PARTICLE TRANSPORT BY MEANS OF A VIBRATING MEMBRANE UNDER VARIABLE TENSION
INTENDED FOR THE DEPARTMENT OF MATHEMATICS AND STATISTICS

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To my wife Beulah and daughter Nichole whom I love dearly and dedicate my work to.
One can have no smaller or greater mastery than mastery of oneself.

–Leonardo Da Vinci
ABSTRACT OF THE THESIS

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The complex dynamical properties of particulate granular systems are yet to be fully understood by the scientific community. The need for a deeper understanding of granular systems is made evident by their commonplace status in both the natural and industrial environments. The formation of patterns within systems of particles subject to transversely driven vibrations is one focus of research in recent years that enhances an understanding of the connection between granular systems and their surroundings. The research suggests it is plausible that particulates of a granular assembly can be segregated and positioned via waves imposed on a medium’s surface. As a study in the non-contact management of granular particles, we investigate a vibrating membrane with variable tension. The model is motivated on grounds that it may broaden an understanding of particle transport through the analysis of the non-uniform membrane’s vibrational characteristics.

The equations of motion of the membrane are developed via the constitutive relations of an elastic membrane as set forth by the theory of partial differential equations. Interesting eigenfunctions satisfying the Helmholtz equation are derived for which their eigenvalues are used to analyze and manipulate the membrane’s motion through a range of tensile forces. The membrane is subjected to external homogenous periodic forcing of frequency $\omega$ and when $\omega$ is the same as a natural frequency of the membrane then resonance occurs. At resonance, standing waves similar to the membrane’s analytical eigenfunctions give rise to quiescent nodal curves. Particles distributed across the membrane’s surface are assumed to accumulate towards and rest along the nodal curves.

Variable tension is an attempt to control node/particle location. By adjusting the tension gradient along the flow axis, node location is shifted to either the left or the right for each setting of the tension gradient. Transient behavior of the wave modes under changes to the tensile force are revealed by analyzing the resonant modes spectral
decomposition. It is observed that while the forcing frequency corresponding to resonant wave modes agrees with predicted frequency response values the wave structure is sensitive to changing tensile conditions while under forced oscillations.
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CHAPTER 1
ON THE DYNAMICAL SYSTEMS COVERED
IN THIS WORK

1.1 VERTICALLY VIBRATED GRANULAR MEDIA

The goal of the theory of dynamical systems is to describe Nature’s changing and sometimes unexpectedly peculiar observable phenomena and attempt to understand the fundamental mechanisms responsible for its complex behavior. A experiment conducted in the late 1700’s serves as a clear example of Nature’s complex and beautiful tendencies. In 1787, the German physicist Ernst Chladni showed how sand particles could be made to self-organize into symmetrical patterns [11]. In his experiments, Chladni showed how the excitation of a regular shaped metal plate with a violin bow would result in various modes of vibration which induced fascinating geometrical patterns of sand along the plate’s vibratory nodal lines. Chladni’s figures, shown in Fig. 1.1, are considered to be the scientific and artistic birthplace of not only the physics of wave motion as studied in this paper but also the modern field of quantum chaos [11, 12]. Different Chladni figures are produced depending on the shape of the excited plate. As explained in Ref. [12], for regularly shaped plates one finds a web of intersecting circles and straight lines such as those in Fig. 1.1. However, irregularly shaped plates give rise to chaotic particle motion typical of the quantum mechanics of chaotic billiards. Although different shapes of plates give different Chladni figures, as shown in Fig. 1.1, any plate will give a characteristic sequence of patterns. Also, Chladni figures are not just confined to pure mathematical pursuits, a current practical application of Chladni figures is their use in the study of resonance and the vibrational characteristics of pianos and string instruments [12].

Still today, granular pattern formation similar to that of Chladni figures continues to intrigue the scientific imagination as novel aspects of pattern formation and the vibratory nature of granular materials have been intensely studied recently. In the research, simplified models are constructed to study various pattern formations of
granular assemblies subject to transversely induced vibrations of various frequencies. Reports include line and hexagon patterns with localized areas of excitation being a general feature of the particulate systems [1, 8]. A striking outcome of some simulations show that although no standing surface waves are present in the simulations, the systems tend to organize themselves into periodic peaks with a dependence on forcing frequency and system size [3]. As a final example, in Ref. [4] it is discovered that the geometry of the container has a dramatic effect on the motion of a large collection of particles wherein placed. When the system is driven by transverse forced vibrations, chaotic evolution is observed resultanting in convection rolls.

The above observations from a broad range of scientists show that there exists many interesting aspects in several configurations of vibrated granular systems. Therefore, in this paper we aim to study not only some of the fundamental physical aspects leading to granular pattern formation but also how these systems may be organized and manipulated via adjustment of the membrane’s tension. As a good starting point for an exploration into these curious and unusual systems we would like to start with a concept as fundamental as the Chladni patterns themselves: vibratory motion.

1.2 The Concept of Vibration

The vibration of systems has been studied since mankind’s early venture into scientific reasoning with origins most likely involving the vibrational characteristics of musical instruments [11]. Nowadays, the concept of vibration has obviously broadened and we consider the theory of vibration as dealing with the study of the oscillatory motion of a medium and its associated forces. In what follows, it will be shown that the aforementioned notion of vibration is by no means described within any self-contained point of view but instead involves an elegant relationship between the physical concepts of energy and motion. Consider first, that by definition, any repetitive motion about a medium’s equilibrium position is called a vibration or oscillation [9]. However, note the phenomena of vibration can be equivalently considered to be a complex evolution of the system’s energy levels as the relationship between the medium and its vibration involves a periodic exchange of potential and kinetic energy [11]. Thus, we have one description of vibration involving forces governed by equations of motion while the latter depicts the
Figure 1.1. Chladni Patterns published by John Tyndall in 1869 and from the UCLA Physics Lab demonstrations web site. The figures illustrate two important features: (1) areas that are vibrating and (2) areas that are not vibrating. When a flat elastic medium is put into vibratory motion, the medium oscillates as a whole and as parts. The non-vibrating quiescent boundaries between the vibrating areas are called nodal lines. If a granular material (shown in white) is put onto the vibrating plate, it collects along the nodal lines. The oscillating areas (black) thus become empty.

system as expressed by equations of state. A simple qualitative chain of reasoning not only shows how these two concepts relate to one another but also sets a mental focus regarding topics to follow later in this manuscript. Let us consider the ideas of motion and energy in vibrating systems a little further.

There are two primary components involved with the exchange of a system’s vibrational energy which are: (1) an elastic element; such as a spring, and (2) an inertial element; such as a particle. The energy exchange occurs when the elastic element stores potential energy and gives it up to the inertial element as kinetic energy [11]. To see the concept of vibration more explicitly, let a particle of mass \( m \) be fastened to the end of a
spring which sits vertically upon a flat surface as depicted in Fig. 1.2. It is easy to imagine the spring element would have dimensions such that its fully stretched position can be measured as residing at a point \( x_{\text{max}} \) and its compressed position measured at a point \(-x_{\text{max}}\). If left at rest the mass should be located at a position, known as the system’s equilibrium point; say \( x_{\text{eq}} = 0 \). As a qualitative statement about the system, we should agree that the spring is of the sorts such that if \( m \) is displaced from \( x_{\text{eq}} \) to some point \( x \in [-x_{\text{max}}, x_{\text{max}}] \) then it \textit{works} to restore \( m \) toward \( x_{\text{eq}} \).

Now, to give a quantitative explanation to the phenomena we can simply repeat the qualitative statement using mathematical symbols. Let \( m \) be displaced to a point \( x \) so that the work \( W \) required to move \( m \) back to \( x_{\text{eq}} \) is

\[
W = \int_{x_{\text{eq}}}^{x_{\text{eq}}} F_i(x) \, dx. \quad (1.1)
\]

In keeping tabs on the energy state of the system we should note that since \( m \) has now been displaced from \( x_{\text{eq}} \) it must follow that energy has been put into the system and this energy is called \textit{potential} energy (P.E.) since the spring now has the potential to move the mass. Consider the force \( F(x) \) to be described as the stress-strain constitutive relation for linear materials, \( F_1(x) = -kx \), where \( k \) is the force constant of the spring. The force \( F_1(x) \) is commonly known as Hook’s Law. Upon completing the integration on Eq. (1.1) we have

\[
P.E. = -\frac{1}{2}kx^2, \quad (1.2)
\]

which is precisely the equation for mechanical potential energy.

However the \textit{motion} of the particle from \( x \) to \( x_{eq} \) is governed by Newton’s second law which reads

\[
F_2(x) = ma = m \frac{dv}{dt}. \quad (1.3)
\]

Inserting Eq. (1.3) into Eq. (1.1) gives

\[
\int_{x}^{x_{\text{eq}}} m \frac{dv}{dt} \, dx = \int_{v}^{0} \frac{dx}{dt} \, dv = m \int_{v}^{0} v \, dv = \frac{1}{2}mv^2, \quad (1.4)
\]

and we obtain the equation for kinetic energy of the mass

\[
\text{K.E.} = -\frac{1}{2}mv^2. \quad (1.5)
\]
Assuming the energy of the system must be conserved for all time implies

\[ K.E. + P.E. = E_{\text{tot}}. \]  

That is, the elastic element has given up the stored potential energy to the inertial element in the form of kinetic energy. Finally, noting that we have two equivalent equations for the force exerted on \( m \) given by Hook’s and Newton’s second law we get the equation of the mass motion through the entire exchange of energy given as

\[ m \frac{d^2x}{dt^2} = -kx. \]  

In summary, we begun with a simple qualitative statement on the spring phenomena and rewrote that statement in its equivalent mathematical form as Eq. (1.1). From there we used force as expressed in terms of Hooks law to achieve Eq. (1.2) known as potential energy. Then using Newton’s second law we where able to arrive at Eq. (1.3) for the mass kinetic energy. Finally, equating the two forces present on the mass we got the equation of the mass motion during the exchange of potential and kinetic energy.

The following types of vibration will be considered in this manuscript:
1. **Free and Forced Vibrations.** Free vibration occurs when a mechanical system is set off with an initial condition and then allowed to evolve freely without the application of any external forcing after time zero [11]. An example of this type of vibration is pulling a child back on a swing and then letting go or hitting a tuning fork and letting it ring. Once set into motion, the mechanical system will then vibrate at one or more of its natural frequencies and damp down to zero. On the other hand, if the system vibrates due to the application of an external force then it is said the system is under forced vibrations. Examples of this type of vibration include a shaking washing machine due to an imbalance or the vibration of a building during an earthquake.

2. **Damped and Undamped Vibrations.** If there is no loss or dissipation of energy, due to friction or other resistance during vibration, the system is said to be undamped. If there is loss of energy due to the presence of damping then the system is said to be damped. A consideration of damping becomes extremely important when a system vibrates near its resonant frequency [11].

### 1.3 Preview

The manuscript is arranged as follows. We start off in Chap. 2 by discussing the mathematical properties of the non-uniform membrane through an initial analysis of the one-dimensional wave equation with variable tension gradient. The analysis leads into Chap. 3 where numerical techniques are employed to fully define the eigenfunctions and eigenvalues. In Chap. 4, the equation for the non-uniform string subject to external periodic forcing and damping is numerically integrated. Using a fourth order Runge-Kutta method and methods from ordinary differential equations many of the temporal characteristics of the non-uniform string are illustrated. In Chap. 5, we get our first glimpse into the physical implications of the non-uniform tension gradient. Implementation of some concepts from theoretical physics allows us to define the energy density of the string and understand vibratory areas in a deeper sense. A side benefit of our energy analysis is it allows a simple and rapid formulation of the equations of motion for the non-uniform membrane. In Chap. 6, numerical experiments are conducted as we see the first implications of tension adjustment on a transversely vibrated surface. Finally,
in Chap. 7, we summarize our results and conclude with some possible avenues for future research.
CHAPTER 2

ANALYSIS OF THE WAVE EQUATION WITH A VARIABLE TENSION GRADIENT

2.1 The Physical Setup of the Non-uniform Membrane

This manuscript will focus on a specific type of mechanical wave known as an elastic wave. Generally, waves can occur whenever a system is disturbed from its equilibrium state and the disturbance travels or propagates from one region of the system to another [9]. Well known examples of mechanical waves are sound waves and water waves and, in the spirit of this paper, granular waves. In regards to vibrational energy as discussed in Chap. 1, the inertial elements of the preceding examples are air molecules, water molecules and particulates of the granular system. The elastic elements differ between these systems too and are determined by the constitutive relations of the medium. The different elastic properties of the examples are air and water’s compressibility and the granular collection’s bulk and shear modulus. However, a global characteristic of mechanical waves, as briefly illustrated earlier, is that their motion is always governed by Newton’s laws and require a material medium for their propagation.

Now, it should be understood that elastic wave motion refers to the motion of a medium having the property such that when displaced, a restorative force proportional to the displacement acts on the medium’s inertial elements to restore them to their original position. Most materials are elastic to some extent so in order to be clear on the elastic medium considered in this paper we shall only assume it to be a flat, somewhat thin and regularly shaped membrane. We do not consider wave propagation through the collection of particles placed upon the membrane’s surface and consider their weight negligible. Thus, once disturbed, the unimpeded physical connection between the membrane’s adjacent elements from which it is composed will allow them to longitudinally influence one after another. Thus, as we have given our membrane the property of elasticity, it
follows that if the elements of a certain segment are perturbed then a wave will propagate across its length.

A wave can be considered to propagate either to infinity or to some finite limit. In either case some condition is usually specified at the extremes and the investigation of such conditions is an important part of any analytical approach to wave motion. As an example of the physical implications of boundary conditions, we again consider Chladni’s experiments. If the plate used in the experiment is clamped at the boundaries then the problem is physically equivalent to the quantum mechanics of a box with infinitely high walls [12]. In the classical demonstration of the pattern experiment as shown in Fig. 2.1, the boundaries are not clamped and instead, the plate is held by the finger tips at a nodal point while the plate is stroked with a bow at a point along one of its sides. Another example of the influence of boundary conditions is given in Ref. [4] where

Figure 2.1. The Chladni experiment from the UCLA Physics Lab demonstrations web site. The figures illustrate effects of forcing and boundary conditions on the plate. Also, note the boundaries here are not fixed whereas with the non-uniform membrane, boundaries are fixed at zero.
the frictional force from the walls of the container containing a large number of particles influences the number of convection rolls present in the system when it is under forced vibration. Here we shall assume the membrane boundaries are not subject to variation and remain fixed for all time.

2.2 THE PDE MODEL OF THE NON-UNIFORM STRING

We begin by considering an elastic membrane, rectangular in shape of length $L$ and with fixed boundaries and a variable tension gradient as depicted in Fig. 2.2. The membrane’s equilibrium position lies in the $(x, y)$ plane. To begin the characterization of the non-uniform membrane we will not consider any motion along the $y$-axis and restrict the degrees of freedom of the system to only wave motion along the $x$-axis. These assumptions lead to examination of the equation of the vibrating non-uniform membrane in only one dimension or that of a non-uniform string. Let time be $t$, the tension of the string at each point along the $x$-axis be given by $T(x)$ and its linear mass density be $\rho_0(x)$. In Ref. [2], the partial differential equation of the vibrating string is given as

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] + \rho_0(x) Q(x, t), \quad (2.1)$$

with initial and boundary conditions given by

I.C.'s : \quad $u(x, 0) = f(x), \quad \frac{\partial u}{\partial x}(x, 0) = g(x), \quad (2.2)$

B.C.'s : \quad $u(0, t) = 0, \quad u(L, t) = 0. \quad (2.3)$

The string is taken to be initially at rest in its equilibrium position. As discussed in the previous section, the string is considered to be a series of adjacent elements whose motion is governed by an ordinary differential equation (ODE) such as Eq. (1.7). The theory of ODEs requires both initial position and initial velocity [6], as given in Eq. (2.2). The function $Q(x, t)$ represents contributions from external forces and for now we will investigate the unforced case in which $Q(x, t) = 0$. Thus, we have the equation

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right].$$
Figure 2.2. Schematic of the variable tension gradient.

The linear mass density of the string is taken to be uniform throughout space so we assume, for a real valued constant \( \rho_0 \), we have \( \rho_0(x) = \rho_0 \) for all \( x \). Dividing the above equation by \( \rho_0 \) we get

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ c^2(x) \frac{\partial u}{\partial x} \right], \quad c^2(x) \equiv \frac{T(x)}{\rho_0}, \quad (2.4)
\]

where \( c(x) \) is the local speed of traveling waves across the string. In the present case \( T(x) \) is not constant and in fact for the entirety of this manuscript we shall primarily study the effects of a linear tension gradient defined as

\[
T(x) = a(x - L/2) + b, \quad 0 \leq x \leq L,
\]

where \( a \) and \( b \) are real constants. Since \( T(x) \) varies across the spatial domain of the membrane we know the speed of wave pulses will vary as well [9].

Continuing with the general problem of solving Eq. (2.4), we proceed via the method of separation of variables and let \( u(x,t) = h(t)\phi(x) \). Inserting this expression into Eq. (2.4) gives

\[
\phi \frac{\partial^2 h}{\partial t^2} = h \frac{\partial}{\partial x} \left[ c^2(x) \frac{\partial \phi}{\partial x} \right].
\]

Multiplying the above by \( 1/\phi h \) gives

\[
\frac{1}{h} \frac{\partial^2 h}{\partial t^2} = \frac{1}{\phi} \frac{\partial}{\partial x} \left[ c \frac{\partial \phi}{\partial x} \right] = -\lambda,
\]

where the number \( \lambda \) is the eigenvalue of the boundary value problem [2] stated as

\[
\frac{d}{dx} \left[ c^2(x) \frac{d\phi}{dx} \right] + \phi \lambda = 0. \quad (2.5)
\]
The form of Eqs. (2.5) and (2.6) show they correspond to a standard Sturm-Liouville eigenvalue problem [2]. We now explore a method for deriving the eigenfunctions for the non-uniform string.

2.3 Analytical Solution of the Eigenfunctions \( \phi(x) \)

In this section we derive an expression of the eigenfunctions which turns out to be a linear combination of both the \( J \) and \( Y \) Bessel functions. Bessel’s differential equation has been investigated for over 150 years [2] but there are only a few examples where Bessel’s functions are used outside of the realm of circular vibrating membranes. Even fewer are demonstrations which involve the use of both the \( J(x) \) and \( Y(x) \) Bessel functions. It is interesting to note that Bessel functions originated in the study of non-spherical systems. According to Ref. [10], Daniel Bernoulli, while investigating vibrations of a homogenous hanging chain, derived an equation which had a solution now well known as the Bessel function of order zero. Thus, the non-uniform vibrating membrane provides an interesting opportunity to gain a deeper understanding of the Bessel function’s mathematical and physical properties.

At this point, it is desired to show how Eq. (2.5) can be manipulated into Bessel’s differential equation of order zero. We begin by stating Bessel’s differential equation of order \( m \) given in Ref. [13] as

\[
z^2 \frac{\partial^2 u}{\partial z^2} + z \frac{\partial u}{\partial z} + (z^2 - m^2) u = 0.
\]

(2.7)

The solutions of Eq. (2.7) are denoted \( J_m(z) \) and \( Y_m(z) \) and called the \( m \)th \( J \) and \( Y \) Bessel functions. Letting \( m = 0 \) we now have Bessel’s equation of order zero

\[
z^2 \frac{\partial^2 u}{\partial z^2} + z \frac{\partial u}{\partial z} + z^2 u = 0.
\]

(2.8)

For reference, plots of the \( J_0(z) \) and \( Y_0(z) \) Bessel functions along with their associated zeros are included in Fig. 2.3.
In the eigenvalue problem (2.5) and (2.6), apply the change of variables
\[ z = 2 \sqrt{a(x - L/2) + b/a} \]
so that
\[ \frac{dz}{dx} = \frac{1}{\sqrt{a(x - L/2) + b/a}}. \] (2.9)
Noting that $az/2 = \sqrt{a(x - L/2) + b}$, Eq. (2.9) can be recast as
\[ \frac{az}{2} \, dz = dx. \]  
(2.10)

Refering back to Eq. (2.5) we can see that
\[ \frac{d}{dx} \left[(a(x - L/2) + b) \frac{d\phi}{dx}\right] + \phi \lambda = 0, \]
can be transformed into
\[ \frac{2}{az} \frac{d}{dz} \left[ \left(\frac{az}{2}\right)^2 \frac{2}{az} \frac{d\phi}{dz}\right] + \phi \lambda = 0. \]

Simplifying the square term inside the brackets gives
\[ \frac{2}{az} \frac{d}{dz} \left[ \frac{az}{2} \frac{d\phi}{dz}\right] + \phi \lambda = 0. \]

Upon expanding the differential operators we have
\[ \frac{2}{az} \left( \frac{az}{2} \frac{d^2 \phi}{dz^2} + \frac{a}{2} \frac{d\phi}{dz} \right) + \phi \lambda = 0, \]
which implies
\[ \frac{d^2 \phi}{dz^2} + \frac{1}{z} \frac{d\phi}{dz} + \phi \lambda = 0. \]

Finally, multiplying through by $z^2$ gives Bessel’s differential equation of order zero
\[ z^2 \frac{d^2 \phi}{dz^2} + z \frac{d\phi}{dz} + z^2 \phi = 0. \]  
(2.11)

By substitution of the transformation $z(x) = 2\sqrt{a(x - L/2) + b}/a$ into the aforementioned solutions $J_0$ and $Y_0$, we see that the two linearly independent solutions to our eigenvalue problem are $J_0 \left( \frac{\sqrt{a}}{a} z(x) \right)$ and $Y_0 \left( \frac{\sqrt{a}}{a} z(x) \right)$ and therefore the eigenfunctions $\phi(x)$ take the form
\[ \phi(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{a}}{a} \sqrt{a(x - L/2) + b} \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{a}}{a} \sqrt{a(x - L/2) + b} \right) \right], \]  
(2.12)

where $C_1$ and $C_2$ are constant coefficients to be determined by boundary conditions.
2.4 DEFINING THE COEFFICIENTS OF THE LINEAR TENSION GRADIENT

Before determining the coefficients $C_1$ and $C_2$, now it is a good time to point out the physical and mathematical implications of the linear tension gradient $T(x)$. For the system to be physical, we must have $T(x) > 0$ for all $x$ such that $0 < x < L$. Thus, we are lead to imposing bounds on either $a$ or $b$. Namely, we require at $x = 0$

$$0 < -\frac{aL}{2} + b,$$

and at $x = L$

$$0 < \frac{aL}{2} + b.$$

Therefore, we shall set $b > 0$ and require $-2b/L < a < 2b/L$.

2.5 THE BOUNDARY VALUE ANALYSIS OF $\phi(x)$

Consider the fixed boundary conditions:

$$\phi(0) = 0, \quad \phi(L) = 0. \quad (2.13)$$

To simplify our equations, let $z_- := T(0) = \sqrt{b - aL/2}$ and $z_+ := T(L) = \sqrt{b + aL/2}$. The boundary conditions in Eq. (2.13) yield the following homogeneous system for $C_1$ and $C_2$

$$C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \right] = 0 \quad (2.14)$$

$$C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \right] = 0 \quad (2.15)$$

Looking closely at the above system we readily see that Eqs. (2.14) and (2.15) form a $2 \times 2$ matrix equation

$$\begin{bmatrix}
J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) & Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \\
J_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) & Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right)
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} = 0. \quad (2.16)$$
From linear algebra, there exists nontrivial solutions $C_1$ and $C_2$ to Eq. (2.16), if and only if the determinant of the matrix is zero. Proceeding by taking the determinant in Eq. (2.16) gives

$$J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) - Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) = 0. \quad (2.17)$$

We now have three equations [(2.14), (2.15) and (2.17)] and three unknowns which tell us the numbers $C_1$, $C_2$ and $\lambda$ are uniquely determined. Explicitly stated, our problem now is to find the unique numbers $C_1$, $C_2$ and $\lambda$ such that

$$C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \right] = 0,$$

$$C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \right] = 0, \quad (2.18)$$

$$J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) - Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) J_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) = 0.$$

From here we can use a root finding method to solve the nonlinear system (2.18).
CHAPTER 3
EIGENVALUES AND COEFFICIENTS OF $\phi(x)$

3.1 The Numerical Solution of the Eigenvalues $\lambda$

In the preceding chapter the analytical solution to the eigenfunctions $\phi(x)$ of the non-uniform wave equation were obtained as an arbitrary linear combination of two independent solutions given as

$$\phi(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) \right] + C_2 \left[ Y_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) \right] ;$$

(3.1)

where

$$T(x) \equiv \sqrt{a(x - L/2 + b)}$$

(3.2)

and $J_0$ and $Y_0$ represent the zeroth $J$ and $Y$ Bessel functions. However, when applying the problem’s boundary conditions we were lead to system (2.18) which must be solved for the three unknown values $C_1$, $C_2$ and $\lambda$. System (2.18) involves not only nonlinear but transcendental equations for which no closed form solution exists. Thus, we are motivated to use numerical techniques to solve the system. Loosely speaking, it would be helpful to employ an algorithm initiated by a guess $v_0$ which leads to a good solution $v$. Expressed mathematically, the idea is to choose $v_0$ such that

$$v_0 = \begin{bmatrix} C_{10} \\ C_{20} \\ \lambda_0 \end{bmatrix} \rightarrow v = \begin{bmatrix} C_1 \\ C_2 \\ \lambda \end{bmatrix} .$$

(3.3)

Upon inspection of system (2.18) it is observed that the third equation

$$r(\lambda) = J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) - Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) J_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) .$$

is independent of the constants $C_1$ and $C_2$ and therefore the eigenvalues $\lambda$ can be solved for using only $r(\lambda)$. A plot of $r(\lambda)$ is given in Fig. 3.1 from which we can see there are
multiple solutions of the problem \( r(\lambda) = 0 \). Each solution will correspond to a different mode of vibration of \( \phi(x) \). To solve for \( \lambda \) such that \( r(\lambda) = 0 \) and for demonstration purposes we employ the following algorithm:

**Algorithm I.** Choose \( \lambda_0 \). Then:

for \( k = 0, 1, 2, \ldots \) Calculate a solution \( p_k \) to the equation

\[
\frac{dr}{d\lambda} p_k = -r(\lambda_k);
\]

(3.4)

Set

\( \lambda_{k+1} = \lambda_k + p_k; \)

end (for)

The above algorithm is known as Newton’s method in one dimension [7]. To begin, we need to to take the derivative of \( r(\lambda) \) with respect to \( \lambda \) which is implicitly given as:

\[
\frac{d}{dz} r(z) = J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \frac{d}{dz} Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) + Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \frac{d}{dz} J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) - Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \frac{d}{dz} J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) - J_0 \left( \frac{2\sqrt{\lambda}}{a} z_- \right) \frac{d}{dz} Y_0 \left( \frac{2\sqrt{\lambda}}{a} z_+ \right) \]

The derivatives of the \( J_m \) and \( Y_m \) Bessel functions can be easily obtained and computed via the known recurrence relations given in Ref. [13] as:

\[
\frac{d}{dz} [z^m J_m(z)] = -z^{-m} J_{m+1}(z), \quad \frac{d}{dz} [z^m Y_m(z)] = -z^{-m} Y_{m+1}(z),
\]

(3.5)

where \( m \in \mathbb{N} \). The results of an implementation of algorithm I are depicted in Fig. 3.1 and they illustrate its convergence properties for different values of initial conditions. If algorithm I is initiated with \( \lambda_0 = .5 \) then the algorithm converges to final value \( \lambda \approx .6334 \). In this instance, the algorithm converged to its closest root. However, if algorithm I is initiated with \( \lambda_0 = .4 \) then the algorithm converges to final value \( \lambda \approx .0674 \). The demonstration shows that with algorithm I an intial value must be
chosen close enough to the desired mode of vibration to ensure the correct convergence. Since we can numerically find $\lambda$, Eqs. (2.14) and (2.15) can be rewritten to form the relationship:
\[
\frac{J_0 \left( \frac{2\sqrt{\lambda} a z}{a} \right)}{Y_0 \left( \frac{2\sqrt{\lambda} a z}{a} \right)} = \frac{J_0 \left( \frac{2\sqrt{\lambda} a z}{a} \right)}{Y_0 \left( \frac{2\sqrt{\lambda} a z}{a} \right)} = -\frac{C_2}{C_1} = k. \tag{3.6}
\]
So that we can write $C_2 = -C_1 k$ where the known number $k$ is taken as:
\[
k = \frac{J_0 \left( \frac{2\sqrt{\lambda} a z}{a} \sqrt{T(x)} \right)}{Y_0 \left( \frac{2\sqrt{\lambda} a z}{a} \sqrt{T(x)} \right)}. \tag{3.7}
\]
Therefore, as long as the eigenvalue corresponding to a mode of vibration is solved for then the eigenfunction can be scaled by a choice of $C_1$. Precisely stated we have:
\[
\phi(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda} a}{a} \sqrt{T(x)} \right) - kY_0 \left( \frac{2\sqrt{\lambda} a}{a} \sqrt{T(x)} \right) \right]. \tag{3.8}
\]
which is the full analytical solution to the sought after eigenfunctions.

Fig. 3.2 depicts a plot of $\phi(x)$ in its third mode of vibration and under different scalings depending on $C_1$. The approach using Newton’s method was successful in deriving solutions to $\lambda$ and can be employed to give highly accurate approximations to $\lambda$ for each desired mode of vibration. However, the success of Newton’s method relies on providing an starting point in the roots basin of attraction for otherwise the algorithm may not converge to the desired point. Therefore, an efficient means for deriving our initial guesses is called for.

### 3.2 Approximating $\phi(x)$ via Finite Difference Methods

Since Newton’s method may not converge correctly when $\lambda_0$ is far from the solution, we desire to obtain approximate guesses close to the actual eigenvalues $\lambda$ for any particular mode of vibration. Also, it would be convenient to get approximations without the use of graphical techniques. Once a numerical solution for $\lambda$ is achieved then simple algebra can be used to scale the coefficient $C_1$. To obtain good approximations for the eigenvalues we shall employ the method of finite differencing. The method relies on
Figure 3.1. Newton’s method applied to finding the roots of $r(\lambda)$. For $a = .18, b = 1$,
(Top): For $\lambda_0 = .5$, Newton’s method is applied and converges to the nearby root $\lambda_3$.
(Bottom): For $\lambda_0 = .4$ Newton’s method overshoots $\lambda_2$ and converges to $\lambda_1$.

discretizing the Helmholtz equation into a matrix form for which we can obtain approximations $\lambda$ with corresponding eigenvectors $v(x) \sim \phi(x)$. Once we have the
numerical solution to $\lambda$ we can use $v(x)$ to check whether the analytical solution $\phi(x)$ in fact corresponds to the mode of vibration thus verifying the employed numerics.

Proceeding with the problem at hand we can expand Eq. (2.5) via the product rule to obtain
\[
\frac{d}{dx} \left[ c(x) \frac{d\phi}{dx} \right] + \phi \lambda = \frac{dc}{dx} \frac{d\phi}{dx} + c(x) \frac{d^2\phi}{dx^2} + \phi \lambda = 0, \tag{3.9}
\]
or
\[
\frac{dc}{dx} \frac{d\phi}{dx} + c(x) \frac{d^2\phi}{dx^2} = -\phi \lambda. \tag{3.10}
\]
Since we are interested in estimating the eigenvalue problem over the interval $I = [0, L]$, let $I$ be partitioned into $N$ points each separated by the distance $\Delta x = L/N$:
\[
x_j = j\Delta x, \quad j = 1, 2, \ldots, N.
\]
Using the above notation, if we let \( x = x_j + \Delta x \) we can express the first and second derivatives as the polynomial approximations:

\[
\frac{d}{dx} [\phi(x_j)] \approx \frac{\phi(x_j + \Delta x) - \phi(x_j - \Delta x)}{2\Delta x} \tag{3.11}
\]

\[
\frac{d^2}{dx^2} [\phi(x_j)] \approx \frac{\phi(x_j + \Delta x) - 2\phi(x_j) + \phi(x_j - \Delta x)}{\Delta x^2} \tag{3.12}
\]

Eqs. (3.11) and (3.12) are known as the centered difference approximations to the first and second derivatives of \( \phi(x) \) at the point \( x_j \) [2]. Introducing the additional notation,

\[
\phi(x_j) \equiv \phi_j,
\]

allows the approximations to \( \phi(x) \) to be expressed in a convenient form. Using Eqs. (3.11) and (3.12), the first and second derivatives of \( \phi(x) \) can then be written in the familiar \( N \times N \) matrix notation. For the first derivative we have:

\[
\frac{d\phi}{dx} \approx \frac{1}{2\Delta x} \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 1 & \cdots & 0 & 0 \\
0 & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
\end{bmatrix} \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\vdots \\
\phi_N \\
\end{bmatrix}. \tag{3.13}
\]

For the second derivative we have:

\[
\frac{d^2\phi}{dx^2} \approx \frac{1}{(\Delta x)^2} \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & \cdots & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & 2 \\
\end{bmatrix} \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\vdots \\
\phi_N \\
\end{bmatrix}. \tag{3.14}
\]

Next, we must ensure the tension gradient is properly distributed across the spatial domain. Now, just as \( \phi(x) \) is discretized over \( L \) we can do the same for \( c(x) \) and \( dc/dx \) to get the column vectors \( c = [c_1 \ c_2 \ \ldots \ c_N] \) and \( c' = [a \ a \ \ldots \ a] \). The following
matrix can now be formed:

\[
c = \begin{bmatrix}
c_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & c_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & c_3 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & c_N
\end{bmatrix},
\]  

(3.15)

so that we can write

\[M_1 = a\phi', \quad M_2 = c\phi''.\]  

(3.16)

Inserting the matrix representations of Eq. (3.16) into Eq. (3.10) lets our problem be reduced to solving the simple linear algebraic eigenvalue equation:

\[A\phi = [M_1 + M_2] \phi = \lambda \phi.\]  

(3.17)

A full discussion of the computation of eigenvalues is outside the scope of this paper but a few words on their numerical solution is offered. For a matrix \(A\) of low dimensionality the exact eigenvalues of \(A\) can be computed by hand in a finite number of steps as the roots of the characteristic polynomial of \(A\). However, the root finding method grows in complexity as the dimensionality of \(A\) gets larger since we become concerned with finding zeros of high order polynomial equations. Thus, as explained in Ref. [14], although eigenvalues and eigenvectors are elegant in definition, their computation quickly becomes no straightforward process and many iterative algorithms exist for their computation. Effective algorithms normally proceed on a two step basis: (1) reduction of the matrix \(A\) to a structured form, then (2) execute an iterative process for final convergence. However, iterative algorithms cannot compute the exact eigenvalues in a finite number of steps and thus, the computing of eigenvalues of high dimensional matrices is in general, an unsolvable problem. However, there are algorithms available that converge very quickly.

We use MATLABs built-in \texttt{eig} function to find our desired approximations to \(\lambda\) and \(\phi(x)\). Figure 3.3 depicts plots of 3 cases of the fifth eigenfunction \(\phi_5\) over the interval \([0, L]\). These plots show several interesting outcomes of the tension gradient on the system dynamics. First, the eigenfunctions are qualitatively similar to Bessel functions which
Figure 3.3. Deflection of the eigenfunction $\phi_5(x)$. (Top): The nodes of the fifth eigenfunction are moved from the right to left by adjusting $a$ from $-0.18$ to $-0.18$. (Bottom): A plot of the range of eigenvalues $\lambda$ for the domain $a \in [-0.18, 0.18]$ The dashed line represents the fifth mode of the classical string: $\lambda = (5\pi/L)^2$.

supports the analysis performed on the problem thus far. Secondly, we can see how as we vary $a$ from its right extreme near $2b/L$ to its left extreme near $-2b/L$ the Bessel
functions transition into sine waves and then transition again into Bessel functions. Third, observing the nodal displacement over the range of $a$ we see how particles, if at rest along the nodes, can be transported across the length of the membranes spatial domain. Finally, the bottom plot of Fig. 3.3 shows the main difference between the eigenvalues of the non-uniform string and the classical string. The bottom plot of Fig. 3.3 is a spread of eigenvalues for the fifth mode with $a \in [-.18, .18]$ whereas for the classical string, the fifth mode is characterized by the single eigenvalue: $\lambda = (5\pi/L)^2$. 
CHAPTER 4
TEMPORAL CHARACTERISTICS OF THE NON-UNIFORM STRING

4.1 The Time Dependent Coefficients $\varphi(t)$

Here we return to an analytical discussion of the non-uniform string. So far we have for any choice of coefficients $a$ and $b$ in the tension gradient $T(x)$, the eigenfunctions $\phi(x)$, and the eigenvalues $\lambda$, which are solved for to a high level of accuracy. However, still unknown is how the non-uniform string will behave under periodic perturbations from its equilibrium position. Although the non-uniform string has unusual eigenfunctions we will see it has behavior very similar to that of the classical string.

Consider the case $Q(x,t) \neq 0$ for which an external force puts the membrane into motion and is described by the equation:

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] + \rho_0(x)Q(x,t) - \beta \frac{\partial u}{\partial t}, \quad (4.1)$$

$$T(x) \equiv a [(x - L/2) + b]. \quad (4.2)$$

where $Q(x,t)$ represents the time and spacially dependent external force and $\beta$ is the damping coefficient. We impose the same boundary conditions as before:

I.C’s : $u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(x,0) = g(x). \quad (4.3)$

B.C’s : $u(0,t) = 0, \quad u(L, t) = 0. \quad (4.4)$

At this point, we know the eigenfunctions $\phi(x)$ of the related homogeneous problem $Q(x,t) = 0$. For generality, for the vibrational modes $i = 1, 2, \ldots$ we designate the related homogenous eigenfunctions as $\phi_i(x)$ which form an orthogonal basis. Thus, any piecewise smooth function, which includes the desired solution $u(x,t)$ of the non-uniform string, may be expressed as a linear combination in terms of an infinite series.
of $\phi_i(x)$:

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_i(t) \phi_i(x) = \varphi_1(t) \phi_1(x) + \varphi_2(t) \phi_2(x) + \varphi_3(t) \phi_3(x) + \cdots,$$  \hspace{1cm} \text{(4.5)}

where the $\varphi_i(t)$ are the time-dependent eigenfunctions. Proceeding with term by term differentiation of Eq. (4.5) with respect to $t$ we get:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=0}^{\infty} \frac{d^2 \varphi_i(t)}{dt^2} \phi_i(x).$$  \hspace{1cm} \text{(4.6)}

For the non-uniform string we proceed a little cautiously. Taking the first derivative of $u$ with respect to $x$ in Eq. (4.5) we get

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} \varphi_i(t) \frac{d \phi_i(x)}{dx}.$$

Then multiplying the above equation through by $T(x)$ it follows

$$T(x) \frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} \varphi_i(t) T(x) \frac{d \phi_i(x)}{dx},$$

and taking the derivative again with respect to $x$ it follows

$$\frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] = \sum_{n=0}^{\infty} \varphi_i(t) \frac{d}{dx} \left[ T(x) \frac{d \phi_i(x)}{dx} \right].$$  \hspace{1cm} \text{(4.7)}

Since the eigenfunctions along with corresponding eigenvalues satisfy:

$$\frac{d}{dx} \left[ T(x) \frac{d \phi_i(x)}{dx} \right] = -\lambda \phi_i,$$  \hspace{1cm} \text{(4.8)}

Eq. (4.6) and (4.8) can be substituted into Eq. (4.1) to obtain

$$\sum_{n=0}^{\infty} \left( \frac{d^2 \varphi_i}{dt^2} + \beta \frac{d \varphi_i}{dt} + \lambda_i \varphi_i \right) = Q(x, t).$$  \hspace{1cm} \text{(4.9)}

Likewise, expanding $Q(x, t)$ in terms of the eigenfunctions gives

$$Q(x, t) = \sum_{n=0}^{\infty} q_i(t) \phi_i(x),$$
and upon multiplication of both sides of the expansion by $\phi_i(x)$ and applying the principle of orthogonality of eigenfunctions one obtains

$$q_i(t) = \frac{\int_0^L Q(x,t)\phi_i(x)dx}{\int_0^L \phi_i(x)^2dx}.$$ 

Letting the forcing function be $Q(x,t) = F_0 \cos(\omega t)$ it follows that

$$q_i(t) = \mu_i F_0 \cos(\omega t); \quad \mu_i = \frac{\int_0^L \phi_i(x)dx}{\int_0^L \phi_i(x)^2dx}, \quad (4.10)$$

so to arrive at the familiar ODE for forced mechanical vibrations:

$$\frac{d^2\varphi_i}{dt^2} + \beta \frac{d\varphi_i}{dt} + \lambda_i \varphi_i = \mu_i F_0 \cos(\omega t). \quad (4.11)$$

Eq. (4.11) appears to be similar to the ODE for the time dependent coefficients of a classical string. However, the quantity $\mu_i(\lambda_i)$ depends on the spatial characteristics of the non-uniform string and has an important role in the excitation of string modes.

### 4.2 Vibrational Modes of the Non-uniform String

To analyze how $\mu_i(\lambda_i)$ influences the excitation of different modes of vibration consider a vibrating homogeneous classical string with both ends fixed at zero for which the boundary conditions imply the eigenfunctions, for some constant $C$, are $\phi_n(x) = C \sin(\sqrt{\lambda_n}x)$ with eigenvalues: $\lambda_n = (n\pi/L)^2$. Now, for the case of a classical string, the computation of $\mu(\lambda)$ from equation (4.10) gives:

$$\mu(\lambda) = \frac{1}{C} \frac{\int_0^L \sin(\sqrt{\lambda}x)dx}{\int_0^L \sin^2(\sqrt{\lambda}x)dx} = \frac{1 - \cos(\sqrt{\lambda}L)}{(1/2)\sqrt{\lambda}L - (1/4) \sin \left(2\sqrt{\lambda}L\right)}. \quad (4.12)$$
The plot of $\mu(\lambda_n)$ in Fig. 4.1 shows that for attempted excitations of even modes of vibration ($\omega = \sqrt{\lambda_n}$; $n$ is even) of a regular string then $\mu(\lambda_n) = 0$. From the theory of ordinary differential equations, the general solution to Eq. (4.11) is:

$$\varphi_i(t) = Ae^{(-\beta/2)t} \sin \left( \frac{\sqrt{4\lambda_i - \beta^2}}{2} t + \psi \right) + \frac{\mu(\lambda_i)F_0}{\sqrt{(\lambda_i - \omega^2)^2 + \beta^2\omega^2}} \sin (\omega t + \theta), \quad (4.13)$$

where $A$ and $\psi$ are constants dependent on I.C.s. Since in this case $\mu(\lambda_n) = 0$, it follows that as $t \to \infty$ then $\varphi_i(t) \to 0$. Therefore, exponentially decaying excitations of the string can be expected as it is perturbed from its equilibrium position. The effect can be physically realized as a cancellation of force due to equally opposing sections of string.

Now consider the eigenfunctions of the non-uniform vibrating string:

$$\phi_i(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) - kY_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) \right]. \quad (4.14)$$

The square term in the denominator of the integral equation of $\mu(\lambda_i)$ is more complicated to compute for the non-uniform string and thus we resort to numerical approximations to $u(\lambda)$ which can be achieved via the Trapezoidal or Simpson’s rule. The plot of $\mu(\lambda_i)$ in this case is shown beside that for the classical string in Fig. 4.1. What Fig. 4.1 illustrates is that while it may be the case $\mu(\lambda) \ll 1$ for the even modes of forcing, nevertheless, the even modes can be excited in the non-uniform string.

Well known from the theory of ordinary differential equations is that the friction term $\beta$ has a stabilizing effect on the string’s amplitude as it is perturbed over time by the forcing function $Q(x,t) = F_0 \cos(\omega t)$. The frequency response curve for the system’s amplitude is given by the function

$$M(\omega) = \frac{1}{\sqrt{(\lambda - \omega^2)^2 + \beta^2\omega^2}}, \quad (4.15)$$

which shows the relationship between $\omega$, $\beta$ and $\lambda$. A plot of $M(\omega)$ for the third vibrational mode of a regular string over different damping coefficients is depicted in Fig. 4.2. As discussed in Ref. [6], for $\omega = 0$ there are no perturbations to the string and we have $M(0) = 1/\lambda$ which makes sense. Also, as $\omega \to \infty$ then $M(\omega) \to 0$ which implies
the inertia of the string limits the extent to which it can respond to extremely rapid vibrations. The local maxima of $M(\omega)$ is located at the frequency $\omega = \sqrt{\frac{\chi}{2}}$ which is called the string’s resonant frequency. As $\beta \to 0$ there is an increase in the frequency response and the resonant frequency shifts to the normal mode of vibration $\lambda$.

4.3 A HYPOTHESIS ON THE NON-UNIFORM STRING’S FREQUENCY RESPONSE

Let us integrate our knowledge into an inquiry of the non-uniform string’s dynamical behavior. In view of what has been discussed so far regarding the non-uniform string’s time dependent behavior when exposed to external forces we may ask the question: “What are the fundamental differences between the classical string and non-uniform string?” We can formulate a hypothesis based on the observations of the previous two sections. In Section 4.1, the ODE of the non-uniform string’s time
Observation I. A uniform tension gradient suppresses even modes of vibration.

Observation II. A non-uniform tension gradient allows even modes of vibration.

Observation III. The function $M(\omega)$ which determines the magnitude of excitement of vibrational wave modes remains consistent between both the uniform and non-uniform strings.

Note that observations I and II above are only the result of the effect from the term $\mu(\lambda)$ which is a spatial and not a temporal property of both strings. Thus, we are lead to hypothesize that the non-uniform string is dynamically similar to that of the classical uniform string and the two systems are separate only by the fact they look
different. Our hypothesis is summarized in the general prediction:

\[(\text{perturbation}) \rightarrow (\text{non-uniform string}) \overset{\text{time}}{\longrightarrow} (\text{steady-state}).\]  

(4.16)

The experimental verification of the above prediction offers an excellent opportunity to check our employed numerical integration scheme. It is desired to observe how well the numerical solution \(u_s(x, t)\) via a 4th order Runge-Kutta method of integration captures the non-uniform string’s analytical solution \(u(x, t)\). If \(u_s(x, t)\) does indeed well approximate \(u(x, t)\) then we shall write \(u_s(x, t) = u(x, t)\) which is our goal and can be achieved via the following test.

Let \(u_s(x, t)\) be the solution of the function \(u(x, t)\) defined by the non-uniform wave equation (1.1) as numerically integrated by our Runge-Kutta method. Then \(u_s(x, t)\) can be decomposed into a summation series involving the eigenbasis \(\phi_k(x)\), \(k = 1, 2, \ldots\) and time-dependent solutions \(\varphi_{sk}(t)\). Therefore, we take

\[u_s(x, t) = \sum_{n=0}^{\infty} \varphi_{sk}(t)\phi_k(x) = \varphi_{s1}(t)\phi_1(x) + \varphi_{s2}(t)\phi_2(x) + \varphi_{s3}(t)\phi_3(x) + \cdots \]  

(4.17)

Multiplying Eq. (4.17) by the eigenfunction \(\phi_f(x)\) corresponding to the intended mode of excitation \(f\) with the forcing frequency \(\omega_f\) and applying orthogonality of eigenfunctions we get

\[\varphi_{sf}(t) = \frac{\int_0^L u_s(x, t)\phi_f dx}{\int_0^L \phi_f^2 dx}. \]  

(4.18)

The above \(\varphi_{sf}(t)\) in Eq. (4.18) is the temporal solution from our numerical integration. Our goal is reached if \(\varphi_{sf}(t) = \varphi_f(t)\) as given in Eq. (4.13) for all \(t\).

We need to construct a good test function, \(\varphi_f(t)\) from Eq. (4.13), that contains coefficients \(A\) and \(\psi\) which depend on an initial condition. Note that if the non-uniform string described by \(u(x, t)\) is given an initial condition \(u(x, 0)\) then by our prediction (4.16) it should converge to a steady state harmonic solution \(u(x, t) \approx \varphi_f(t)\phi_f(x)\). Also, we would like to observe the effects of tension adjustment on the non-uniform string. Thus, we set our initial condition as the string in a vibrational mode \(i\), in resonance for some \(a_i\) then numerically integrate the wave equation with damping \(\beta\) and forced at a
frequency \( w_f = \sqrt{\lambda_f - \beta/2} \) where \( \lambda_f \) corresponds to \( a_f \) and observe how well \( \varphi_{sf}(t) \) approximates \( \varphi_f(t) \).

The above scheme is easy to implement and will follow by the direct application of the theory of ordinary differential equations. To begin, we shall specify our eigenbasis by choosing any \( C_1 \) and setting

\[
\phi(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) - kY_0 \left( \frac{2\sqrt{\lambda}}{a} \sqrt{T(x)} \right) \right].
\]  

To illustrate the effect of tension adjustment on the string’s dynamics we shall use our initial condition as a simulation of the string in resonance at a particular mode, say \( i \). So, we choose \( a_i \) with its corresponding \( \lambda_i \) so that if we expand \( u(x, t) \) into its spectral decomposition, then for \( t \gg 1 \) we have

\[
u(x, t) \approx \varphi_i(t)\phi_i(x) \approx M(\omega_i)\mu(\lambda_i) \sin(\omega_i t + \theta_i)\phi_i(x),
\]  

since by Eqn. (4.13), as \( t \to \infty \)

\[
\varphi_i(t) \to \frac{\mu(\lambda_i)F_0}{\sqrt{(\lambda_i - \omega_i^2)^2 + \beta^2\omega_i^2}} \sin(\omega_i t + \theta_i),
\]  

where \( \tan(\theta_i) = (\lambda_i - \omega_i^2)/\beta\omega_i \) and

\[
\mu(\lambda_i) = \frac{\int_0^L \phi_i(x) dx}{\int_0^L \phi_i(x)^2 dx}.
\]  

Noting that \(-1 \leq \sin(\omega t + \theta) \leq 1\) for all \( t \), we set the string’s initial condition as

\[
u(x, 0) = M(\omega_i)\mu(\lambda_i)\phi_i(x),
\]  

Next, we will set \( a_f = -a_i \) and force the string at a frequency \( \omega_f = \sqrt{\lambda_f - \beta/2} \) and observe the string’s temporal behavior, as it evolves to its final state,

\[
u(x, t) \approx \varphi_f(t)\phi_f(x), \text{ corresponds to that predicted by the solution to the ODE in Eq. (4.11). Now, we find the initial condition for } \varphi_i(t) \text{ by setting}
\]

\[
u(x, 0) = M(\omega_i)\mu(\lambda_i)\phi_i(x) = \sum_{n=0}^{\infty} \varphi_k(0)\phi_k(x).
\]  

(4.24)
Multiplying through by $\phi_f(x)$ and applying orthogonality to the R.H.S of the above equation yields

$$\varphi_f(0) = M(\omega_i)\mu(\lambda_i) \frac{\int_0^L \phi_i(x)\phi_f(x)dx}{\int_0^L \phi_f(x)^2dx}. \tag{4.25}$$

From here the numbers $A$ and $\psi$ for $\varphi_f(t)$ can be solved for using elementary methods as outlined in Ref. [6]. We note that $\phi_i(x)\phi_f(x) \neq 0$ since $\phi_i(x)$ and $\phi_f(x)$ are eigenfunctions within the same mode.

Figure 4.3 shows the success of the scheme outlined above. At $t = 0$ the integration starts at a nonzero value and evolves over time. From the onset of the experiment the numerical solution agrees remarkably well with the analytical solution: $\varphi_{sf}(t) = \varphi_f(t)$. Therefore, indeed we have

$$u_s(x, t) = u(x, t). \tag{4.26}$$

The uniform convergence of $u(x, t)$ to its steady-state solution over time also agrees very well with the prediction set forth in Eq. (4.18). Thus, we have verified our prediction that perturbations to the non-uniform string converge to a steady-state just like a classical string. To expand our results to a more general remark about the non-uniform string we calculate $|\varphi_{sf}(t)|$ in order to test the non-uniform string’s predicted behavior over a range of forcing frequencies. Both even and odd modes of vibration were excited. Figure 4.4 shows again remarkable agreement between the system as it is integrated over time and the amplitude of the steady state response as predicted by the frequency response function $M(\omega)$.

### 4.4 Concluding Remarks on the Non-uniform String’s Transient Behavior

In the previous sections the response of the non-uniform wave equation to perturbations from an external force was analyzed. What was shown was that the wave’s response was similar to that of an ordinary wave over a medium of homogenous tension. The nonhomogeneous waves response comprised of two separate time dependent solutions $y_h(t)$ and $y_p(t)$ were $y_h(t)$ was the transient response and $y_p(t)$ was the steady state
Figure 4.3. The time-dependent eigenfunction $\varphi_3(t)$ in a simulation of varying the tension gradient. (Top): The initial condition is selected with $a = -0.18$. (Center): The steady state response of the string with $a = 0.18$ as it is forced over $t \in [0, 500]$. (Bottom): A comparison of the analytical solution of $\varphi_3(t)$ from Eqn. (4.13) (solid black) and as calculated from the Runge-Kutta method (black circles). The calculated results agree remarkably well with the analytical solution.

response. We found that the steady state response was a harmonic which oscillated at the frequency of excitation and whose amplitude of vibration is strongly dependent on both
Figure 4.4. The steady state amplitude and frequency response $M(\omega)$. (Top): With $a = 0.18, b = 1$ the vibrational mode 3. (Bottom): The vibrational mode 4. The solid black line is the function $M(\omega)$ and the black circles are the maximum amplitude of the non-uniform wave for the final 60 iterations of its time integration of $t \in [0, 1000]$.

the frequency of excitation and spacial properties of the system. More so, the steady state response was independent of the initial conditions which contrasts the transient response of the wave that depends on everything to include initial conditions. What will be seen in
a Chap. 6 is how the non-uniform string’s transient behaviour, which it inherits from a
classical string, affects the means of which the nodes can be controlled via the adjustment
of the medium’s tension gradient.

As final remarks, we first note that for a particular forcing frequency and any
initial condition, if given enough time it is mathematically guaranteed that the wave will
eventually settle to its steady state response. However, the purpose here is nodal control.
That is: we aim to maintain the quiescent parts of the wave and move them about the
surface. Thus, to reach our goal, the wave’s transient behavior must be controlled in order
to maintain an ordered resonant configuration throughout a range of movement.

Consider the string’s transient solution:

\[ \varphi_{ih}(t) = A e^{(-\beta/2m)t} \sin \left( \frac{\sqrt{4\lambda_i - \beta^2}}{2m} t + \psi \right). \]  

(4.27)

Observe that there are three variables at our disposal: (1) time \( t \), (2) the damping
coefficient \( \beta \) and (3) the eigenvalues \( \lambda \). The effect of time has been discussed partially but
will also play part in the eigenvalue adjustment as we manipulate the medium’s tension
gradient. The damping coefficient \( \beta \) depends on the material properties of the medium
and will be uncontrollable once it is selected. One way to minimize the transient effects
of the non-uniform wave (or a regular wave) is to get the transient term to approach zero
quickly as possible. Observing the damping term \( e^{-\beta t} \) we see that the larger the damping
term the closer to zero it is which equates to shorter transient times. Additionally,
observing the frequency response curve of the wave we see that smaller damping
coefficients correspond to sharper resonance curves. Thus, system stability should be
enhanced by selecting an elastic material of moderate damping.

To begin our discussion on the manipulation of (3), the sharpness of the resonance
curve means that perturbations about a resonance frequency via eigenvalue manipulation
will give rise to significant changes in the steady state wave amplitude of \( \varphi(t) \) which is
physically determined by the energy of the system. Since each point along the
non-uniform wave is modeled as a mechanical harmonic oscillator with its own steady
state amplitude then we can suppose there should be a steady state energy distribution
about the wave.
CHAPTER 5
ON THE ENERGY DISTRIBUTION OF THE
NON-UNIFORM STRING

5.1 INTRODUCTION

Let us reflect once more on the experiments conducted by Ernst Chladni. The patterns of particulates that appeared on the plate resulted from sand gathering along the quiescent nodal lines which have no vertical movement while the plate is under vibration. The nodal areas are contrasted with the vibratory areas whose motion resides between a maximum and minimum amplitude of displacement. If one observes any point along the string they will see that at its maximal amplitude it must momentarily come to rest before changing its direction of motion. At the instant of rest it can be assumed there is no kinetic energy at any of the points. Therefore, if one examines the distribution of energy across the surface of the medium then the nodal lines will be those areas which are constantly absent of kinetic energy. In this chapter we will explore the notion of kinetic and potential energy as it is distributed along the length of the non-uniform string.

5.2 THE NON-UNIFORM STRING’S POTENTIAL ENERGY

A mathematical argument lets one find the potential energy along the non-uniform string. In the derivation of the string’s energy we shall assume the string as being composed of a uniformly distributed mass density \( \rho \) and a series of infinitesimal elements of length \( \Delta x \), whose mass is equal to \( \rho \Delta x \). Each string element has zero total energy at equilibrium and small deflections move disturbances about space in a frictionless manner. Finally, we shall assume that the tension \( T(x) \) is linearly variable along the string. The potential energy at any position \( x \) of the strings length \( L \) as described by \( u(x, t) \), is only the work required by an external force to move an element of the string from position \( u = 0 \) to a position \( u = u(x, t) \).
Figure 5.1. Analysis of the motion of a segment of string. Left: As the string moves from position $\overline{AC}$ to $\overline{AB}$ it makes a series of stops and we consider the point mass along the line $\overline{BC}$ whose motion is described by $\beta_k u(x_0 + \Delta x)$, $0 \leq \beta \leq 1$. Right: The free body diagram of the body forces on the point mass.

Considering the segment of string depicted in the Fig. 5.1 of length $\Delta x$ we can imagine the center of mass of our infinitesimal element moving vertically on the line $\overline{BC}$. The string segment initially positioned at $\overline{DE}$ is characterized by $u = 0$ and must be moved to the position $\overline{AB}$ characterized by $u = u(x,t)$. First, we shall move the string to position $\overline{AC}$ then to position $\overline{AB}$. The energy required to move the string from $\overline{DE}$ to $\overline{AC}$ is zero since the tensile force is perpendicular to the movement. For the movement of the string from $\overline{AC}$ to $\overline{AB}$ it is necessary to stretch the segment’s length from $\Delta x$ to $h = \sqrt{\Delta x^2 + \Delta u^2}$.

We proceed using an argument similar to that found in Ref. [5]. When the segment of string has a wave imparted into it and the segment moves and stretches then it makes a series of stops annotated as $\beta u(x,t)$. In this manner, as $\beta$ ranges from 0 to 1, we can imagine our element making stops at $\beta u(x_0 + \Delta x)$ along the line $\overline{BC}$, beginning at $u = 0$ and terminating at $u = u(x_0 + \Delta x)$. Let $T(x_0)$ be the tensile force our element feels towards the direction of $x_0$ and $T(x_0 + \Delta x)$ be the tensile force felt towards the point $x_0 + 2\Delta x$. Note that the string is designed in a fashion for which all its elements feel a mutually attracting tensile force which increases as the distance between the elements
increase. Also, as our element is moved vertically, the vertical component of the stretch or tensile force acts to restore the point mass towards the equilibrium position. Thus, we shall designate the vertical component of the tensile forces acting on the element as

$$F_{y1} = T(x_0 + \Delta x) \sin (\theta(x_0 + \Delta x)),$$  \hspace{1cm} (5.1)

and

$$F_{y2} = -T(x_0) \sin (\theta(x_0)).$$ \hspace{1cm} (5.2)

Summing the tensile forces gives the total body forces acting on the segment as

$$\sum F_y(x_0 + \Delta x, t) = T(x_0 + \Delta x) \sin (\theta(x_0 + \Delta x, t)) - T(x_0) \sin (\theta(x_0, t)),$$ \hspace{1cm} (5.3)

where \(y\) denotes forces acting in the \(y\)-direction. Noting the point \(x_0\) is chosen randomly and using Taylor’s expansion on the R.H.S. of Eq. (5.1) we have:

$$T(x + \Delta x) \sin (\theta(x + \Delta x, t)) - T(x) \sin (\theta(x, t)) = \frac{\partial}{\partial x} [T(x) \sin(\theta(x, t))] \, dx + O(dx^2).$$ \hspace{1cm} (5.4)

Observing Fig. 5.1, and applying the common trigonometric identity for the sine function, it follows

$$\sin(\theta(x, t)) = \frac{\Delta(\beta u)}{\Delta x^2 + (\Delta(\beta u))^2} = \frac{\Delta(\beta u)/\Delta x}{\sqrt{1 + (\Delta(\beta u)/\Delta x)^2}} = \frac{\partial(\beta u)/\partial x}{\sqrt{1 + (\Delta(\beta u)/\Delta x)^2}},$$

and since the deflections are small, it follows \(\Delta(\beta u)/\Delta x \ll 1\). Therefore to second order we have

$$\sin(\theta(x, t)) = \frac{\partial(\beta u)}{\partial x}.$$ \hspace{1cm} (5.5)

Writing \(F_{tot}(x, t) = \sum F_y(x_0 + \Delta x, t)\) and inserting Eq. (5.3) into (5.2) we obtain

$$F_{tot}(x, t) = \frac{\partial}{\partial x} \left[ T(x) \frac{\partial(\beta u)}{\partial x} \right] \, dx.$$ \hspace{1cm} (5.6)

The above equation is no surprise for it is the right hand side of the non-uniform wave equation with \(Q(x, t) = 0\) and \(\beta = 0\).

Applying the notion of the work accomplished by a variable force from Ref. [9] it is pointed out that \(F_{tot}(x, t)\) is a variable force over \(\beta u\), but is approximately constant
over a small interval $d(\beta u)$. Thus, the little amount of work, $dW$, required to move the segment of string from $\beta u$ to $(\beta + d\beta)u$ is

$$dW = F_{\text{tot}}(x, t) \, d(\beta u) = u F_{\text{tot}}(x, t) \, d\beta. \quad (5.7)$$

The infinitesimal work, $dU$, required to move the string from $u = 0$ to $u = u(x, t)$ is obtained by integrating Eq. (5.5) from $\beta = 0$ to $\beta = 1$ and is expressed as

$$dU = \int_0^1 u F_{\text{tot}}(x, t) \, d\beta = \int_0^1 u \frac{\partial}{\partial x} \left[ T(x) \frac{\partial(\beta u)}{\partial x} \right] \, dx \, d\beta$$

$$= \int_0^1 u \frac{\partial}{\partial x} \left[ T(x) \frac{\partial(u)}{\partial x} \right] \, dx \, d\beta = u \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] \, dx \int_0^1 \beta \, d\beta$$

$$= \frac{1}{2} u \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] \, dx.$$

Thus, the density of potential energy of the non-uniform string is

$$dU = \frac{1}{2} u \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] \, dx. \quad (5.8)$$

As for the kinetic energy, let the equilibrium position of the string be positioned at the location $DE$ and $u(x, t)$ denote the location of the string at point $x$ at time $t$ from equilibrium. Then at time $t$ the kinetic energy of the string element at the point $x_0 + \Delta x$ of mass $m = \rho \Delta x$ is

$$dK = \frac{1}{2} m \left( \frac{\partial u}{\partial t} \right)^2 = \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 \, dx. \quad (5.9)$$

Where $dK/dx$ is the kinetic energy density of the non-uniform string.

The total mechanical energy $dE_{\text{tot}}$ from the strain and kinetic energies of the string at any point $x$ is therefore

$$dE_{\text{tot}} = dK + dU = \frac{1}{2} \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 - u \frac{\partial}{\partial x} \left( c^2 \frac{\partial u}{\partial x} \right) \right] \, dx, \quad (5.10)$$

where $c^2 = T(x)/\rho$. 

5.3 Application of the Results to a Resonating Non-uniform String

To show how the results of the proceeding section correctly predicts the energy dynamics of a resonant non-uniform wave, once again consider the non-uniform string’s expansion in terms of the eigenfunctions $\phi_i(x)$ written as

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_i(t)\phi_i(x) = \varphi_1(t)\phi_1(x) + \varphi_2(t)\phi_2(x) + \varphi_3(t)\phi_3(x) + \cdots$$

As the string resonates at one of its normal frequencies $\omega_n = \sqrt{\lambda_n - \beta^2/2}$ we have

$$u(x, t) \approx \varphi_n(t)\phi_n(x).$$

(5.11)

Inserting Eq. (5.9) into Eq. (5.8) we can compute $dE_{\text{tot}}$ as

$$dE_{\text{tot}} = \frac{1}{2} \rho \left[ \left( \frac{d\varphi_n}{dt} \right)^2 \phi_n^2(x) - \varphi_n^2(t)\phi_n(x) \frac{d}{dx} \left( c^2 \frac{d\phi_n}{dx} \right) \right] dx.$$

(5.12)

Since the eigenfunction $\phi_n(x)$ satisfies the Helmholtz equation

$$\frac{d}{dx} \left[ c^2(x) \frac{d\phi_n}{dx} \right] + \phi_n \lambda_n = 0.$$

(5.13)

Eq. (5.10) can be rewrote as

$$dE_{\text{tot}} = \frac{1}{2} \rho \left[ \left( \frac{d\varphi_n}{dx} \right)^2 \phi_n^2(x) + \varphi_n^2(t)\phi_n^2(x)\lambda_n \right] dx.$$

(5.14)

$$\Rightarrow \frac{1}{2} \rho \left[ \left( \frac{d\varphi_n}{dx} \right)^2 + \varphi_n^2(t)\lambda_n \right] \phi_n^2(x) dx.$$

(5.15)

From the results of Chap. 4, as the string vibrates at its resonant frequency at $t \gg 1$ we know

$$\varphi_i(t) \approx \frac{\mu(\lambda_i)F_0}{\sqrt{(\lambda - \omega^2)^2 + \beta^2\omega^2}} \sin(\omega t + \theta).$$

(5.16)

Inserting Eq. (5.14) into Eq. (5.13) gives

$$dE_{\text{tot}} = \frac{1}{2} \rho \left( \frac{\mu(\lambda_i)F_0}{\sqrt{(\lambda - \omega^2)^2 + \beta^2\omega^2}} \right)^2 \phi_n^2(x) \left[ \omega^2 \cos^2(\omega t + \theta) + \lambda_n \sin^2(\omega t + \theta) \right] dx.$$

(5.17)
Figure 5.2. Energy distribution of the non-uniform string. In both plots, (Top): The non-uniform string as it resonates in its third vibrational mode for $a = -1.18, b = 1$. (Bottom): The wave’s energy at various points (black circles) compared with Eq. (5.16) (solid black).

For resonant forced vibrations we have $\omega^2 = \lambda_n - \beta/2$ so that Eq. (5.15) becomes

$$dE_{tot} = \frac{1}{2} \rho (\mu(\lambda_n) F_0)^2 \phi_n^2(x) \left[ \frac{\lambda_n - \beta}{\omega^2} \cos^2(\omega t + \theta) \right] dx.$$  (5.18)
Note the point mass’ total energy varies in time and diverges as $\beta \to 0$.

Figure 5.2 is the results of resonating the non-uniform string in its third vibrational mode for $a = -0.18$ and $b = 1$. As the non-uniform string is initially perturbed at its resonance frequency, its energy density begins to swell around the neighborhood of the anti-nodes as a wave is propagated through its length. At the nodes, where no vibration occurs, the total energy remains zero. For $t \gg 1$ the strings energy density exponentially approaches and stabilizes on the curve described by Eq. (5.16). The results show that at resonance, the string’s energy density is precisely that as predicted in Eq. (5.16).

### 5.4 TOTAL ENERGY OF THE NON-UNIFORM STRING

In this section we set the stage to expand our analysis to a two dimensional non-uniform membrane. Integrating Eq. (5.10) from $0$ to $L$ gives

$$E_{\text{tot}} = \frac{1}{2} \rho \int_0^L \left[ \left( \frac{\partial u}{\partial t} \right)^2 - u \frac{\partial}{\partial x} \left( c^2 \frac{\partial u}{\partial x} \right) \right] dx, \quad (5.19)$$

$$= \frac{1}{2} \rho \int_0^L \left( \frac{\partial u}{\partial t} \right)^2 dx - \frac{1}{2} \rho \int_0^L u \frac{\partial}{\partial x} \left( c^2 \frac{\partial u}{\partial x} \right) dx. \quad (5.20)$$

We can use integration by parts on the expression for the strain energy in Eq. (5.20) by setting

$$\xi = u, \quad d\eta = \frac{\partial}{\partial x} \left( c^2 \frac{\partial u}{\partial x} \right) dx = \partial \left( c^2 \frac{\partial u}{\partial x} \right) u,$$

so that

$$d\xi = du, \quad \eta = \int \partial \left( c^2 \frac{\partial u}{\partial x} \right) = c^2 \frac{\partial u}{\partial x}.$$
Now we have

\[ \int_0^L u \frac{\partial}{\partial x} \left( c^2 \frac{\partial u}{\partial x} \right) \, dx = uc^2 \frac{\partial u}{\partial x} \bigg|_0^L - \int_0^L c^2 \frac{\partial u}{\partial x} \, du = uc^2 \frac{\partial u}{\partial x} \bigg|_0^L - \int_0^L c^2 \left( \frac{\partial u}{\partial x} \right)^2 \, dx. \]

Figure 5.3. The forced non-uniform string. For \( a = -0.18, \ b = 1, \ \beta = 0.15 \) and \( \lambda_3 \approx 0.6334 \) the non-uniform string is periodically forced at \( \omega_3 \). The plots show the string (solid black) approaching its maximum amplitude (dashed black) as it resonates.
Figure 5.4. Total energy of the non-uniform string. For $a = -0.18$, $b = 1$, $\beta = 0.15$ and $
abla_3 \approx 0.6334$ the non-uniform string is periodically forced at $\omega_3$. The total energy (solid black) of the wave is calculated at each time step as it is exposed to forced vibrations. The wave’s energy approaches its maximum value (dashed black) as it approaches its maximum amplitude.

Inserting the above equation into the right integrand of Eq. (5.20) gives

$$E_{tot} = \frac{1}{2} \rho \int_0^L \left( \frac{\partial u}{\partial t} \right)^2 \, dx + \frac{1}{2} \rho \int_0^L c^2 \left( \frac{\partial u}{\partial x} \right)^2 \, dx - uc^2 \frac{\partial u}{\partial x} \bigg|_0^L,$$

(5.21)

$$E_{tot} = \frac{1}{2} \rho \int_0^L \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] \, dx - uc^2 \frac{\partial u}{\partial x} \bigg|_0^L.$$

(5.22)

Noting the boundary conditions $u(0, L) = \frac{du}{dt} \bigg|_0 = 0$ we have

$$E_{tot} = \frac{1}{2} \rho \int_0^L \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] \, dx,$$

(5.23)

which is the well known equation for the total energy of the string. In Fig. 5.3 is a plot of the string’s total mechanical energy as calculated from Eq. (5.23) while it is subjected
periodic forcing corresponding to its third mode of vibration. The string's energy begins at $E_{\text{tot}} = 0$ at $t = 0$ and grows exponentially over time. A slight oscillation is observed in the $E_{\text{tot}}$ which is attributed to the damping coefficient $\beta$ and the forcing. As the string approaches resonance, the total mechanical energy approaches and stabilizes about its limiting value as indicated by the dashed line.
CHAPTER 6

THE EFFECTS OF TENSION ADJUSTMENT
ON A TRANSVERSELY VIBRATED SURFACE

The purpose of this chapter is to give an experimental analysis of the manipulation of the non-uniform string’s modal formation by means of a variable tension gradient. Explicitly stated, a single numerical experiment is carried out to determine the response of a vibrating non-uniform string to the variation of tensile forces. In preparation for our numerical analysis a few calculations to determine the string’s sensitivity to forcing amplitude and frequency is performed. The tension will be treated as a functional independent variable consisting of the two independent tension parameters $a$ and $b$. Output data will be collected via computer runs for selected parameters $a$ and $b$. We shall use a 4th order Runge-Kutta scheme to solve the non-uniform wave equation.

We begin by restating some of the key concepts used throughout this manuscript for ease to the reader to follow the method of reasoning in constructing the numerical models. We will begin by restating the non-uniform wave equation and its eigenfunctions and corresponding eigenvalues. Afterwards, we will illustrate how the wave’s frequency response and energy density can be used to arrive at a model response that can be used to measure the departure of the model from established baselines as parameters are varied over time. The goal is to arrive at a rate of change in the tension parameters so that the system becomes adiabatically invariant to tension adjustment for all time.

6.1 DESIGN OF THE NUMERICAL EXPERIMENT

Let time be $t$, the tension of the string at each point along the $x$-axis be given by $T(x)$ and its linear mass density be $\rho_0(x)$. In Ref. [2], the partial differential equation of the vibrating string is given as

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] + \rho_0(x) Q(x, t),$$

(6.1)
With initial and boundary conditions given by

\[
\text{I.C.'s : } \quad u(x, 0) = f(x), \quad \frac{\partial u}{\partial x}(x, 0) = g(x), \quad (6.2)
\]

\[
\text{B.C.'s : } \quad u(0, t) = 0, \quad u(L, t) = 0. \quad (6.3)
\]

The string is taken to be initially at rest in its equilibrium position. As discovered in Chap. 2, the eigenfunctions, \( \phi_n(x) \), of the non-uniform wave equation are

\[
\phi_n(x) = C_1 \left[ J_0 \left( \frac{2\sqrt{\lambda_n} a}{\sqrt{T(x)}} \right) - k Y_0 \left( \frac{2\sqrt{\lambda_n} a}{\sqrt{T(x)}} \right) \right], \quad (6.4)
\]

and from Chap. 4 we found the time-dependent coefficients to be

\[
\varphi_n(t) = A e^{(-\beta/2)t} \sin \left( \frac{\sqrt{4\lambda_n - \beta^2}}{2} t + \psi \right) + \frac{\mu(\lambda_n) F_0}{\sqrt{(\lambda_n - \omega_n^2)^2 + \beta^2 \omega_n^2}} \sin (\omega t + \theta), \quad (6.5)
\]

where \( \tan(\theta_n) = (\lambda_n - \omega_n^2) / (\beta \omega_n) \) and

\[
\mu(\lambda_n) = \frac{L}{\int_0^L \phi_n(x) \, dx} - \frac{0}{\int_0^L \phi_n(x)^2 \, dx}. \quad (6.6)
\]

From Eq. (6.5) it follows that as \( t \to \infty \)

\[
\varphi_n(t) \to \frac{\mu(\lambda_n) F_0}{\sqrt{(\lambda_n - \omega_n^2)^2 + \beta^2 \omega_n^2}} \sin(\omega_n t + \theta_n). \quad (6.7)
\]

The integer \( n \) denotes a particular mode of vibration with corresponding eigenvalue \( \lambda_n \). Previously in Chap. 3 it was shown that for any particular mode \( n \) the value of \( \lambda_n \) spans a range of values throughout a domain of \( a \in [-2b/L, 2b/L] \) where the parameter \( b \) remains fixed. For each \( a \), its corresponding eigenvalue \( \lambda_n \) is computed via the techniques discussed in Chap. 3. In Chap. 5 the total mechanical energy for the non-uniform string given as the integral equation

\[
E_{\text{tot}} = \frac{1}{2} \rho \int_0^L \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] \, dx. \quad (6.8)
\]
Figure 6.1. Total string energy vs. the parameter $a$. A plot of the string’s steady state energy for the third mode of vibration over the domain $a \in [-.18, .18]$ and $b = 1$.

Now we use the ideas of modal vibration, steady state conditions and energy to establish a baseline to evaluate the string’s response to tension adjustment. For each $a_i \in [-2b/L, 2b/L]$ with $i = 1, 2, \ldots, 40$ we select $\lambda_n$ and calculate the forcing frequency as $\omega_n = \sqrt{\lambda_n - \beta/2}$. Next, for the forcing function defined as

$$Q(x, t) = F_0 \cos(\omega t),$$

we integrate, via a 4th order Runge-Kutta method, Eq. (6.1) for $t \in [0, 1000]$ so that the wave is on its resonance attractor as described by Eq. (6.7). During the integration scheme we compute $E_{\text{tot}}$ at each time step. After the run we compute $\text{avg} \{E_{\text{tot}}\}$ for the final 200 seconds. The result is in Fig. 6.1 which shows the string’s optimal energy baseline for each $a$ as it vibrates on its attractor for a given mode. Figure 6.1 demonstrates the influence of the parameter $a$ on the total energy of the wave’s steady state solution with the string’s minimum energy at $a = \pm 2b/L$. The string’s maximum energy state is at $a = 0$. 
6.2 RESULTS

We chose to test the dependence of the numerical model’s total energy output on perturbations to the parameter $a$ and determine whether the perturbations influence the energy output to any significant degree. Other model parameters are held fixed. As before, we periodically force the non-uniform string with a choice of $a_i \in [-2b/L, 2b/L]$ with $i = 1, 2, \ldots, 40$. Then, at $t = 1000$ we perturb $a$ by an amount $\epsilon f(t)$. Explicitly stated, at $t > 1000$

$$a = a_0 + \epsilon f(t), \quad (6.10)$$

where $a_0 = -2b/L$, signifying the string beginning in its extreme right deformed state and perturbed left. The results of the perturbations with $\epsilon = 10^{-5}$ and $f(t) = H(t - 850)(t - 850)$, where $H$ is the Heaviside function, is included in Fig. 6.2. The plots show that when the tension adjustment is introduced to the system then the magnitude of strings energy decays rapidly from its steady state. Thus, even if $a$ is changed by a fairly small amount at each time step the system’s energy is greatly decreased in a short time span without making much headway towards shifting the strings nodes.

6.3 EXPLAINING THE STRING’S RESPONSE TO TENSION ADJUSTMENT

We want to more closely examine the temporal effects of perturbations of the parameter $a$ on the non-uniform string. As given by Eq. (6.5), the non-uniform string’s temporal motion is described in terms of a one dimensional damped harmonic oscillator given by

$$\frac{d^2 \varphi_n}{dt^2} + \beta \frac{d \varphi_n}{dt} + \lambda_n \varphi_n = \mu(\lambda_n) F_0 \cos(\omega t). \quad (6.11)$$

Some modifications to Eq. (6.11) can be made in order to simplify our analysis. First, note that when $t \gg 1$ the combined effect of the damping coefficient and forcing term in Eq. (6.11) is the resonant harmonic wave in Eq. (6.7) which oscillates with a near constant amplitude on its attractor. Let us proceed forth by setting $\beta = 0$ and $F_0 = 0$ in Eq. (6.11) to yield the familiar equation

$$\frac{d^2 \varphi}{dt^2} + \lambda_n \varphi = 0, \quad (6.12)$$
Figure 6.2. Total string energy for perturbations to the parameter $a$. At $t = 850$ the parameter $a$ is perturbed by $\epsilon f(t)$ where $\epsilon = 10^{-5}$ and $f(t) = H(t - 850)(t - 850)$. When the string’s tension is continuously adjusted its energy drops substantially.

which is a harmonic oscillator of frequency $\omega_n = \sqrt{\lambda_n}$ with constant amplitude. To well pose the discussion at hand, let us further impose the initial conditions

$$\varphi(0) = 0,$$
$$\frac{d}{dt}\varphi(0) = \sqrt{\lambda_0}.$$

In the non-uniform string system, if the parameter $a$ is perturbed by setting $a = a_0 + \epsilon f(t)$ then we model the perturbation to the string’s eigenvalues as $\lambda = \lambda_0 + \epsilon g(t)$. Thus, the relation $\omega = \sqrt{\lambda}$ can be extended to define the time dependent frequency

$$\omega^2(\epsilon t) = \lambda_0 + \epsilon g(t), \quad (6.13)$$

and refine Eq. (6.12) as

$$\frac{d^2\varphi}{dt^2} + \omega_n^2(\epsilon t)\varphi = 0. \quad (6.14)$$

Equation (6.14) describes a simple harmonic oscillator whose frequency changes over time. The physical interpretation of Eq. (6.14) is considered to describe small amplitude
oscillations of a pendulum with a slowly varying length. A numerical solution to Eq. (6.14) shows that the amplitude of the solution’s oscillations changes with respect to time. Therefore, we have simplified the equation of motion of the non-uniform string without removing the decay phenomena we are interested in studying.

To study the dynamics of Eq. (6.14) we shall consider its motion over two time scales. First, for \( \epsilon \ll 1 \) we shall define the slow \( O(\epsilon) \) time scale by setting \( \tau = \epsilon t \) so that if \( g(t) = O(t) \) then \( \omega^2(\epsilon t) \) changes slowly over time. If \( \epsilon \ll 1 \) then \( \omega(\epsilon t) \) may be taken to be quasi-static and one may take a first order solution to Eq. (6.14) as

\[
\varphi(t) \approx A(t) \sin(\omega(\epsilon t)t) .
\] (6.15)

Proceeding with the assumption of Eq. (6.15) we define the parameter

\[
\theta = \omega(\epsilon t)t = \omega(\tau)t,
\]

for which multiplication by \( \epsilon \) yields

\[
\epsilon \theta = \omega(\tau)\epsilon t = \zeta(\tau),
\] (6.16)

and thus define \( \theta \) as the fast \( O(1) \) time variable as

\[
\theta = \frac{\zeta(\tau)}{\epsilon}.
\] (6.17)

Returning back to the mechanics of two time scales, we expand the solution of Eq. (6.14) in a power series as

\[
\varphi(\theta, \tau) = \sum_{k=0}^{\infty} \epsilon^k \varphi_k(\theta, \tau) = \varphi_0(\theta, \tau) + \epsilon \varphi_1(\theta, \tau) + \epsilon^2 \varphi_2(\theta, \tau) + \cdots .
\] (6.18)

By the aforementioned definitions of \( \tau \) and \( \theta \), the time derivatives of \( \varphi \) are transformed by using the chain rule of differentiation as:

\[
\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial \theta} \frac{\partial \theta}{\partial t} + \frac{\partial \varphi}{\partial \tau} \frac{\partial \tau}{\partial t}.
\]

Since \( \theta = \zeta(\tau)/\epsilon \) it follows

\[
\frac{\partial \theta}{\partial t} = \frac{\partial \zeta}{\partial \epsilon \partial t} = \frac{\partial \zeta}{\partial \epsilon \partial t} = \frac{\partial \zeta}{\partial \tau}.
\]
Hence,
\[
\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial \theta} \frac{\partial \zeta}{\partial \tau} + \epsilon \frac{\partial \varphi}{\partial \tau},
\]
and using compact notation we have
\[
\frac{d\varphi}{dt} = \zeta \partial_\theta + \epsilon \partial_\tau. \tag{6.19}
\]
Next, we compute the second derivatives again using the chain rule:
\[
\frac{d^2 \varphi}{dt^2} = \frac{\partial \dot{\varphi}}{\partial \theta} \frac{\partial \zeta}{\partial \tau} + \epsilon \frac{\partial \dot{\varphi}}{\partial \tau} + 
\]
\[
\left[ \frac{\partial^2 \varphi}{\partial \theta^2} \left( \frac{\partial \zeta}{\partial \tau} \right) + \epsilon \frac{\partial \varphi}{\partial \tau \partial \theta} \right] \frac{\partial \zeta}{\partial \tau} + \epsilon \left[ \frac{\partial \varphi}{\partial \tau \partial \theta} \left( \frac{\partial \zeta}{\partial \tau} \right) + \frac{\partial \varphi}{\partial \theta} \frac{\partial^2 \zeta}{\partial \tau^2} \right] + \epsilon^2 \frac{\partial^2 \varphi}{\partial \tau^2}.
\]
Simplifying the above and using compact notation we have
\[
\frac{d^2 \varphi}{dt^2} = (\zeta_t)^2 \varphi_{\theta \theta} + \epsilon \left( 2 \zeta_t \varphi_{\theta \tau} + \zeta_{\tau \tau} \varphi_\theta \right) + \epsilon^2 \varphi_{\tau \tau}. \tag{6.20}
\]
Inserting Eq.(6.18) into Eq.(6.20) and then Eq.(6.20) into Eq.(6.14) one achieves
\[
(\zeta_t)^2 \varphi_{\theta \theta} + \omega^2 \varphi_0 + \epsilon (\zeta_t)^2 \varphi_{1 \theta \theta} + \epsilon \omega^2 \varphi_1 + \epsilon (2 \zeta_t \varphi_{\theta \theta} + \zeta_{\tau \tau} \varphi_0) + O(\epsilon^2) = 0. \tag{6.21}
\]
Collecting powers of $\epsilon$ yields a pair of differential equations:
\[
O(1) : \quad (\zeta_t)^2 \varphi_{\theta \theta} + \omega^2 \varphi_0 = 0. \tag{6.22}
\]
\[
O(\epsilon) : \quad (\zeta_t)^2 \varphi_{1 \theta \theta} + \omega^2 \varphi_1 + (2 \zeta_t \varphi_{\theta \theta} + \zeta_{\tau \tau} \varphi_0) = 0. \tag{6.23}
\]
At this point, it is noted that instead of explicitly setting $\theta = \omega(t \epsilon)$ we are justified in implicitly defining $\theta = \zeta(\tau)/\epsilon$ and enforcing
\[
\frac{d\zeta}{d\tau} - \omega = 0, \tag{6.24}
\]
with the initial condition defined by
\[
\zeta(0) = 0. \tag{6.25}
\]
Solving the above IVP by separation of variables we obtain
\[
\zeta(\tau) = \int \omega(\tau) d\tau = \int (\lambda_0 + \tau)^{\frac{3}{2}} d\tau
= \frac{2}{3} (\lambda_0 + \tau)^{\frac{3}{2}} + C(0),
\]
for which the I.C. in Eq. (6.25) implies $C(0) = -\frac{2}{3}(\lambda_0)^{\frac{3}{2}}$. Thus,

$$\theta = \frac{2}{3}(\lambda_0 + \tau)^{\frac{3}{2}} - \frac{2}{3}(\lambda_0)^{\frac{3}{2}}. \quad (6.26)$$

Returning to the main problem, Eq. (6.24) implies

$$(\zeta_\tau)^2 = \omega^2,$$

so the general solution to $\varphi_0$ is

$$\varphi_0 = A(\tau) \sin \theta + B(\tau) \cos \theta. \quad (6.27)$$

To determine $A(\tau)$ and $B(\tau)$, we need to go to the next higher order of $\epsilon$. Substituting Eq. (6.27) into Eq. (6.23) gives

$$\omega^2 \left( \varphi_{1\theta\theta} + \varphi_1 \right) + \left( 2\omega \dot{A} + \omega_\tau A \right) \cos \theta - \left( 2\omega \dot{B} + \omega_\tau B \right) \sin \theta = 0, \quad (6.28)$$

where the dot denotes differentiation with respect to $\tau$. Since we want approximations free from secular terms, in Eq. (6.28) we set the coefficients of the resonant terms to zero. Thus, we require

$$2\omega \dot{A} + \omega_\tau A = 0, \quad (6.29)$$
$$2\omega \dot{B} + \omega_\tau B = 0. \quad (6.30)$$

Noticing the chain rule in the above ODE’s it follows

$$\left( \omega A^2 \right)_\tau = 0, \quad (6.31)$$
$$\left( \omega B^2 \right)_\tau = 0, \quad (6.32)$$

so that Eqs. (6.31) and (6.32) can be solved to give

$$\left( \omega A^2 \right) = \omega(0)A(0)^2 = \text{constant}, \quad (6.33)$$
$$\left( \omega B^2 \right) = \omega(0)B(0)^2 = \text{constant}. \quad (6.34)$$

Thus, the amplitude of the oscillator is proportional to $\omega^{-1/2}$ as the frequency changes. To complete the analysis we must determine the initial values $A(0)$, $B(0)$ and $\omega(0)$. The
initial values are determined by Eqs. (6.13), (6.18) and (6.19) and the given initial conditions \( \varphi(0) = 0 \) and \( \dot{\varphi}(0) = \sqrt{\lambda_0} \). At \( t = 0 \), Eq. (6.13) gives \( \omega(0) = \sqrt{\lambda_0} \) and Eq. (6.18) gives \( 0 = \varphi_0(0, 0) + \epsilon \varphi_1(0, 0) + O(\epsilon^2) \). To satisfy the initial condition on \( \varphi \) for all sufficiently small \( \epsilon \), we must have

\[
\varphi_0(0, 0) = 0 \quad \text{and} \quad \varphi_1(0, 0) = 0. \tag{6.35}
\]

Then Eq. (6.19) gives,

\[
\sqrt{\lambda_0} = \partial_\tau \zeta(0, 0) \partial_\theta \varphi_0(0, 0) + \epsilon \partial_\tau \zeta(0, 0) \partial_\theta \varphi_1(0, 0) + \epsilon \partial_\tau \varphi_0(0, 0) \tag{6.36}
\]

For which \( \partial_\tau \zeta(0, 0) = \sqrt{\lambda_0} \) implies

\[
\sqrt{\lambda_0} = \sqrt{\lambda_0} \partial_\theta \varphi_0(0, 0) + \epsilon \left( \sqrt{\lambda_0} \partial_\theta \varphi_0(0, 0) + \partial_\tau \varphi_0(0, 0) \right), \tag{6.37}
\]

so that

\[
\partial_\theta \varphi_0(0, 0) = 1, \tag{6.38}
\]

and \( \sqrt{\lambda_0} \partial_\theta \varphi_1(0, 0) + \partial_\tau \varphi_0(0, 0) = 0 \). Combining Eq. (6.35) with Eq. (6.27) we find

\( B(0) = 0 \); hence by Eq. (6.34), \( B(\tau) \equiv 0 \). Similarly, Eq. (6.38) and Eq. (6.27) imply

\( A(0) = 1 \), so Eq. (6.33) gives \( A(\tau) = \sqrt{\omega_0/\omega} \). Finally, Eq. (6.27) becomes

\[
\varphi_0(\theta, \tau) = \sqrt{\omega_0/\omega} \sin \theta. \tag{6.39}
\]

Hence

\[
\varphi(\theta, \tau) = \sqrt{\omega_0/\omega} \sin \theta + O(\epsilon), \tag{6.40}
\]

\[
= \sqrt{\omega_0/\omega} \sin \left[ \frac{2}{3}(\lambda_0 + \tau) + \frac{2}{3}(\lambda_0)^{1/2} \right] + O(\epsilon) \tag{6.41}
\]

is the predicted approximate solution. Figure 1 shows the results of the above analysis where, over a long time scale, the two curves are almost indistinguishable, even though \( \epsilon \) is not that small.

Thus, a good explanation has been given for the non-uniform string’s undesirable response to tension adjustment. We see the results of the numerical experiment can be
explained in a pure mathematical sense. At the temporal level of the model there exists an equation for a damped harmonic oscillator. Even if the damping coefficient were removed and the tension adjusted so to perturb the system’s eigenvalues, a decay in the system’s amplitude would still be observed because of the properties of the harmonic oscillator as demonstrated via a multiple time scale analysis.

Figure 6.3. Comparison of numerical and two-timing solutions to the equation $\varphi_{tt} + \omega^2(\epsilon t)\varphi = 0$ for $\epsilon = 1/10$. 
In summary, we have analyzed a proposed method for which particulate matter may be moved about a horizontal surface. The standard wave equation with a variable tension gradient was used as a model for which particle movement may be accomplished. The governing equations of motion were solved analytically via the method of separation of variables. It was found the non-uniform string’s eigenfunctions are an interesting combination of both the $J$ and $Y$ Bessel functions. The eigenvalues for any particular mode were dependent upon the tension parameters and spread out over a range of values.

By analyzing the various oscillatory mode shapes we see that by adjusting the tension parameters then any mode shape could be deformed either to the left or to the right. The non-uniform string’s temporal characteristic bore a strong similarity to that of a classical string. Frequency response curves for the system can be generated that predict the steady-state amplitude response of the system under periodic forcing. The 4th order Runge-Kutta method of integration proved to be an efficient and accurate means of obtaining numerical solutions to the system which compared very well to some of the analytical predictions of the model’s behavior.

The energy density of the system was derived using some methods from theoretical physics as well as some other classical techniques. The system’s eigenfunction decomposition for a particular mode can be inserted into the energy density function which provides a steady state energy distribution along the length of string. Numerical simulations of the total energy of the system over a range of tension parameters were used to compare the system’s state of order with an optimal energy state. An important numerical experiment was designed based on the analytical results of the manuscript. Through experiment, the effect of tension adjustment on the horizontal string as it is periodically forced over time was understood. We observed the following effects on the vibrating non-uniform string when perturbations were introduced:

- Tensile perturbations had a dramatic negative effect on the system’s energy state.
The string’s energy state decreased even though the magnitude of perturbations were of low order. Continuous tensile adjustment over time resulted in a continuous degradation of the system’s energy which correlates with less defined string nodes.

Therefore, based on the results of the manuscript the following conclusions and recommendations are proposed. From the above observations we conclude that, in its current state of development, tensile adjustment should not be attempted as a viable means for manipulating nodal patterns. However, the results of this manuscript lay a good foundation for future exploration in the correction of the one-dimensional model’s deficiencies. As a proposed method of correction, methods by which the string is forced can be explored. By adjusting the forcing amplitude along with frequency while the string’s tension is adjusted then the appropriate amount of energy may be restored to compensate for the system’s inherent decay. Also, the model’s design can easily be extended to higher dimensions. For example, the model could be expanded into a two-dimensional membrane with analytical solutions easily obtainable from the functional analysis performed within this manuscript. In the non-uniform membrane model, the energy potentials of a string for each axis can be applied to a Lagrangian and the principle of least action to arrive at the non-uniform membrane’s equations of motion. The membrane eigenfunction’s would then be a product solution of the eigenfunctions found here. Eigenvalues would again be dependent on tensile parameters and mode shapes could be deformed either to the left or to the right. Additionally, many if not all numerical simulations and experiments can be easily modified and expanded to higher dimensions such as for a membrane.
BIBLIOGRAPHY


