V, E) \) represent the associated bipartite graph. For any positive integer \( p \) and \( P \in \psi_{m,n,p} \), the graph \( G_1 = (V_1, E_1) \) representing the matrix \( H^TP \) is a \( p \)-lift of our graph \( G = (V, E) \) described by \( H \). This proof allowed us to understand the Bethe permanent definition \( B_{\text{perm}}(H) \triangleq \sqrt{\langle \text{perm}(H^TP) \rangle_{P \in \psi_{m,p}}} \), where the angular brackets \( \langle \rangle \) represent the arithmetic average of \( \text{perm}(H^TP) \) over all \( P \in \psi_{m,p} \) as stated in [2]. While trying to calculate the Bethe permanent of the matrix \( H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), we realized that for each \( p \)-lift we obtained a block matrix of the form \( \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \), where \( P, Q, R \) and \( S \) are \( p \times p \) permutation matrices. We were able to prove the following: Let \( P, Q, R \) be \( n \times n \) permutation matrices. Let \( S \) be a \( n \times n \) matrix with positive entries. Then the following holds:

\[
\text{perm} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \text{perm} \begin{bmatrix} I & I \\ I & T \end{bmatrix}, \quad \text{where} \quad T = PR^{-1}SQ^{-1}.
\]

Utilizing this result, we were then able to show that for a permutation matrix \( P \) and real number \( q \),

\[
\text{perm} \begin{bmatrix} I & I \\ I & qP \end{bmatrix} = \text{perm}(I + qP).
\]

Let \( \sigma \) be the permutation associated to \( P \) and let \( a_k \) be the number of cycles of size \( k \) in the factorization of \( \sigma \) into disjoint cycles. Then,

\[
\text{perm}(I + qP) = \prod_{k=1}^{n} (1 + q^k)^{a_k}.
\]
All that was left to calculate the Bethe permanent was to calculate the sum over all possible permutation matrices. To find this result we needed to create a partition of $S_n$, the symmetric group of $n$ elements. For a set $A$ with $1 \in A \subseteq [n]$ define

$$nG_A = \{\sigma \in S_n : \text{orb}_\sigma(1) = A\}$$

$$nH_k = \bigcup_{A:|A|=k} nG_A.$$ 

We then had the following Corollary: Let $n \in \mathbb{N}$, $k$ be a positive integer such that $k \leq n$. Then

$$\sum_{\sigma \in nH_k} \text{perm}(I + qP_\sigma) = \binom{n-1}{k-1} (k-1)! (1 + q^k) \sum_{\tau \in S_{n-k}} \text{perm}(I + qP_\tau)$$

Our final step produced the Bethe permanent of the $2 \times 2$ all ones matrix and we obtained the main result of this thesis. Let $q \in \mathbb{R}$, then,

$$\sum_{P \in \mathcal{R}_n} \text{perm} \left[ \begin{array}{cc} I & I \\ I & qP \end{array} \right] = n! \times \sum_{i=0}^{n} q^i.$$ 

Lastly, we showed that for a $2 \times n$ binary matrix $H$ with no zero rows or columns, the degree $p$ Bethe permanent vector and the Bethe permanent vector as stated in Definition 4.38 are pseudocodewords.

Through all of our work we were still unable to prove some of the material in Smarandache’s paper. We were unable to find any interesting pseudocodewords. We would like to continue this work and find the Bethe permanent for matrices larger than $2 \times 2$. Then we would like to prove that the Bethe permanent vector produces a pseudocodeword. As claimed in [2], for any $m \times n$ with $m \geq n$ binary matrix $H$ with no zero rows or columns, and for any size $(m + 1)$ subset $\beta$ of $[n]$, the degree $p$ Bethe permanent vector $w^{\beta}_{B,p}$ and $w^{\beta}_B$ are pseudocodewords of $H$. We are hoping that larger cases might produce more useful results.
BIBLIOGRAPHY


