ON THE ASYMPTOTIC STABILITY OF THE ZERO SOLUTION
FOR A LINEAR DIFFERENTIAL EQUATION WITH TWO DELAYS

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Mathematical models incorporating discrete time delays are utilized in scientific applications, because delays are intrinsic in the structure of many biological, physical, economic, and control systems. The following thesis is a study of a linear, scalar delay differential equation with two delays. The stability analysis given in this thesis uses the definitions, theorems and framework given by Mahaffy, Zak, and Joiner. After scaling the system has four model parameters: three coefficients and one delay term. For an interval of fixed delay values, the two- and three-dimensional regions of stability were located using the theory and definitions provided by Mahaffy, et al., in conjunction with the computational tools and routines developed during the research portion of this thesis. The regions of stability investigated displayed interesting geometric characteristics, many of which are summarized and illustrated in this work. One of those features are self-intersecting bifurcation surfaces that result in regions of stability that create “spur-like” formations on the main stability surface. Additionally, a complete characterization of the unbounded stability surface of the two-delay equation for a particular fixed delay is provided in this thesis. The evidence explained within also provides the basis for seeking a proof that the region of stability for that fixed delay is substantially larger than an already established guaranteed minimum.
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CHAPTER 1
INTRODUCTION

Delay differential equations (DDEs) are used in a variety of applications, and understanding their stability properties is a complex and important problem. It has been observed that a differential equation with two delays, which are rationally related, has a region of stability that is larger than one with delays nearby that are irrationally related. This thesis investigates the stability region of the following linear, two-delay differential equation:

\[ \dot{y}(t) + a y(t) + b y(t - r_1) + c y(t - r_2) = 0, \]  

(1.1)
as the coefficient parameters \(a, b, c \in \mathbb{R}\) vary and the ratio of the delays, \(\frac{r_2}{r_1}\), varies over the interval \([0.20, 0.25]\).

The study of Eqn. (1.1) and its stability properties may seem like a fairly simple thing to do upon a first glance at the equation. Equilibrium solutions of (1.1) and its associated characteristic equation are easily found. However, the characteristic equation is a particularly nasty, infinite dimensional, transcendental equation. Tracking the roots of an exponential polynomial of this type, and whether or not those roots have negative real parts provides significant challenges. Describing and illustrating or visualizing the complex labyrinth of bifurcation surfaces that intersect to form the five-dimensional stability region of Eqn. (1.1) seems to present an impossibility. In order to gain a better understanding of the stability space of (1.1), the geometric study of two-, three- and four-dimensional subspaces of Eqn. (1.1) and the rich dynamics manifest by (1.1) are required.

Zaron [48] has shown that when the ratio of the delays is a constant, there is a minimum region of stability in the coefficient-parameter space of Eqn. (1.1). Numerical evidence suggests that for many values of \(\frac{r_2}{r_1}\), the stability region asymptotically approaches this pyramidal-shaped region, for large \(a\). However, for some delay ratios, the region of stability remains significantly larger than this minimum region. As an example, this paper argues and presents numerical evidence that when \(\frac{r_2}{r_1} = 0.25\), then the region of stability remains over 26.86% larger. Our study of the two- and three-dimensional stability boundary for delay ratios approaching 0.25 from the left help motivate and explain the rationale behind this conjecture. The numerical study presented here of the stability region when \(\frac{r_2}{r_1}\) is a fixed value from the interval \([0.20, 0.25]\) investigates distortions in the surface away from the minimum region which jut out from three-dimensional stability surfaces, dubbed stability
spurs by Mahaffy, Zak, and Joiner [34, 35]. The evidence presented here suggests a finite number of these stability spurs are closely correlated to a range of $\frac{r_2}{r_1}$ values.

As an introduction to this geometric study, pictures of a section of the three-dimensional stability surface associated with $\frac{r_2}{r_1} = 0.20$ are shown in Figure 1.1. The minimum region of stability is the checkered, black pyramidal shape within the larger stability surface. Three stability spurs can be seen poking out from the surface. The distortion in the shape away from the minimum stability pyramid is evident.

![Figure 1.1. The stability region in the coefficient-parameter space of Eqn. (1.1).](image)

In Chapter 2, interesting examples of how two-delay differential equations are relevant to the modeling of scientific processes are discussed. A short summary is given of previous works found in the literature that focused on the stability analysis of Eqn. (1.1). Definitions, theorems and guiding principles from the work of Mahaffy, Zak and Joiner [34, 35] that facilitate the eigenvalue analysis of the characteristic equation are provided in Chapter 3. Key definitions are illustrated. Chapter 4 presents results from specific case studies of stability spurs and asymptotic regions of stability. Conjectures are formed based on the data that was gathered in the research phase of this thesis. A focus on the asymptotic region of stability when the delay ratio is 0.25 is at the heart of this thesis. We will conclude in Chapter 5 with a discussion of how the results from Chapter 4 might be extended to other rational delay ratios. In addition, goals are outlined of the direction future work might take.
CHAPTER 2
BACKGROUND

Differential equations incorporating time delays have been used to model a wide range of scientific processes. Delays occur naturally and frequently in time rates of production, transportation, storage, ingestion, incubation and regeneration processes, to name a few.

Species population models, excluding environmental factors and interspecies interaction, have featured discrete time delays to explain for regeneration periods and natality lag. Hutchinson [26] added a delay to the self-regulatory term in the one-species logistic population model. In order to derive a more biologically realistic model, many have introduced a second delay, so that the self-regulatory term depends on two points in past time. Braddock and van den Driessche [11] and later Gopalsamy [17] extended Hutchinson’s [26] equation by simply adding a second time lag. MacDonald [31, 32] and Murdoch et al. [40] constructed interspecies competition or parasitoid-host models admitting multiple delays to offset for different maturation times. Cooke and Yorke [13] and MacDonald [32] modeled the transmission of a gonorrhea epidemic with two delays explaining differing incubation times amongst males and females.

Mathematical physiologists have also incorporated two time delays into their development and disease models; for example, Bélair, Mackey, and Mahaffy [6] use lags in their modeling of erythropoiesis (red blood cell production). MacDonald [29, 30] used lags to model the production of granulocytes, a category of white blood cells. Bélair and Mackey [4] modeled mammalian platelet production with a two-delay differential equation. Beuter, Bélair and Labrie [8] studied neurological disease with two-delay models. Motor-control differential equations, employing non-linear variable delays, have been analyzed by Bélair and Campbell [2, 12], Bélair, Campbell and Van Den Driessche [3] Beuter and Bélair [8], as well as Glass, Bueter, and Larocque [16], Gopalsamy and Leung [18], Guzelis, Cuneyt and Chua [19], and Mohamad and Gopalsamy [39].

Delay differential equations have also been used in the modeling of both mechanical and robotic control [23] systems. Both Marriot et al. [36] and Mizuno & Ikeda [38] combined multiple-time delay compartments within control loops for their optics problems. In the realm of economics, delay differential equations can pertain to the study of the price dynamics affecting a single commodity. Economic cobweb models used for optimal production decisions have time delays explaining the “known lag between the initiation of production decisions and the delivery of goods.” [33] Bélair and Mackey [5] built delays into their
supply and demand price functions to specify how consumers’ memories of past prices factored into the market price model of a particular product. Howroyd and Russell [25] consider a system of two competing firms, both of which can predict each other’s optimal production output, based upon current production levels and the previously known production levels of competitors.

The complicated dynamics that result from the construction of these models can be examined by conducting linear stability analysis. This thesis considers the stability of Eqn. (1.1), which depends on its five parameters and can be determined by locating the complex roots of the characteristic equation: \( \lambda + a + be^{-r_1 \lambda} + ce^{-r_2 \lambda} = 0 \). Bellman and Cooke [7] credited Hayes [24] with having been the first to conduct the bifurcation analysis for the one-delay problem \((c = 0)\). Hayes [24] proved two lemmas about the location of the roots of the characteristic equation \( \lambda = be^{\lambda} \), from which conditions on \( a \) and \( b \) were derived so that the roots of the equation lay in the left half of the complex plane. The explicit representation of the stability set of parameters for the one-delay equation was given by Boese [9, 10], Hale [20] and El’sgolt’s and Norkin [14] (pages 141-142). More recently, Sakata [45] and Matsunaga [37] extended the results of [20, 24] to investigate the asymptotic stability of the one-delay equation with \( b \) being an \( n \times n \) matrix.

The boundary of the stability region for Eqn. (1.1) has been studied by many researchers [1, 15, 21, 22, 28, 35, 41]. Several authors [11, 20, 27, 28, 41, 42, 43, 44, 46, 47] have studied the bifurcation dynamics of a special case of the two-delay problem where \( a = 0 \). Hale and Huang [21] performed a stability analysis of (1.1), where they fixed the parameters, \( a \), \( b \), and \( c \), then constructed the boundary of stability in the \( r_1 r_2 \) delay space. Braddock and van den Driessche [11] completely determined the stability boundary for the case \( b = c \), and partially extended the results outside that special case. Nussbaum [41] and Ruiz-Claeyssen [44] both determined the zeros of Eqn. (1.1) for \( a = 0 \), \( R = 1 \), and with other specific parameter values.

Most of these analyses have studied the two-dimensional stability structure of either Eqn. (1.1) or a version of Eqn. (1.1) with one of the delays scaled to unity, with one parameter equal to zero or fixing some of the parameters. Often the two-dimensional analyses result in observing disconnected stability regions for (1.1). Elsken [15] has proved that the stability region of Eqn. (1.1) is connected in the \( abc \)-parameter space with fixed \( r_1 \) and \( r_2 \). After scaling out one of the delays, Mahaffy, Zak and Joiner [34, 35] studied Eqn. (1.1) for an interval of delay ratio values, examining two-dimensional stability regions and developing complete three-dimensional bifurcation surfaces in the coefficient-parameter space. That work provides the primary background for this thesis. Mahaffy et al. [34, 35] showed interesting results, which suggested further study.
CHAPTER 3
TOOLS AND TECHNIQUES FOR ANALYSIS

In the introduction we presented the first-order linear constant-coefficient differential equation with two delays. A re-scaling of the time variable by replacing $t$ with $r_1 \tau$, and making the substitutions $R = r_2/r_1$, $A = r_1 a$, $B = r_1 b$, and $C = r_1 c$ yields the following model equation:

$$\dot{y}(\tau) + A y(\tau) + B y(\tau - 1) + C y(\tau - R) = 0.$$  

(3.1)

If $A + B + C \neq 0$, then (3.1) has the unique equilibrium solution $y = 0$. If $A + B + C = 0$, then $y = k$ ($k \in \mathbb{R}$) is an equilibrium solution of Eqn. (3.1). Notice that if $r_1 > r_2$ in Eqn. (1.1), then $0 < R < 1$. This thesis studies the region of stability associated with the equilibria of Eqn. (3.1) in the $A, B, C$ and $R$ parameter space, $\mathbb{R}^3 \times (0, 1)$. If the delay ratio, $R$, is a fixed constant, then a connected, unbounded stability surface, formed by the intersection of many bifurcation surfaces, is found in the three-dimensional coefficient-parameter space of Eqn. (3.1). The complex shape of that stability surface does not resemble any commonly known geometric shape. Mahaffy, Zak, and Joiner [34, 35] provide a framework for the analysis of such a surface. The focus in this chapter is to outline that framework, so that this thesis is mostly self-contained.

The stability properties of the equilibrium solutions of Eqn. (3.1) are determined by the eigenvalues of the system. The characteristic equation associated with those eigenvalues is found after substitution of the solution $y = \xi e^{\lambda \tau}$ into Eqn. (3.1), giving

$$\lambda + A + B e^{-\lambda} + C e^{-\lambda R} = 0.$$  

(3.2)

This is an exponential polynomial, which has infinitely many solutions, as one would expect because a DDE is infinite dimensional. From stability theory, it is known that the zero solution of Eqn. (3.1) is asymptotically stable whenever all the roots (eigenvalues) of Eqn. (3.2) have negative real parts. When $\lambda = 0$, any point on the plane $A + B + C = 0$ satisfies the characteristic equation. Therefore, the plane provides one boundary where a real eigenvalue, $\lambda$, crosses between positive and negative, so it creates a bifurcation surface. Above the real root-crossing plane is a guaranteed region of stability, given by the following theorem.

**Theorem 3.1. Minimum Region of Stability (MRS)** For $A > |B| + |C|$, all solutions $\lambda$ to Eqn. (3.2) have $\Re(\lambda) < 0$, which implies that Eqn. (3.1) is asymptotically stable inside the pyramidal-shaped region centered about the positive A-axis, independent of $R$. 

The proof of the theorem can be found in both Zaron [48]. The actual region of stability can be much larger than this guaranteed minimum when \( R \) is rationally dependent. Moreover, it is important to note that since one face of the MRS is located on the \( A + B + C = 0 \) plane, one part of the stability boundary always runs along a portion of the zero root-crossing surface. A loss of the stability of Eqn. (3.1) also comes from complex eigenvalues (with non-zero imaginary part) whose real part change from negative to positive. Thus, determining the image of the imaginary axis in the \( ABC \) parameter space assists in locating the actual boundary of stability, which must be found numerically.

Upon substitution of \( \lambda = \omega i \ (\omega \in \mathbb{R}) \), Eqn. (3.2) can be simplified to:

\[
[A + B \cos(\omega) + C \cos(\omega R)] + i [\omega - B \sin(\omega) - C \sin(\omega R)] = 0.
\]

Equating real and imaginary parts renders the following parametric equations in \( \omega \):

\[
B(\omega) = \frac{A \sin(\omega R) + \omega \cos(\omega R)}{\sin(\omega(1 - R))},
\]

\[
C(\omega) = -\frac{A \sin(\omega) + \omega \cos(\omega)}{\sin(\omega(1 - R))},
\]

where \( \frac{(j-1)\pi}{1-R} < \omega < \frac{j\pi}{1-R} \), and \( j \in \mathbb{Z}^+ \). Because both \( B(\omega) \) and \( C(\omega) \) are even functions in \( \omega \), one need only consider \( j \in \mathbb{Z}^+ \). In addition, as a result of the symmetry of Eqns. (3.3) and (3.4), their definitions describe the image of \( \lambda = \pm \omega i \) in the parameter space of Eqn. (3.1). The singularities for \( B(\omega) \) and \( C(\omega) \) at \( \omega = \frac{j\pi}{1-R} \) lead to the following definition for bifurcation surfaces.

**Definition 3.1.** When a value of \( R \) in the interval \((0, 1)\) is chosen, **Bifurcation Surface** \( j \) is determined by Eqns. (3.3) and (3.4), and is defined parametrically for \( \frac{(j-1)\pi}{1-R} < \omega < \frac{j\pi}{1-R} \) and \( A \in \mathbb{R} \). This creates a separate parameterized surface representing pure imaginary, conjugate solutions of the characteristic equation, (3.2), which can be sketched in the \( ABC \) coefficient-parameter space of Eqn. (3.1), for each positive integer, \( j \).

With the value of \( R (0 < R < 1) \) fixed in place, the real root-crossing plane, \( A + B + C = 0 \), and bifurcation surfaces, given by Eqns. (3.3) and (3.4), intersect in the \( ABC \) space forming the continuous three-dimensional region of stability associated with Eqn. (3.1). Because the MRS is centered on the \( A \)-axis, we often choose to fix \( A \) and view the stability region in the \( BC \) cross section. The perspective allows \( A \) to be increased incrementally to study the complicated boundary of the stability region. Thus, we have the related definition:

**Definition 3.2.** **Bifurcation Curve** \( j \) is determined by Eqns. (3.3) and (3.4), and is defined for \( \frac{(j-1)\pi}{1-R} < \omega < \frac{j\pi}{1-R} \). With the values of \( R \) and \( A \) fixed constant, as \( \omega \) is allowed to sweep through its interval, a separate parameterized curve can be drawn on the \( BC \) plane for each \( j \).
For most values of $A$, the curves in the $BC$ plane generated by (3.3) and (3.4) tend to infinity parallel to the lines $B + C = 0$ or $B - C = 0$, because of the singularities that exist for Eqns. (3.3) and (3.4) when $\omega = \frac{j\pi}{1-R}$. Bifurcation Curve 1, however, presents a special case whereby one end tends to infinity when $\omega = \frac{j\pi}{1-R}$, but taking a limit of $(B(\omega), C(\omega))$ as $\omega \to 0^+$ shows that Curve 1 intersects the real root-crossing plane at $B = (AR + 1)/(1 - R)$ and $C = -(A + 1)/(1 - R)$, which is the line

$$\frac{A + 1}{1 - R} = \frac{B - 1}{R} = -C$$

in $ABC$ parameter space.

For any fixed values of $A$ and $R$, the D-decomposition method given by El’sgol’ts [14] provides a systematic way of determining the finite number of curves that comprise the boundary of stability in the $BC$ plane, as well as the dimension of the unstable manifold for Eqn. (3.1). Figure 3.1 shows a relatively simple example illustrating the D-decomposition partitioning method used to determine the boundary of stability. A picture of the two-dimensional stability region, with fixed values $R = 0.25$ and $A = 5$, is given by the area bounded by $A + B + C = 0$ (purple), Curve 1 (blue), Curve 2 (green) and Curve 3 (black). Since both $B(-\omega) = B(\omega)$ and $C(-\omega) = C(\omega)$ for Eqns. (3.3) and (3.4), crossing a bifurcation curve (blue, green, and black) results in the gain or loss of two complex conjugate eigenvalues with positive real part. Crossing $A + B + C = 0$ (purple) results in the gain or loss of one real, positive eigenvalue. In addition, Figure 3.1 shows several non-generic crossings at points where bifurcation curves intersect. For example, where Curves 1 and 3 intersect, two pairs of eigenvalues cross the imaginary axis. The numbers in each bounded region shown in the figure indicate the dimension of the unstable manifold for Eqn. (3.1).

The two-dimensional analysis in the $BC$ cross sections extends easily to the three-dimensional $ABC$ parameter space. The bifurcation surfaces defined by Eqns. (3.3) and (3.4) form boundaries that partition the three-dimensional space into regions with the same number of eigenvalues with positive real part. By carefully following changes in the boundaries of the $BC$ cross sections, the three-dimensional structure of the stability region (no eigenvalues with positive real part) is found as $A$ varies.

Once a value of $R$ is selected from $(R_0, 1)$, where $R_0 \approx 0.0117$, Mahaffy, Zak and Joiner [34, 35] provide us with a place to begin locating the boundary of stability.

**Theorem 3.2** (Starting Point). If $R > R_0$, then the stability surface comes to a point at $(A_0, B_0, C_0) = \left( \frac{R_0 + 1}{R}, \frac{R}{R - 1}, \frac{R}{R(1 - R)} \right)$, and Eqn. (3.1) is unstable for $A < A_0$.

The proof of Theorem 3 can be found in [34]. At the Starting Point of a stability surface, bifurcation Curve 1 intersects $A + B + C = 0$ in one location, given by Eqn. (3.5). However, when $A$ is increased slightly from $A_0$, Curve 1 intersects the plane a second time.
Figure 3.1. D-decomposition method example.

Consequently, a closed area, the region of stability, is formed by the portions of Curve 1 and the real root-crossing plane in between the two intersections.

Figure 3.2a shows a $BC$ cross-section plot of the stability surface at the Starting Point, $A_0$, for the surface corresponding to $R = 0.25$. The two-dimensional plot faces the negative $A$ direction. Curve 1 (blue) intersects the zero root-crossing plane (purple) at the point, $(B_0, C_0)$, generated by Eqn. (3.5). As $\omega$ increases from zero through the interval $[0, \frac{\pi}{1-R})$, the points of Curve 1 are sequentially drawn in the $BC$ plane. The resulting trajectory of Curve 1 starts at $A + B + C = 0$, and then monotonically decreases to negative infinity in both the $B$ and $C$ directions, parallel to $B - C = 0$. Curve 2 (green), parameterized by the $\omega$ interval $\left(\frac{\pi}{1-R}, \frac{2\pi}{1-R}\right)$, has limits

$$\lim_{\omega \rightarrow \frac{\pi}{1-R}^+} (B(\omega), C(\omega)) = (\infty, \infty) \quad \text{and} \quad \lim_{\omega \rightarrow \frac{2\pi}{1-R}^-} (B(\omega), C(\omega)) = (\infty, -\infty).$$

It can be observed that when $A = A_0$, as $\omega$ increases through the interval $\left(\frac{\pi}{1-R}, \frac{2\pi}{1-R}\right)$, Curve 2 simultaneously decreases from infinity, parallel to $B - C = 0$, in both the $B$ and $C$ directions; and comes into view near the Starting Point, where the curve clearly reaches a minimum $B$ value before heading off to $(\infty, -\infty)$.

For all $R \in (R_0, 0.5)$, as $A$ increases from $A_0$, the boundary of stability is given by a portion of the plane and Curve 1, at least for some range of $A$, until $A$ reaches a critical value, marking the cusp point of Curve 2 (Figure 3.2b). Subsequently appearing in the $BC$ plane is the development of a second stable region, provided by Curve 2, disjoint from the larger region provided by Curve 1 and the plane (see Figures 3.2c and 3.2d). This smaller disconnected region of stability seen in the $BC$ plane can be viewed in the $ABC$ parameter
space as a stability spur [34, 35]—a cone-like shape—connected to and protruding out from the larger surface. This smaller disjoint region in the $BC$ plane is constructed exclusively by Curve 2, which self-intersects for a range of $A$ values.

Definition 3.3 (Stability Spur). If Bifurcation Surface $j + 1$ self-intersects above the zero-root crossing plane as $A$ increases, with the Cusp Point of Spur $j$ denoted $A^p_j$, then the quasi-cone-shaped stability spur has its cross-sectional area monotonically increase with $A$ until $A$ reaches a transitional value, $A^*_j$. For $A = A^*_j$, Stability Spur $j$ connects with the larger stability surface, via the degeneracy line given by Eqn (3.8). The one-dimensional distance $A^*_j - A^p_j$ is referred to as Spur $j$’s length.

The stability spur joins the main stability region bounded by Bifurcation Surface 1 and the $A + B + C = 0$ plane at $A = A^*_j$, which we define to be the first transition (Figure 3.2e).

Definition 3.4 (Transition). There are critical values of $A$ corresponding to where Eqns. (3.3) and (3.4) become indeterminate at $\omega = \frac{j\pi}{1-R}$. These transitional values of $A$ are denoted by $A^*_j$, where

$$A^*_j = -\frac{j\pi}{1-R} \cdot \cot \left( \frac{j\pi}{1-R} \right), \quad j = 1, 2, \ldots . \tag{3.6}$$
At a transition, Curves \( j \) and \( (j + 1) \) coincide at the specific point \((B_j^*, C_j^*)\), where

\[
B_j^* = (-1)^j \frac{(1 - R) \cos \left( \frac{jR\pi}{1 - R} \right) - jR\pi \csc \left( \frac{jR\pi}{1 - R} \right)}{(1 - R)^2}
\]

\[
C_j^* = -(-1)^j \frac{(1 - R) \cos \left( \frac{j\pi}{1 - R} \right) - j\pi \csc \left( \frac{j\pi}{1 - R} \right)}{(1 - R)^2}.
\]

All along the line

\[
(B - B_j^*) + (-1)^j(C - C_j^*) = 0, \quad A = A_j^*,
\]

there are two roots of (3.2) on the imaginary axis with \( \lambda = \pm \frac{j\pi}{1 - r} \). If (bifurcation) Surface \( j \) is on the boundary of the stability region for \( A \) slightly less than \( A_j^* \), then Eqn. (3.8) becomes part of the stability region’s boundary at Transition \( j \) (i.e., once \( A \) is increased to \( A = A_j^* \)), and Surface \( j + 1 \) enters the boundary of the stability region via the stability spur.

One can readily see from Eqns. (3.6) and (3.7) that for fixed \( R \), the system is subjected to a countably infinite number of transitions. However, it is important to remember that not all of them will be stability-boundary-changing transitions. Transitions, which occur outside of the stability surface, affect the organization of the bifurcation curves. But boundary-changing transitions come in two forms: spur-connecting transitions and smooth transitions. Both types of transitions create the greatest distortions of the stability surface away from the MRS.

**Definition 3.5 (Smooth Transition).** Transitions along the boundary of stability that do not connect stability spurs—because Curve \( j + 1 \) does not self-intersect, or because Curve \( j + 1 \) self-intersects under the plane—are referred to as smooth transitions. Smooth transitions are characterized by the smooth replacement of Surface \( j \) along the boundary with Eqn. (3.8), once \( A \) is elevated to \( A = A_j^* \). Instantly following Smooth Transition \( j \), (i.e., for \( A > A_j^* \) ) Surface \( j + 1 \) smoothly replaces Eqn. (3.8) along the boundary.

Figure 3.3 provides a sequence of pictures of the stability region before, during and after the smooth transition that takes place for \( R = 0.26 \) when \( A_5^* \approx 22.15 \). Figure 3.3a shows the boundary of stability before \( A_5^* \) is given by a portion of Curves 5 (dark green), 1 (blue), 3 (black), 2 (green) and \( A + B + C = 0 \) (purple). When the smooth transition takes place (Figure 3.3b), Eqn. (3.8) becomes part of the boundary of stability. Subsequently, Curve 6 (orange) takes the place of Eqn. (3.8) along the boundary, the instant \( A \) is incremented past \( A_5^* \). The smooth nature of the transition can be seen in Figure 3.3 as not causing a drastic change to the stability region.

There are a couple of other ways for bifurcation surfaces to enter (or leave) the boundary of the main stability region as \( A \) increases. We define these means of altering the
Figure 3.3. Smooth transition, \( A^*_5 \).

boundary as *transferrals* and *tangencies*, which relate to higher frequency eigenvalues becoming part of the boundary (or being lost) as \( A \) increases.

**Definition 3.6.** The **transferral** value of \( A = A^z_{i,j} \) is the value of \( A \) corresponding to the intersection of bifurcation Surface (Curve) \( j \) with bifurcation Surface (Curve) \( i \) at the \( A + B + C = 0 \) plane with the bifurcation Surface (Curve) \( j \), entering the boundary of the stability region for some range of \( A > A^z_{i,j} \).

As a result of the transferral occurring, bifurcation Surface (Curve) \( i \) does not cease to provide a part of the boundary, rather it merely gives over a share of the boundary for some range of \( A > A^z_{i,j} \). Figure 3.4 gives an example of the stability region before, during and shortly after a transferral takes place for \( R = 0.25 \). The boundary of stability before \( A^z_{1,6} \) (Figure 3.4a) is formed by portions of the \( A + B + C = 0 \) plane (purple) and bifurcation Curves 1 (blue), 2 (green) and 3 (black). When \( A \) is incremented to \( A^z_{1,6} \) (Figure 3.4c), Curves 1 (blue) and 6 (orange) intersect at the plane. Afterwards (Figure 3.4d), the point of intersection between Curves 1 and 6 slides upward and away from the plane.

Additionally, for some values of \( R \) the stability surface can undergo a reverse *transferral*, \( \tilde{A}^z_{j,i} \), which is a transferral characterized by bifurcation Surface (Curve) \( j \) leaving the boundary, or a transferring back over to the bifurcation Surface (Curve) \( i \) the portion of the boundary originally taken by the Surface (Curve) \( j \) surface at \( A^z_{i,j} (\prec \tilde{A}^z_{j,i}) \).

The most frequently observed and last of the four modes manifesting change along the stability boundary, as \( A \) is propagated past the starting point, \( A_0 \), occurs postliminary to the tangency of two surfaces.

**Definition 3.7.** The value of \( A \) corresponding to the **tangency** of two surfaces \( i \) and \( j \) is denoted \( A^{t,i}_i \). Bifurcation Surface (Curve) \( j \) becomes tangent to bifurcation Surface (Curve) \( i \); where Surface (Curve) \( i^{th} \) is a part of the stability boundary preliminary to \( A = A^t_{i,j} \). As \( A \) increases from \( A^t_{i,j} \), the new \((j^{th})\) bifurcation surface is incorporated into the boundary of the
Figure 3.4. Transferral 1, $A^{z}_{1,6}$.

stability region, separating segments of the bifurcation surface to which it was tangent. However, many times as $A$ is increased Surface (Curve) $j$, the same surface (curve) which entered the boundary through tangency $A^{t}_{i,j}$, can be seen leaving the stability boundary via a reverse tangency, denoted $\tilde{A}^{t}_{j,i}$.

Figure 3.5 shows how Curve 9 (gray) enters the boundary of stability by way of the tangency, $A^{t}_{3,9}$, for the stability surface associated with $R = 0.25$. Prior to the tangency (Figure 3.5a), Curve 9 is away from the boundary, and portions of Curves 1 (blue), 2 (green), 3 (black) and 6 (orange) can be found along the stability boundary. However, Curve 9 undergoes a transition (Figure 3.5b) with Curve 8 (pink) outside of the stability boundary, subsequently changing the end behavior of both curves, so that Curve 9 is near the stability boundary. As $A$ approaches $A^{t}_{3,9}$, Curves 3 (black) and 9 (gray) approach one another until
becoming tangent (Figure 3.5c). Afterwards (Figure 3.5d), Curve 9 takes on a portion of the boundary in between its two intercepts with Curve 3.

Of great importance to the study of global asymptotic change in stability surfaces, the main focus of this thesis, is that for rational $R$, due to the periodic nature of the sinusoidal functions, the bifurcation curves order themselves into families of curves.

**Definition 3.8.** For $A$ fixed, take $R = \frac{k}{n}$ and $j = n - k$. From Eqns. (3.3) and (3.4), one can see that the singularities occur at $\frac{n i \pi}{j}$, $i = 0, 1, \ldots$. The bifurcation Surface $i$ with $\frac{n (i-1) \pi}{j} < \omega < \frac{n i \pi}{j}$ satisfies:

$$B_i(\omega) = \frac{A \sin(\frac{k \omega}{n}) + \omega \cos(\frac{k \omega}{n})}{\sin(\frac{j \omega}{n})}, \quad C_i(\omega) = -\frac{A \sin(\omega) + \omega \cos(\omega)}{\sin(\frac{j \omega}{n})}$$
Now consider the bifurcation Surface \((i + 2j)\) with \(\mu = \omega + 2n\pi\), then

\[
B_{i+2j}(\mu) = \frac{A \sin\left(\frac{\mu}{n}\right) + \mu \cos\left(\frac{\mu}{n}\right)}{\sin\left(\frac{\mu}{n}\right)} = \frac{A \sin\left(\frac{\omega}{n}\right) + (\omega + 2n\pi) \cos\left(\frac{\omega}{n}\right)}{\sin\left(\frac{\omega}{n}\right)}
\]

\[
C_{i+2j}(\mu) = -\frac{A \sin(\omega) + (\omega + 2n\pi) \cos(\omega)}{\sin\left(\frac{\omega}{n}\right)}
\]

These equations show that \(B_{i+2j}(\mu)\) follows the same trajectory as \(B_i(\omega)\) with a shift of \(2n\pi \cos\left(\frac{\omega}{n}\right) \sin\left(\frac{j\omega}{n}\right)\) for \(\omega \in \left(\frac{(j-1)\pi}{1-R}, \frac{j\pi}{1-R}\right)\), while \(C_{i+2j}(\mu)\) follows the same trajectory as \(C_i(\omega)\) with a shift of \(2n\pi \cos(\omega) \sin\left(\frac{j\omega}{n}\right)\) over the same values of \(\omega\). This related behavior of bifurcation surfaces separated by \(\omega = 2n\pi\) creates \(2j\) families of curves in the \(BC\) plane for fixed \(A\). Thus, there is a quasi-periodicity among the bifurcation surfaces when \(R\) is rational.

For example, when \(R = \frac{1}{2}\) there are two families of curves in the \(BC\) plane, when \(R = \frac{1}{3}\) and \(R = \frac{1}{4}\) there are four and six families, respectively. The organization of the bifurcation curves into families appears to be especially significant in the asymptotic structure of the stability region as \(A\) becomes large.

Having briefly defined and described most of the terms and definitions that will be called upon in the remainder of this thesis, I present an example of a stability surface in the \(ABC\) coefficient space with \(R\) fixed at 0.25 in the next section. The discussion given alongside pictures of the stability region aims to help the reader further comprehend the nature of the four mechanisms that create change on the stability surface (as \(A\) is increased from \(A_0\)), and how those mechanisms influence its shape. Following the example given in Section 3.1 is a discussion of the family structure of bifurcation curves for \(R = 0.25\). How that structure assists in the asymptotic determination of the surface as \(A \to \infty\) is pointed out, then the main results of this thesis study are brought forth in Chapter 4.

### 3.1 Example of a Stability Surface with Delay Ratio \(R = 0.25\)

In this section, a detailed description is supplied along with illustrations of the stability region with \(R\) fixed at 0.25, using the definitions and theorems given in the previous section. Finite sections \((A \in [A_0, 100] \text{ and } A \in [A_0, 12])\) of the unbounded three-dimensional stability surface are presented in Figure 3.6. A \(BC\) cross-section plot of the surface for \(A \in [A_0, 100]\) is displayed in Figure 3.7, to highlight the five major changes that occur on the surface for \(R = 0.25\). Two spur-connecting transitions, one transferral, and two different tangencies occur between the bifurcation curves.

From Theorem 3 it is ascertained that for \(R = 0.25\), \((A_0, B_0, C_0) = (-5, -\frac{1}{3}, \frac{16}{3})\) is the Starting Point of the stability surface. For \(A < A_0\), Eqn. (3.1) is unstable. Along with the
Starting Point, it is known from Eqn. (3.5) that as $\omega \to 0^+$, Curve 1 intersects the $A + B + C = 0$ plane at $\omega = 0$. As $A$ sweeps forward past $A_0 = -5$, the first bifurcation curve intersects the plane a second time, forming a bounded stability region on the $BC$ plane. Figures 3.2a-f show the progression of the surface in the $BC$ cross section as $A$ increases from $A_0$ to a point just past the first transition at $A^*_1 \approx -2.4184$. For $A \in [A_0, A^*_1]$, with $A^*_1 \approx -2.7023$, the stability region is bounded by a portion of the plane $A + B + C = 0$ (purple) and the Curve 1 (blue). The $BC$ cross section of the boundary for $A = A^*_1$ can be
seen in Figure 3.2b. After reaching $A_1^p$, but before the transition $A_1^\ast$, a smaller stability region (Figures 3.2c and 3.2d) disjoint from the larger region is given by the self-intersection of Curve 2 (green), which forms a stable spur in $ABC$ parameter space with length $A_1^\ast - A_1^p \approx 0.2839$.

As $A$ evolves through $A \in (A_1^p, A_1^\ast]$, the area of the spur monotonically grows larger in the $BC$ cross section. Additionally, Curves 1 (blue) and 2 (green) are stretched towards one another (Figures 3.2c and 3.2d), until $A$ reaches $A_1^\ast \approx -2.41840$ (Figure 3.2e). The spur-connecting transition is one of four modalities by which it is observed the two-dimensional stability boundary becomes altered. At this transitional value of $A$, Curves 1 and 2 intersect at the point $(B_1^\ast, C_1^\ast)$, given by Eqn. (3.7). The same singular value of $\omega$ that generates the point of transition also begets the degeneracy line $C - B = \frac{3\pi}{2}$ (given by Eqn. 3.8). The region of stability at the first transition (Figure 3.2d) is composed of the plane, a portion of the degeneracy line, $C - B = \frac{3\pi}{2}$, Curve 1 and a portion of Curve 2. Part of Curve 2 and the degeneracy line connect the preceding stability spur to the larger region of stability in $ABC$ parameter space.

The advancement of the pictures (a-f) provided by Figure 3.2 also demonstrates a characteristic that follows all transitions: the switching of the end behavior, along the degeneracy line, of the two bifurcation curves $j$ and $(j + 1)$ undergoing the transition. For example, notice that prior to the transition, one end of the Curve 2 (green) tends towards positive infinity in both the $B$ and $C$ directions. At the same time, the end behavior of Curve 1 (blue) tends towards negative infinity in both the $B$ and $C$ directions. When the transition occurs, both these ends converge on the point of transition. Afterward, the aforementioned ends swap or switch infinities along the degeneracy line.

Figure 3.8 shows cross-sectional plots of the boundary after the first transition. The overall area of the stability region monotonically grows with $A$ (Figures 3.8a, 3.8b and 3.8c). When the value of $A$ sweeps past $A = 0$, the minimum region of stability (MRS), supplied by Theorem 3, comes into existence. The MRS is depicted in Figure 3.8 by the black dashed square. One can easily see that the actual stability region is significantly larger than the MRS pyramid. The reason is that $A$ is approaching another spur-connecting point of transition, $A_2^\ast$ (Figures 3.8d and 3.8e), causing Curve 2 (green) to be drawn towards Curve 3 (black). The disconnected region formed by the self-intersection of Curve 3 adjoins to the main stability surface at the transition $A_2^\ast \approx 4.83$. In comparison to Spur 1, Spur 2 has a minuscule volume and much shorter spur length of $A_2^\ast - A_2^p \approx 0.0304$. Moreover, the second spur on the surface for $R = 0.25$ is the last stable spur that will appear on the surface for all $A \in [A_0, \infty)$, since the transition $A_j^\ast$ (all $j = 3, 4, \ldots$) occurs prior to the $j^{th}$ curve composing a portion of the boundary.
Figure 3.8. Transition 2, $A^*_2$.

The subsequent change to the two-dimensional stability region for $A > A^*_2$ is administered by the transferral of a portion of the boundary from Curve 1 to Curve 6 at $A = A^*_{1,6} \approx 13.3$. When the value of $A$ is exactly at $A^*_{1,6}$, Curves 1 and 6 intersect at some point along the $A + B + C = 0$ plane. The evolution of the boundary before, during, and after the first transferral is featured in Figure 3.4. In all four illustrations, Curves 1 (blue), 2 (green), 3 (black), 5 (dark green) and 6 (orange) are among the closest bifurcation curves to the existing stability region. These curves compete with the plane for a share of the stability boundary, while the area of the stability region retains its monotonicity. Additionally, the boundary can be seen bulging out from the MRS (dashed line), defined by Theorem 3. Figure 3.4.a shows that before the transferral, at $A = 10$, the boundary is furnished by part of the plane and Curves 1 (blue), 2 (green), and 3 (black).

Preceding the transferral $A^*_{1,6} \approx 13.3$ (Figure 3.4c) is the transition $A^*_5 \approx 12.092$ (Figure 3.4b) occurring outside of the stability boundary, marked by the intersection of Curves 5 (dark green) and 6 (orange) below the plane. There is a value of $A$, for which Curve 6 begins to self-intersect, but a stable spur is not formed because Curve 6 self-intersects underneath the plane, so a positive real eigenvalue exists. The transition $A^*_5$ does not directly
influence change on the stability surface, but it creates a bulge in the stability region away from the MRS, similar to $A_2^*$. Shortly after the transition $A_5^*$, Curve 6 (orange) enters the stability boundary by way of the transferral $A_{1,6}^*$ (Figure 3.4c). After the transferral $A_{1,6}^* \approx 13.3$ (Figure 3.4d), parts of Curves 1, 2, 3, and 6 compose the boundary of stability, with Curve 1 (blue) transferring over more of the boundary to Curve 6 (orange) with rising $A$.

As $A$ further increases, the next alteration to the boundary is induced by another type of surface change defined in Section 3, the tangency (denoted $A_{i,j}^t$) of two surfaces. The first tangency located along the surface associated with $R = 0.25$ is the tangency of Curves 3 and 9 at $A_{3,9}^t \approx 49$, located above the $B$-axis. The boundary near this tangency event is illustrated in Figures 3.5a-d, where the picture of the boundary before the tangency (Figure 3.5a) is identical to Figure 3.4d, but with a larger axis window to show that Curves 8 (pink) and 9 (gray) are both marching towards a transition point outside of the boundary when $A = 16$.

Figure 3.5b displays the boundary at the transition $A_8^* \approx 19.347$, after which Curve 9 roughly parallels Curve 3. The moment $A$ catches up to a point just past the tangency at $A_{3,9}^t \approx 49$, evidenced in Figure 3.5c, Curve 9 (gray) begins to take over a share of the boundary from Curve 3 (black). The portion taken over by Curve 9 from Curve 3 is then further enlarged with increasing $A$, for a range of $A$.

The last and final change along the stability surface for $R = 0.25$ and $A \in [A_0, 100]$ is another tangency $A_{6,12}^t \approx 78.95$ (Figure 3.7) being supplied by Curves 6 (orange) and 12 (red). Notice that like the previous tangency, $A_{3,9}^t$, the bifurcation curve numbers are separated by six. As noted after Definition 3.8, $R = 0.25$ has six families of curves, and the tangencies are occurring amongst these family members.

Prior to the tangency, which appears below the $B$-axis, a transition $A_{12}^*$, accompanied by the switching of infinities, along the degeneracy line, of the end behaviors of Curves 11 and 12 has to occur, so that Curve 12 may then be in position to compete for a portion of the boundary. Since Curve 11 was not on the boundary of stability prior to the transition, and the self-intersection of Curve 12, if it did happen, occurred underneath the plane, the transition does not directly affect the boundary. The transition’s only contribution to the next surface change event, the tangency, was the infinity switching of the endpoint of Curve 12. Consequently, the placement of Curve 12 due to the swapping in end behaviors that accompanies $A_{12}^*$ allows Curve 12, which would otherwise have been bounded away from the stability region (due to the familial ordering and monotonicity of the curves), to leverage the next resulting change to the boundary.

In summary, the four mechanisms that bring about change along the boundary of a stability surface are the transition, transferral, and tangency of two surfaces; and the self-intersection of a single surface which creates a stability spur. For any particular value of
$R (0 < R < 1)$, the system is subjected to an infinite number of transitions, but not all of them create change along the boundary of stability. In addition, not all transitions which directly bring change to the boundary are accompanied by the attaching of a stability spur onto the larger stability surface. Instead, smooth transitions, $A_j^*$, when they arise, are marked by the smooth replacement along the boundary, in the $BC$ cross section, of Curve $j$ with Curve $j + 1$. Nevertheless, the influence of the transition upon the transferral and tangency was pointed out in the discussion in this section and demonstrated with the accompanying figures. On the other hand, transitions that contribute direct change to the boundary have a tendency to—but don’t always—connect stability spurs to the larger stability surface. Not enough is known about the existence and uniqueness of stability spurs, one of the topics explored and discussed in the numerical section of this thesis. In the next section, followed by the main results of this thesis, there will appear illustrations and an explanation of how the family structure of bifurcation curves aids in the asymptotic determination of the boundary for $R = 0.25$. How that determination can be extended to three-dimensional stability surfaces for rational $R \in [0.22, 0.28]$, as well as for $R \in (R_0, 1/2]$ will come thereafter.

### 3.2 Family Structure of Bifurcation Surfaces for $R = 0.25$

The bifurcation curves for $R = 0.25$ shown in Figure 3.9 have six characteristic shapes and a periodic curve ordering, which partitions the curves into six families. This distinct ordering facilitates the construction of the stability region.

In this section, a careful examination of the geometric structure of the six families of curves for $R = 0.25$ is undertaken. The analysis is similar to that found in Appendix C of Mahaffy, et al. [34]. A rotation of 45° simplifies the dissection of the families of curves, so the following change of coordinates is brought forth:

\[
X = B + C = -A \cos \left( \frac{\omega (1+R)}{2} \right) + \omega \sin \left( \frac{\omega (1+R)}{2} \right) \cos \left( \frac{\omega (1-R)}{2} \right)
\]

\[
Y = B - C = A \sin \left( \frac{\omega (1+R)}{2} \right) + \omega \cos \left( \frac{\omega (1+R)}{2} \right) \sin \left( \frac{\omega (1-R)}{2} \right).
\]  

(3.9)

These equations are derived from Eqns. (3.3) and (3.4). For $R = \frac{1}{4}$, Eqn. (3.9) produces:

\[
X = -A \cos \left( \frac{5}{8} \omega \right) + \omega \sin \left( \frac{5}{8} \omega \right) \cos \left( \frac{5}{8} \omega \right)
\]
Figure 3.9. Six family structure of $R = 0.25$.

The $X$ and $Y$ intercepts will also be utilized to help show the ordering of certain branches of bifurcation curves:

$$(X, Y) = \left(\frac{\omega}{\cos \left(\frac{5}{8} \omega\right) \sin \left(\frac{5}{8} \omega\right)}, 0\right) \quad \text{where } \omega \cot \left(\frac{5}{8} \omega\right) = -A, \quad (3.12)$$

and

$$(X, Y) = \left(0, \frac{\omega}{\sin \left(\frac{5}{8} \omega\right) \cos \left(\frac{5}{8} \omega\right)}\right) \quad \text{where } \omega \tan \left(\frac{5}{8} \omega\right) = A. \quad (3.13)$$

Figure 3.9 individually shows the six bifurcation curve families in the rotated coordinate system for $R = 0.25$. We observe that the end behaviors of the lower-ordered curves can vary
from the higher-order curves due to transitions, and this is affected by $A$. All six families plotted on the same graph can be seen in Figure 3.10. Twenty six members from each family are shown in each of the pictures in Figures 3.9 and 3.10 because plotting several more than twenty six members from each family on the same graph would have several overlapping curves making it more difficult to discern the different end behaviors. The formulae above are used to show some of the properties intrafamilial members share.

![Figure 3.9 with more curves.](image1)

![At the transition, $A^*_5$](image2)

**Figure 3.10. Six families of bifurcation curves.**

For instance, consider the first family of bifurcation curves (blue) presented in Figure 3.9a. These curves are formed by examining $8k\pi < \omega < 8k\pi + \frac{4}{3}\pi$, $k = 0, 1, \ldots$ The following limits are readily obtained from Eqn. (3.10):

\[
\begin{align*}
\lim_{\omega \to 8k\pi^+} X(\omega) &= -A, \\
\lim_{\omega \to 8(k+1)\pi^+} Y(\omega) &= +\infty, \\
\lim_{\omega \to 8k\pi + 4\pi/3^-} X(\omega) &= +\infty, \\
\lim_{\omega \to 8k\pi + 4\pi/3^-} Y(\omega) &= \frac{1}{6} \left[ 3A - 4\pi\sqrt{3}(1 + 6k) \right], \quad \text{with all limits having } k = 0, 1, \ldots
\end{align*}
\]

Figure 3.9a shows that $\frac{dY}{dX} < 0$. This suggests that Eqn. (3.11) can be used to show the monotonicity of the first family of curves on the interval $8k\pi < \omega < 8k\pi + \frac{4}{3}\pi$, $k = 0, 1, \ldots$, for $A > A^*_1$.

In addition, from the one delay problem we know that Curve 1 passes closest to the point $(X, Y) = (A, A)$. The asymptotic limits above show that each member of the first family must intersect the other members of this family at least once, but from the location of
Curve 1 this intersection requires that \( X > A \). As a result all members of this family except for Curve 1 are located away from the boundary of stability. The family lies outside of the region bounded by a portion of bifurcations Curves 1 and 2 and the line \((B + C) = X = -A\) for \( A \in (A_1^*, A_2^*)\) and for \( A > A_2^*\) the first family lies outside of the region bounded by a portion of Curves 1 and 3 and the line \((B + C) = X = -A\). Only Curve 1 composes a portion of the boundary for a range of \( A \).

A similar result can be acquired for the fourth family of curves (red), represented by Figure 3.9b, where \( 8k\pi + 4\pi < \omega < 8k\pi + \frac{16\pi}{3}, k = 0, 1, \ldots \) The fourth family is nearly a reflection of the first. The following limits are obtained from Eqn. (3.10):

\[
\lim_{\omega \to 8k\pi + 4\pi^+} X(\omega) = +\infty, \quad \lim_{\omega \to 8k\pi + 4\pi^+} Y(\omega) = -A, \quad k = 0, 1, \ldots
\]

\[
\lim_{\omega \to 8k\pi + 16\pi/3^+} X(\omega) = \frac{1}{6} \left[ 3A - 8\pi\sqrt{3}(2 + 3k) \right], \quad \lim_{\omega \to 8k\pi + 16\pi/3^+} Y(\omega) = +\infty, \quad k = 0, 1, \ldots
\]

In arguments similar to the ones used for the first family of bifurcation curves, Eqn. (3.11) yields \( \frac{dY}{dX} < 0 \) for \( A > 0 \) on the interval \( 8k\pi + 4\pi < \omega < 8k\pi + \frac{16\pi}{3}, k = 0, 1, \ldots \) The monotonicity and positioning of these curves prevents them from joining the boundary of the stability region. In particular, this family lies outside the region bounded by Curves 4 and 5 for \( A \in [A_0, A_0^*] \); and outside the region bounded by Curves 4 and 6 for \( A > A_5^* \approx 12.1 \). As \( A \) increases from \( A_0 \), because of the ordering of curves, family four never contributes to the boundary of stability.

The remaining four families of curves play a more important role in determining the boundary of stability than do families one and four. In particular, it has been observed that the third and sixth families (Figures 3.9e and 3.9f) are the only families which cause change along the boundary in the form of intrafamilial tangencies for rising \( A \) past \( A_1^* \). Families two and three (Figures 3.9c and 3.9e) can be seen to always be in transition with one another outside of the boundary of stability. The same can be said about families five and six (Figures 3.9d and 3.9f). Because the ends of the bifurcation curves swap positions, it could perhaps be recognized that these transitions eventually render families two and five away from the boundary when \( A \) reaches infinity, and allow for the positioning of more and more members of families three and six to have tangency relations.

The transitions between families two and three and families five and six don’t seem to induce the intrafamilial tangencies amongst families three and six, but it appears (from my experience viewing plots of curves) that families three and six would be bounded away from the boundary of stability when \( A \to \infty \) otherwise. Moreover, the near symmetry of families two and three to families five and six explains why the occurrences of intrafamilial tangencies have a propensity to alternate between families with increasing \( A \). It was also observed that
the transition branches remain in the same approximate location relative to the MRS and boundary of stability. Also the rate at which the families transition with one another seems to be constant as \( A \) grows. In support of the preceding arguments and conjectures, consider the relationship of family two to family three, which is scrutinized below.

The second family of bifurcation curves are defined for
\[
\frac{4}{3}\pi + 8k\pi < \omega < \frac{8}{3}\pi + 8k\pi, \quad k = 0, 1, \ldots
\]
The following limits are derived from Eqn. (3.10):
\[
\lim_{\omega \to 8k\pi + 4\pi/3^+} X(\omega) = -\infty, \quad \lim_{\omega \to 8k\pi + 4\pi/3^+} Y(\omega) = \frac{1}{6} \left[ 3A - 4\pi\sqrt{3}(1 + 6k) \right],
\]
\[
\lim_{\omega \to 8k\pi + 8\pi/3^+} X(\omega) = \frac{1}{6} \left[ 3A - 8\pi\sqrt{3}(1 + 3k) \right] \quad \text{and}
\]
\[
\lim_{\omega \to 8k\pi + 8\pi/3^-} Y(\omega) = \begin{cases} 
-\infty & \text{if } k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \\
\infty & \text{if } k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi),
\end{cases}
\]
with all limits having again \( k = 0, 1, \ldots \) Eqn. (3.11) shows that as \( \omega \to 8k\pi + \frac{4\pi^+}{3} \), \( \frac{dY}{dX} \to 0 \); while as \( \omega \to 8\pi k + \frac{8\pi^-}{3} \), \( \frac{dY}{dX} \to \pm\infty \) depending upon whether \( k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \) or not. This confirms the horizontal and vertical nature of the end behavior of each curve.

Furthermore, as \( \omega \to 8\pi k + \frac{4\pi^+}{3} \), the numerator of the leading factor of (3.11) tends to
\(-\left(\frac{5}{2} A\sqrt{3} + 20\pi \right. \left. k + 4 \sqrt{3} + \frac{10}{3}\pi \right) < 0 \); but as \( \omega \to 8\pi k + \frac{8\pi^-}{3} \) it approaches
\(-\frac{3}{2} A\sqrt{3} + 12\pi k + 4 \pi \), which is positive for \( k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \), demonstrating that the second family of bifurcation curves have \( \frac{dY}{dX} = 0 \) at some point in the interval at the very least for the higher-ordered members satisfying \( k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \). Additionally, Eqn. (3.12) can be used to see that for large \( k \), when \( \omega \to 8\pi k + \frac{8\pi^-}{3} \), \( X \to \frac{16}{9}\pi\sqrt{3}(3k + 1) \), so that the curves pass through the \( X \) axis as parallel lines separated by \( 8\pi \) for large \( \omega \). This gives a clear ordering for large \( \omega \).

Consequently the second family of bifurcation curves have a distinctive “L” shape. Provided \( k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \), Eqn. (3.11) can be shown to be negative so that each curve is monotone decreasing. Additionally, as \( A \) increases more and more curves belonging to family two satisfy \( k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \), so that asymptotically all members of this family will be monotone decreasing and away from the boundary of stability as \( A \to \infty \). However, curves satisfying \( k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \) begin by decreasing to a minimum value of \( Y \). Shortly after the \( Y \) value begins to increase from the minimum the maximum \( X \) value is reached the instant the \( Y \) value shoots off to positive infinity.
As for the third family of bifurcation curves, they are defined for 
\[ 8k\pi + \frac{8}{3}\pi < \omega < 8k\pi + 4\pi, \ k = 0, 1, \ldots \] The following limits come from Eqn. (3.10):

\[
\lim_{\omega \to 8k\pi + 8\pi/3^+} X(\omega) = \frac{1}{6} \left[ 3A - 8\pi \sqrt{3}(1 + 3k) \right],
\]

\[
\lim_{\omega \to 8k\pi + 8\pi/3^+} Y(\omega) = \begin{cases} 
-\infty & \text{if } k > \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \\
\infty & \text{if } k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi), 
\end{cases}
\]

\[
\lim_{\omega \to 8k\pi + 4\pi^-} X(\omega) = -\infty \quad \text{and} \quad \lim_{\omega \to 8k\pi + 4\pi^-} Y(\omega) = -A \quad \text{with all limits having } k = 0, 1, \ldots.
\]

In a fashion similar to that of the second family, Eqn. (3.11) shows that as \( \omega \to 8k\pi + 4\pi^- \), \( \frac{dY}{dX} \to 0 \). As \( \omega \to 8\pi k + \frac{8\pi}{3}^+ \), \( \frac{dY}{dX} \to \pm\infty \) depending upon whether \( k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \) or not. Moreover, as \( \omega \to 8\pi k + \frac{4\pi}{3}^+ \), the numerator of the leading factor of (3.11) can be used to show that there is again a sign change in \( \frac{dY}{d\omega} \) throughout each \( \omega \) interval satisfying \( k < \frac{1}{24\pi}(3A\sqrt{3} - 8\pi) \). This demonstrates that the lower ordered members of family three reach a minimum \( Y \) value.

Similar observations, limit statements and relationships between families two and three carry over to and can be found for families five (Figure 3.9d) and six (Figure 3.9d). The larger \( A \) mass \( A^*_2 \), the less relevant family five becomes to the boundary. More work with these families may help establish analytic results about how the family structure contributes to the asymptotic nature of the stability boundary, a topic of the next chapter.
CHAPTER 4
NUMERICAL RESULTS

This chapter examines details of numerical studies of Eqn. (3.1), as the delay, \( R \), varies. In particular, stability spurs are studied in the region \( R \in [0.20, 0.25] \). Later, in Section 4.2, the focus is on describing the unbounded stability surface for \( R = 0.25 \). The reader may recall from the introduction that certain rational delays have unusually large regions of stability. As an introduction to Chapter 4, Figure 4.1 is used to provide the reader with a graphical summary of the changes that occur to the stability boundary, for each stability surface associated with \( R \) in the aforementioned interval of interest, as \( A \) is increased from the Starting Point (Theorem 3), up to \( A = 200 \). The type of change occurring along the boundary of stability is labeled, in each case, with the notation defined in Chapter 3 for transitions, transferrals, and tangencies. The Starting Point curve (cyan), three spur-connecting transition curves (red), and smooth transition curves (brown or pink), were determined by Theorem 3 and Eqn. (3.6). Each transferral (blue) and tangency curve (black or green) was determined using Newton’s Method and the point-wise continuity of the characteristic equation (3.2).

Figure 4.1. Graphical summary of surface changes.
Following a vertical line shows exactly what occurs on the boundary of the stability surface for a given $R$. Visual verification of the actual occurrence of each boundary changing event depicted in Figure 4.1, for several surfaces in the interval $[0.20, 0.25]$ and $A \in [A_0, 200]$, was painstakingly conducted during the creation of the figure, using computational tools and skills that were developed during the research portion of this thesis. A graphical user interface, dynamically displaying changes to the boundary in the $BC$ cross section for variable values of $A$ and $R$, was built to rapidly validate the construction of Figure 4.1. A more detailed discussion is given later of the figure and how it describes the four-dimensional parameter space of Eqn. (3.1) for a limited range of $A$.

### 4.1 Stability Spurs

Recall from Chapter 3 that whenever the value of $R$ is selected from the interval $(0, 1)$, the real root crossing plane, $A + B + C = 0$, and bifurcation surfaces, given by Eqns. (3.3) and (3.4), intersect in the $ABC$ coefficient-parameter space of Eqn. (3.1) to form an unbounded stability surface. While studying stability surfaces associated with the two-delay equation, Mahaffy, Zak, and Joiner [34, 35] noticed that a stability spur, as defined in the previous chapter, can appear on the connected, three-dimensional stability surface. Present numerical evidence suggests the existence of only three stability spurs for surfaces corresponding to $R \in (0.20, 0.25)$, which is the parameter range discussed in this chapter.

The size and location of these spurs vary with $R$. The discussion in this section provides a detailed description of the stability surface near the Starting Point, $A_0$, and the three stability spurs. Discussion of either transferrals or tangencies are purposely kept to a minimum to highlight our study of the stability spurs.

As $A$ is decreased from infinity, each stability surface associated with the $R$ interval $(0.20, 0.25)$ comes to a point, the Starting Point, $(A_0, B_0, C_0)$, which is furnished by Theorem 3. As the value of $R$ is swept through $[0.20, 0.25]$, the Starting Point travels from $(A_0, B_0, C_0) = (-6, -\frac{1}{4}, \frac{25}{4})$ to $(A_0, B_0, C_0) = (-5, -\frac{1}{3}, \frac{16}{3})$ along the curve having smooth parametrization $(A_0(R), B_0(R), C_0(R)) \equiv \left(-\frac{R+1}{R}, \frac{R}{R-1}, \frac{1}{R(1-R)}\right)$. Therefore, the locations of all the Starting Points of surfaces contained in $[0.20, 0.25]$ are near one another in the $ABC$ coefficient-parameter space of the two-delay equation (Eqn. (3.1)). Additionally, this places the set of Starting Points associated with $R \in [0.20, 0.25]$ both relatively near and offset from the apex of the MRS, located at the origin in $\mathbb{R}^3$.

Figure 3.2a provides an archetypic representation of a $BC$ cross-section plot of the stability surface at the Starting Point, $A_0$, for all surfaces associated with $[0.20, 0.25]$. When $A$ is increased slightly from $A_0$, Curve 1 (blue) can be seen intersecting the plane (purple) in two locations: once at the point, $(B(0), C(0))$, generated by $\omega = 0$, and a second time at a point, $(B(\omega_1), C(\omega_1))$, spawned by a pair of eigenvalues, $\lambda = \pm i\omega_1$, where $\omega_1 \in (0, \frac{\pi}{1-R})$. 
Consequently, a closed area, the region of stability, is formed by the portions of Curve 1 and the plane in between the two intersection points. A representative $BC$ cross-section plot of the stability region for values of $A$ slightly greater than $A_0$ is given by Figure 3.2b. From the frame of reference given in Figure 3.2b—facing the negative $A$ direction—the zero eigenvalue, Eqn. (3.5), produces the intersection point (of Curve 1 and the plane) located to the right of the $C$-axis, while the point $(B(\omega_1), C(\omega_1))$ is located to the left of the $C$-axis.

Figure 4.2 shows representative pictures of the three-dimensional stability surface that is formed near $A_0$, for all surfaces associated with $R \in [0.20, 0.25]$. The portion of the stability surface formed by $A + B + C = 0$ (purple) and Curve 1 (blue) for the $A$ interval $[A_0, A^*_1]$ resembles a horn-like shape. Recall that traversing from the interior of this stability region across the $A + B + C = 0$ plane (purple) results in a single real eigenvalue of Eqn. (3.1) becoming positive; while traversing the Surface 1 (blue) from the interior results in a pair of eigenvalues of Eqn. (3.1) having positive real part. These eigenvalues have frequency $\omega \in (0, \omega_1)$ for $A \in (A_0, A^*_1)$. Looking from the positive $A$ axis towards $A_0$, the frequency increases in the counterclockwise direction. Figure 4.3a shows the smooth increase in $\omega_1$ as $A \to A^*_1$.

![Figure 4.2. The stability surface near the Starting Point.](image)

The cusp point of Spur 1 is denoted in the Figure 4.2a as $A^p_1$. For $A > A^p_1$, Spur 1 (green), manifested by the self-intersection of Surface 2, is fleshed out along with the larger, main region of stability given by Surface 1 and the plane, until the value of $A$ arrives at $A = A^*_1$, Transition 1. Spur 1 connects, via the degeneracy line (supplied in Chapter 3 by Eqn. (3.8)) at $A^*_1$, to the main region of stability assembled by Surface 1 and the plane.
The set of ordered triples, \( \{(A, B(\omega_0), C(\omega_0))\} \), generated by the zero eigenvalue, \( \omega = 0 \), as \( A \) is increased through \([A_0, A^*_1]\), constructs a line segment in the \( ABC \) space, given by Eqn. (3.5). This line segment forms a corner of the aforementioned section of the stability surface, visible in Figure 4.2 as the right-most intersection of the purple and blue surfaces. As \( A \) is increased through \([A_0, A^*_1]\), the left-most intersection of the purple and blue surfaces produced by \( \pm \omega_1 \) is simultaneously swept through \([0, \frac{\pi}{1-R}]\), and the corresponding set of points \( \{(A, B(\omega_1), C(\omega_1))\} \) traces a non-linear curve segment in the \( ABC \) space.

Additionally, Curve 1 has a maximum point which intersects \((B^*_1, C^*_1)\) at \( A^*_1 \). Representative \( BC \) cross-section plots of the stability boundary associated with all surfaces in the \( R \) interval \([0.20, 0.25]\) before, during, and after the first spur-connecting transition is provided in Figure 3.2a-f.

The following statistics associated with Spur 1 demonstrate the shrinking nature of Spur 1 as a function of \( R \). Transition 1 is one of the three red, spur-connecting transition curves displayed in Figure 4.1. The range of \( A^*_1 \) for \( R \in [0.20, 0.25] \) is approximately \([-3.9270, -2.4184] \). In Chapter 3, the length of Spur 1 was defined as the one-dimensional distance function: \( A^*_1(R) - A^p_1(R) \), where \( A^p_1 \) is numerically defined in Appendix B of [34]. It was determined that the range of spur length associated with Spur 1 over \( R \in [0.20, 0.25] \) is approximately \([0.2840, 0.4064]\), where \( A^*_1(0.20) - A^p_1(0.20) \approx 0.4064 \) and \( A^*_1(0.25) - A^p_1(0.25) \approx 0.2840 \). Thus, Spur 1’s length decreases 30% over \( R \in [0.20, 0.25] \).

The spur length as a function of \( R \in [0.20, 0.25] \), is provided in Figure 4.3b (green).

Additionally, in the \( BC \) cross section when \( A = A^*_1 \) (Figure 3.2e), the ratio of the larger area, constructed by the real root-crossing plane, degeneracy line and Curve 1, to the area...
fabricated by Spur 1, was found to increase through the approximate interval
\[ [4.6648, 13.1237] \], as \( R \) was swept through the interval \([0.20, 0.25] \).

Obtaining the graph of a stability spur was a non-trivial process and several difficulties
were encountered. One could obtain numerically, using Newton’s Method, the eigenvalue, \( \omega^p_j \),
and the value of \( A = A^p_j \) locating the spur’s cusp point. In order to find an approximation of
\( A^p_j \) for the Newton Algorithm, a Matlab program was constructed allowing us to dynamically
view the two-dimensional plot of the bifurcation curve providing the spur over an \( A \) interval.
Once a reasonable value of \( A \) locating the cusp was found, a static plot of the curve was
inspected for an approximate location of the cusp point at \((B(\omega^p_j), C(\omega^p_j))\). After noting the
approximate values of \( B(\omega^p_j) \) and \( C(\omega^p_j) \), Matlab was used to generate tables of values of
\( \omega, B, \) and \( C \), so that the approximate eigenvalue associated with the noted ordered pair could
be found. This eigenvalue was then used as an initial approximation of \( \omega^p_j \) for the Newton
Algorithm. Once the values of \( \omega^p_j \) and \( A = A^p_j \) were found for \( R = 0.25 \), Matlab was
programmed to determine the values of \( \omega^p_j \) and \( A = A^p_j \) over the \( R \) interval \([0.20, 0.25] \) for the
spur length calculation.

The greatest difficulty to overcome before plotting a spur was finding the pair of
eigenvalues that created the self intersection point for the entire (discrete) range of \( A \) in
\([A^1_p, A^2_p]\). The distance between the eigenvalues creating the intersection point is zero at \( A^p_j \)
(and monotonically increases with \( A \) until the transition) making it hard to find both
eigenvalues for values of \( A \) close to \( A^p_j \). (The algorithm was unable to produce 2 different
eigenvalues.) In addition, one of the eigenvalues creating the intersection point rapidly
approaches \( \omega = \frac{j\pi}{R} \) as \( A \) approaches \( A^*_j \), causing a singularity in the Jacobian Matrix used in
the Newton Algorithm. Consequently, the root-finding method always broke down before
reaching \( A^*_j \). For most stability spurs that we plotted, the eigenvalues at the self-intersection
point were determined for a range of \( A \) values representing 70-96% of the spur length’s
length. As a result of not being able overcome these difficulties, quadrature algorithms
approximating spur volume were not implemented, as was originally hoped.

Following the first transition, bifurcation Surfaces 1 and 2 intersect each other and
\( A + B + C = 0 \) to form the three-dimensional stability surface for the range of \( A \) values in
the interval \([A^p_1, A^p_2]\) —for all surfaces associated with \( R \in [0.20, 0.25] \). A representative \( BC \)
cross-section plot of the boundary, facing the negative \( A \) direction, is given by Figure 3.2f and
Figure 3.8a. The stability boundary is comprised of, starting at the zero eigenvalue
intersection of Curve 1 (blue) and the plane (purple) right of the origin, and working along the
boundary in a counterclockwise fashion: Curve 1 until it intersects Curve 2 (green) at the
point generated by \( \omega_1 \), followed by Curve 2 between its intersection with Curve 1 at \( \omega_1 \) and
the plane on the left, and, finally, the section of the plane between its intersection with Curve 2
and the point generated by \( \omega_0 \). Curve 2 (green), concave down and having end behaviors tending to infinity parallel to \( B + C = 0 \) and \( B - C = 0 \), also has an absolute maximum point in the \( BC \) cross section. The trajectory traced out in the \( ABC \) coefficient-parameter space by this maximum point as \( A \) is swept through the interval \((A^*_1, A^*_2]\), is stretched up and over from the 1st to the 2nd quadrant with increasing \( A \), towards its intersection with the second point of transition, \((A^*_3, B^*_3, C^*_3)\). This behavior can easily be seen when comparing Figure 3.2f with Figure 3.8a. Figure 3.8 provides the reader with pictures of the stability boundary in the \( BC \) cross section before, during and after \( A^*_2 \).

Prototypical two- and three-dimensional depictions of the finite portion of the stability surface (associated with each surface in the \( R \) interval \([0.20, 0.25]\)) near the Starting Point and Spurs 1 and 2 are provided by Figure 4.4. Traces of the MRS pyramid, (black), were included in each figure to give the reader a perspective of how much of, and the way in which, the actual boundary bulges out from the MRS, both before and shortly after the MRS comes into existence (at \( A = 0 \)). Once \( A \) is elevated to \( A^p_2 \), the tip, or cusp point of Spur 2 (black), provided by the self-intersection of Surface 3, is manifest. Thereafter, for rising \( A > A^p_2 \), Spur 2 is fabricated and connects itself to the larger stability surface at \( A = A^*_2 \). For all surfaces in \((0.20, 0.25]\) though, the MRS pyramid (black), comes into existence, beginning with its apex (at the origin, \((A, B, C) = (0, 0, 0)\)), prior to \( A = A^p_2 \).

The range of Transition 2, \( A^*_2 = A^*_2(R) \), corresponding to domain \( R \in [0.20, 0.25] \), is approximately \([0, 4.8368]\). \( A^*_2 \) is the second of three red-colored, spur-connecting transition curves illustrated in Figure 4.1. One can easily see from the figure that the range of Transition 2 in the \( R \) region of interest is on the same order of magnitude as the range of Transition 1. Yet, Spur 2 (black) is much tinier in size than Spur 1 (green). The range of spur length associated with Spur 2 over \( R \in [0.20, 0.25] \) was found to be \([0.0305, 0.1136]\), where \( A^*_2(0.20) - A^p_2(0.20) \approx 0.1136 \) and \( A^*_2(0.25) - A^p_2(0.25) \approx 0.0305 \). Therefore, Spur 2’s length decreases 73% over \( R \in [0.20, 0.25] \), shrinking a lot more rapidly over the \( R \) interval than Spur 1’s length. Furthermore, in the \( BC \) cross section when \( A = A^*_2 \), the ratio of the larger stability region area to the area fabricated by Spur 2, was calculated. The ratio was found to increase through the approximate interval \([332, 6744]\), as \( R \) was swept through the interval \([0.20, 0.25]\), demonstrating that the actual size of Spur 2 in comparison to the larger surface is insignificant.

Following Transition 2, as can be easily seen in Figure 3.8d, the two-dimensional boundary of the stability is composed of a portion of \( A + B + C = 0 \) and bifurcation Surfaces 1 (blue), 2 (green) and 3 (black). More specifically, the stability boundary, as seen in the \( BC \) cross section (facing the negative \( A \) direction), is comprised, starting at the zero eigenvalue and working along the boundary in a counterclockwise fashion, of Curve 1 (blue)
Figure 4.4. Surface with two spurs.

intersecting the plane (purple) at Eqn. (3.5), Curve 1 intersecting Curve 3 (black) at $\omega_1$, Curve 3 intersecting Curve 2 (green) and, finally, Curve 2 intercepting the plane on the left. One can tell by inspection of the stability surface depicted by Figure 4.4, that Surface 3 (black) displaces more of Surface 2 (green) along the boundary with rising $A$ past $A_2^*$. Additionally, the distortion of the stability surface away from the MRS caused by the second spur-connecting transition is apparent.

Figure 4.1 shows a third spur is attached along the surface right away for $R = 0.20$, around $A_3^* \approx 12.78$, but as $R$ is increased, so does the value of $A_3^*$, and in turn, the range of $A$ between spur-connecting transitions $A_2^*$ and $A_3^*$. In fact, Transition 3 is asymptotic with $A_3^* \to \infty$ at $R = 0.25$. The range of $A_3^* = A_3^*(R)$ for $R \in [0.20, 0.25)$ is approximately $[12.7810, \infty)$. All the remaining changes to the stability surface for $R = 0.25$ occur through
tangencies, reverse tangencies, or one reverse transferral. None of these events dramatically change the shape of the boundary of the stability region. What is significant is that the asymptote at \( R = 0.25 \) for \( A_3^* \) facilitates a delay in the onset of the third spur until \( A_3^* = \infty \), causing the distortion in the boundary away from the MRS to persist, while nearby delays do not have this distortion.

Figure 4.5 shows the cross-sectional boundary of stability at \( A_3^* \) for \( R = 0.20 \) and shortly after \( A_3^* \). The MRS pyramid, contained within the larger closed region formed by the aforementioned curves, is illustrated with black, dashed line segments. It is clear from both pictures that Curve 2 (green) has translated nearly all the way off of the stability surface. The two-dimensional boundary of stability at \( A_3^* \), beginning at Eqn. (3.5) and keeping with the counterclockwise accounting, consists of Curve 1 (blue) until it intersects with Curve 3 (black) in the 1st quadrant. Then, a section of Curve 3 is found along the boundary until intersecting with Eqn. (3.8) (dashed black line) and Curve 4 (the red stability spur). Eqn. (3.8) forms the boundary in the 2nd quadrant until intersecting Curve 2 (green). Finally, we see the real root crossing plane, \( A + B + C = 0 \), take on a tiny section of the boundary in the 2nd quadrant; and \( A + B + C = 0 \) exclusively bounds the 3rd quadrant portion of the stability region.

![Figure 4.5. Stability region at and after the third transition.](image)

Afterwards (Figure 4.5b), Curve 4 replaces the degeneracy line along the boundary. The size of the third spur (not visible in Figure 4.5b) on the \( A \) interval \([A_3^p, A_3^*]\), in relation to the larger surface fabricated by the real root-crossing plane and Surfaces 1, 2, and 3 for \( A < A_3^* \) is almost negligible. Figure 1.1 provides a view of the stability region of Eqn. (3.1) for \( R = 0.20 \) and \( A \leq 21 \). The finite portion of the surface past \( A_3^* \) and up to the first
transferral can be seen. It is worth mentioning the range of spur length associated with Spur 3 over \( R \in [0.20, 0.25] \) is \([0, 0.0110]\), where \( A_3^*(0.20) - A_3^p(0.20) \approx 0.0110 \) and \( A_3^*(0.25) - A_3^p(0.25) \approx 0.0110 \). Spur 3’s length decreases 100% over \( R \in [0.20, 0.25] \). Figure 4.3b compares the spur lengths associated with the three spur-connecting transitions, \( A_1^*, A_2^* \) and \( A_3^* \).

During this numerical study, the trend amongst stability spurs pointed out by Mahaffy, Zak, and Joiner [34, 35], that each additional spur occurring along the stability surface with rising \( A \) possesses a corresponding spur length shorter than that of the previous spur, held. Counterexamples to the following two conjectures regarding the existence of stability spurs were not found.

**Conjecture 1:** Suppose a particular surface has \( j \) stable spurs. Then 
\[
(A_1^* - A_1^p) > (A_2^* - A_2^p) > \cdots > (A_{j-1}^* - A_{j-1}^p) > (A_j^* - A_j^p).
\] In other words, each spur added to the surface with rising \( A \) will possess a corresponding spur length shorter than that of the previous spur, and in turn, a smaller volume.

**Conjecture 2:** For each natural number \( j \), there are exactly \( j \) stable spurs on each unbounded stability surface in the \( R \) interval \([\frac{1}{j+2}, \frac{1}{j+1}]\), where \( 0 < R < \frac{1}{2} \).

### 4.2 Asymptotics

In this section, observations and results from the numerical study and application of the definitions given in Chapter 3 are summarized. How the extra region of stability outside of the MRS associated with \( R = 0.25 \) evolves as \( A \to \infty \) was of the greatest interest. Numerical evidence suggests the two-dimensional region of stability is 26.86% times larger than the MRS for large \( A \). Case studies of finite sections of stability surfaces corresponding to \( R = 0.249 \) and \( R = 0.2495 \) are presented here that will help explain the asymptotic region of stability for \( R = 0.25 \). Figure 4.6, a centerpiece of the numerical work done for this part of the thesis, provides a complex overview of the transitions, transferrals and tangencies that occur along the boundary of stability, as a function of \( A \), for surfaces associated with \( R \) values chosen from the interval \([0.247, 0.251]\), for the range of \( A \in [A_0, 1000] \).

Both Figures 4.6 and 4.1 depict tangency curves in either black or green, depending upon whether the tangency occurs in the 1st or 4th quadrant of the \( BC \) space, for any \( R \) in the intervals shown in the figures. For \( R = 0.25 \), the difference in color also helps distinguish which of the six families provides the next tangency: either Family 3 (green) or Family 6 (black). In the \( BC \) cross section, it was observed that Families 2 and 3 and, respectively, Families 5 and 6 transition with one another outside of the stability boundary with rising \( A > A_2^* \). Shortly after transitioning with their associated cohorts from Families 2 and 5, bifurcation curves from Families 3 and 6 first enter the two-dimensional boundary of stability.
as one transferral and several tangencies. Additionally, notice from Figures 4.6 and 4.1 that the tangencies provided by Families 3 and 6 come onto the stability surface by alternating in occurrence as $A$ is increased beyond $A_{1,6}$.

### 4.2.1 Case Study: $R = 0.249$

A description of the finite portion of the connected, three-dimensional stability surface associated with $R = 0.249$ for $A \in [A_0, 1000]$ is presented in this section. With the value of $R$ fixed at 0.249, the real root crossing plane, $A + B + C = 0$, and bifurcation surfaces, given by Eqns. (3.3) and (3.4), intersect in the $ABC$ coefficient-parameter space of Eqn. (3.1) to form
the stability surface. The discussion given in this section explains in great detail, how those surfaces enter or leave the stability boundary, via transitions, transferrals and tangencies, as \( A \) is increased in a continuous fashion through the interval \([A_0, 1000]\), as summarized in Figure 4.6.

In the previous section, we discussed the evolution of stable spurs connecting to the main stability region for \( R \in [0.20, .025] \). For \( R = 0.249 \), as seen in Figure 4.6, the first two events affecting the stability region are the addition of two stable spurs. The main stability surface begins at the Starting Point \((A_0, B_0, C_0) = (-5.01606, -0.33156, 5.34762)\), then the first stable spur begins at \( A_1^p \approx -2.7326 \), adjoining to the main surface at \( A_1^* \approx -2.4464 \). A second stable spur begins at \( A_2^p \approx 4.70671 \) and connects at \( A_2^* \approx 4.71 \). This sequence of events was discussed thoroughly in the previous section, and by continuity of the characteristic equation (3.2), the figures showing expansion of the stability region by the stability spurs carries to our case \( R = 0.249 \).

Subsequent to \( A_2^* \) occurring, Figures 4.1 and 4.6 indicate Curve 6 supplies the first transferral, \( A_{1.6}^* \approx 13.3 \). Representative \( BC \) cross-section plots of the region of stability for \( R = 0.249 \) before, during and after \( A_{1.6}^* \) are visible in Figures 3.4a-d. As \( A \) is increased, so is the overall size and volume of the stability surface, as well as the MRS pyramid contained within. Prior to Transferral 1 (Figures 3.4a and 3.4b), and also when \( A = A_{1.6}^* \) (Figure 3.4c), a larger, almost triangular-shaped subregion of stability in the 4th quadrant, outside of the MRS, is formed by Curve 1 and the real root-crossing plane. Additionally, subsequent to \( A_2^* \), Curve 3 (black) provides an extra portion of the stability region below the MRS sides given in the 1st and 2nd quadrants.

With rising \( A \) past \( A_{1.6}^*, \) as the two-dimensional region of stability expands along with its MRS, for a range of \( A \) Curve 6 (orange) can be seen cutting off the quasi-triangular section of the stability region in the 4th quadrant (see Figures 3.5 and 3.4d). Simultaneously, the maximum point of Curve 3 (black), located in the 2nd quadrant, begins translating towards the eventual point of transition associated with \( A_3^* \), located in the 1st quadrant. Curve 3 begins to converge to the MRS in the 2nd quadrant. Curve 2 sluggishly continues to leave the two-dimensional boundary of stability. For a small range of \( A \) following \( A_{1.6}^* \) (Figure 3.4d), sections of \( A + B + C = 0 \) (purple) and Curves 1 (blue), 2 (green), 3 (black) and 6 (orange) form the boundary of the stability region.

Following the transferral, Figures 4.1 and 4.6 indicate that higher-ordered bifurcation curves enter the two-dimensional stability boundary with rising \( A \), in the form of tangencies (black or green curves) alternating in location between the 1st and 4th quadrants. Table 4.1 summarizes the values of \( A \) locating these tangencies for \( R = 0.249 \). The family structure of \( R = 0.25 \) appears preserved for the surface corresponding to \( R = 0.249 \) up to \( A \approx 500; \)
Table 4.1. Boundary Changing Events for $R = 0.249$ on $A \in [A_0, 750]$

<table>
<thead>
<tr>
<th>surface change</th>
<th>$A$</th>
<th>reverse tangency</th>
<th>$\tilde{A}_{3.39} \approx 559.216$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$-5.016$</td>
<td>$A_0^{3,39,33}$</td>
<td></td>
</tr>
<tr>
<td>spur 1</td>
<td>$[A_1^0, A_1^+] \approx [-2.7326, -2.4464]$</td>
<td>reverse tangency</td>
<td>$A_0^{3,36,30} \approx 622.341$</td>
</tr>
<tr>
<td>spur 2</td>
<td>$[A_2^0, A_2^+] \approx [4.7067098, 4.70671]$</td>
<td>reverse tangency</td>
<td>$A_0^{3,33,27} \approx 655.407$</td>
</tr>
<tr>
<td>transferral</td>
<td>$A_1^{1,6} \approx 13.3$</td>
<td>reverse tangency</td>
<td>$A_0^{3,30,24} \approx 678.811$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_3^{3,9} \approx 49.4$</td>
<td>reverse tangency</td>
<td>$A_0^{3,27,21} \approx 696.727$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_6^{1,12} \approx 80.216$</td>
<td>reverse tangency</td>
<td>$A_0^{3,24,18} \approx 710.883$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_9^{1,15} \approx 108.4$</td>
<td>reverse tangency</td>
<td>$A_0^{3,21,15} \approx 722.187$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{12}^{1,18} \approx 142.479$</td>
<td>reverse tangency</td>
<td>$A_0^{1,18,12} \approx 731.176$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{15}^{1,21} \approx 174.915$</td>
<td>reverse tangency</td>
<td>$A_0^{1,15,9} \approx 738.192$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{18}^{1,24} \approx 208.787$</td>
<td>reverse tangency</td>
<td>$A_0^{1,12,6} \approx 743.460$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{21}^{1,27} \approx 244.699$</td>
<td>reverse tangency</td>
<td>$A_0^{1,9,3} \approx 747.134$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{24}^{1,30} \approx 283.613$</td>
<td>reverse transferral</td>
<td>$A_0^{6,1} \approx 749.4$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{27}^{1,33} \approx 327.299$</td>
<td>$Spur 3$</td>
<td>$A_0^{6} \approx 749.93$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{30}^{1,36} \approx 379.973$</td>
<td>transferral</td>
<td>$A_0^{7,1} \approx 749.94$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{33}^{1,39} \approx 462.063$</td>
<td>tangency</td>
<td>$A_0^{4,10} \approx 749.953$</td>
</tr>
<tr>
<td>tangency</td>
<td>$A_{36}^{1,43} \approx 750.044$</td>
<td>tangency</td>
<td>$A_0^{7,13} \approx 750.044$</td>
</tr>
</tbody>
</table>

although we know from Definition 3.8 that $R = 0.249$ has 1502 families. The first eleven tangency curves on the $RA$ plane in Figure 4.6 above transferral curve $A_1^{1,6}$ (blue) reach minimum $R$ values. The branch of each tangency curve above its minimum $R$ value represents a reverse tangency curve.

The first tangency for $R = 0.249$ is $A_3^{3,9} \approx 49.4$, marking the entry of bifurcation Surface 9 onto the stability surface. The boundary near this tangency along with the two-dimensional region of stability is well depicted by Figures 3.5a-d. The gap in the 2nd quadrant between Curve 3 and the MRS continues to shrink as $A$ approaches the first tangency, so that the extra portion of the region of stability that is formed outside of the MRS
is mostly given in the 1st and 4th quadrants. The extra stability region outside of the MRS in the 4th quadrant, once provided by Curve 1 and the plane, appears further cut off by Curve 6 (orange) shortly after $A_{1,6}$ (Figure 3.5b) and leading up to $A_{3,9}$ (Figure 3.5c). Yet, after the first tangency this region appears to enlarge in proportion to the MRS (Figure 3.5d). Additionally, the portion of Curve 9 along the boundary in the 1st quadrant does not drastically shrink the enlarged lobe that forms above the 1st quadrant face of the MRS.

After the tangency between Curves 3 and 9 occurs in the 1st quadrant, a tangency between Curves 6 and 12 occurs in the 4th quadrant, when $A_{6,12} \approx 80.2164$. Curve 12 subsequently enters the boundary of stability in the same fashion as Curve 9, only as an almost symmetric reflection about the B-axis. Following $A_{6,12}$, Curve 12 displaces more of Curve 6; then another tangency occurs along the boundary in the 1st quadrant between Curves 9 and 15, at $A'_{9,15} \approx 108.4$. As one increases A more, Curve 18 enters the boundary of stability, again in the form of a tangency in the 4th quadrant, when $A'_{12,18} \approx 142.4797$.

Figure 4.7b shows the 4th quadrant region of stability at $A = 150$, and a portion of the 3rd quadrant region, above $A + B + C = 0$ (purple). The real root-crossing plane and Curves 6, 12, and 18 (orange) form the boundary with Curve 1 (blue). Directly under the plane in the 3rd quadrant, Curve 6 has below it Curve 12, and under Curve 12 is Curve 18. Each of these curves intersects $A + B + C = 0$ in the 4th quadrant. Of the three intersection points with $A + B + C = 0$ in the 4th quadrant, the C-coordinate of the point of intersection is the greatest for Curve 6, followed by Curve 12 and then Curve 18.

The boundary of stability at $A = 150$, in keeping with the counterclockwise accounting, is given by $A + B + C = 0$ until intersecting Curve 6. Curve 6 contributes a small section to the boundary then intersects Curve 12, and Curve 12 intersects Curve 18. Curve 18, having entered the stability boundary at $A'_{12,18} \approx 142.4797$, constructs a portion of the boundary before intersecting Curve 12 once more. Finally, in Figure 4.7b we see Curve 12 along the boundary before intersecting Curve 6, and then the intersection of Curve 6 with Curve 1 near the positive B-axis. Curve 1 then intersects Curve 3 (Figure 4.7a) in the 2nd quadrant, near the corner, $(B, C) = (150, 0)$, of the MRS.

Working along the boundary in the 1st quadrant, we see that Curves 3, 9 and 15 (black) have a similar arrangement and an almost B-axis symmetry to Curves 6, 12, and 18 (orange). A section of Curve 3 constructs the boundary before intersecting Curve 9, which constructs the boundary until intersecting Curve 15. Then, near the top corner of the MRS, $(0, 150)$, these curves intersect once more, with Curve 15 making the boundary until intersecting Curve 9; followed by Curve 9 until it intersects Curve 3. Curves 3, 9 and 15 enter the 2nd quadrant separated by a distance. Curve 3 is the black curve in the 2nd quadrant that is nearly on top of the 2nd quadrant face of the MRS. Above Curve 3 in the 2nd quadrant is
Figure 4.7. Pictures of the region of stability for $R = 0.249$. 

(a) 9 curves, including $A + B + C = 0$. 
(b) 9 curves, including $A + B + C = 0$. 
(c) 16 curves, including $A + B + C = 0$. 
(d) 8 curves, including $A + B + C = 0$. 
(e) Bottom Corner 
(f) Top Corner.
Figure 4.7c shows a picture of the two-dimensional boundary when $A$ is increased to 500, after seven more tangencies take place. Surfaces 21, 24, 27, 30, 33, 36, and 39 intersect with the larger stability surface from $[A_0, 174.9]$ so that sixteen bifurcation curves can be found along the boundary in the $BC$ plane at $A = 500$. The arrangement, ordering and almost symmetry of the curves along the stability boundary remains the same with each added pair of bifurcation curves. Despite the tangencies that take place, the extra regions of stability outside the MRS in the $1^{st}$ and $4^{th}$ quadrants grow larger in proportion to the MRS. However, if one zooms in on Curve 3, located below the other black curves in the $2^{nd}$ quadrant and above the $2^{nd}$ quadrant face of the MRS, one would see a smaller region of stability above the MRS not visible in Figure 4.7c. Curve 3 is the black curve in Figure 4.7c in the $1^{st}$ quadrant with the highest maximum. Tangencies $A_{18,21}^t, \ldots, A_{33,39}^t$ occurred below this maximum.

Between $A_0$ and $A = 500$, both Figure 4.6 and Table 4.1 indicate that eleven tangency points $A_{3,9}^t, A_{6,12}^t, A_{9,15}^t, \ldots, A_{27,33}^t$ and $A_{33,39}^t$ can be found on the stability surface for $R = 0.249$. Those tangency points mark the location of where Surfaces 9, 12, 15, \ldots, and 39 first take over a portion of the stability boundary with rising $A$. Tangency curves $A_{3,9}^t, A_{6,12}^t, A_{9,15}^t, \ldots, A_{27,33}^t$ and $A_{33,39}^t$ (Figure 4.6), influenced by the vertical asymptote that transitions $A_{3n}^t$ have at $R = 0.25$, curve back on themselves in the $RA$ plane. For $R = 0.249$, this means that as $A$ is increased from 500, the stability surface sheds the eleven bifurcation surfaces that took over a portion of the stability boundary on $[49, 500]$. Curves 39, 36, 33, \ldots, 12 and 9 leave the two-dimensional boundary, via reverse tangencies, occurring in the following sequential order: $\tilde{A}_{39,33}^t, \tilde{A}_{36,30}^t, \tilde{A}_{33,27}^t, \ldots, \tilde{A}_{12,6}^t, \tilde{A}_{9,3}^t$. Somewhere in the range of $[A_{33,39}^t, \tilde{A}_{39,33}^t]$ the family structure of $R = 0.25$ is lost.

A picture of the boundary at $A = 735$, after $\tilde{A}_{18,12}^t$ (Curves 21, 24, 27, 30, 33, 36, and 39 have come off of the stability surface) is given in Figure 4.7d. Several thousand bifurcation curves were graphed in the $BC$ plane at $A = 735$, and curves were then removed from the graph until it was verified that only $A + B + C = 0$ and Curves 1, 2, 3, 6, 9, 12, and 15 composed the boundary at $A = 735$. At this point in the evolution of the boundary, the distortion in the stability surface away from the MRS in the $1^{st}$ and $4^{th}$ quadrants is seen constructing almost triangular-shaped, quasi-symmetric sections above and below the MRS. The top and bottom corners of these regions in Figure 4.7d have a black square around them. If one zooms in to see the boundary in those squares, Figures 4.7e and 4.7f result.

These figures identify the curves on the boundary of stability with numerical labels. Similar to the boundary at $A = 150$ (Figure 4.7b), at $A = 735$, $A + B + C = 0$ makes up the
stability boundary in the 3rd and 4th quadrants until intersecting Curve 6 in the 4th quadrant. Curve 6 takes on a tiny part of the boundary, but then intersects Curve 12 near the plane. Curve 12 constructs a large part of the boundary in the 4th quadrant until intersecting Curve 6, which intersects Curve 1 near the corner of the MRS, \((B, C) = (735, 0)\). Curve 1 fashions a small part of the stability boundary in the 4th and 1st quadrants until intersecting Curve 3 near \((735, 0)\). Curve 3 picks up a part of the boundary until intersecting Curve 9. Working around the boundary in a counterclockwise fashion from that point (Figure 4.7f), we see Curve 9 intersect Curve 15 near the top corner of the stability region, followed by Curve 15 intersecting Curve 9 once more, and Curve 9 intersecting Curve 3. Curve 3 makes up the stability boundary in the 2nd quadrant until intersecting Curve 2 (green). Curve 2 constructs a seemingly negligible part of the boundary before intersecting the real root-crossing plane.

The sequence of reverse tangencies, \(\{\tilde{A}_{9,3}^t\}_{n=1}^{n=11}\), that occur for \(R = 0.249\) between \(A = 500\) and \(A = 750\) (Table 4.1), mark where the sequence of bifurcations Surfaces \(\{3j\}_{j=1}^{j=13}\) peel off of the stability surface. Very shortly after the last of these reverse tangencies, \(\tilde{A}_{9,3}^t \approx 747.1\), the reverse transferral, \(\tilde{A}_{6,1}^t \approx 749.4\), occurs. \(\tilde{A}_{6,1}^t\) marks the departure of Curve 6 from the two-dimensional boundary of stability, after Curve 6 intersects the zero eigenvalue intersection of Curve 1 at the real root-crossing plane. Next, \(A_{3}^* \approx 749.93\) occurs, marking the connection of a third, miniscule spur and the subsequent departure of Curve 3 from the boundary. Figure 4.8a shows the \(BC\) cross-section plot of the boundary at Transition 3. Only Curves 1 (blue), 2 (green), 3 (black), 4 (not visible), Eqn. (3.8) (dashed line) and the real root-crossing plane (purple) construct the boundary at \(A_{3}^*\).

Comparing Figure 4.8a to a picture of the boundary at Transition 3 for \(R = 0.20\) (Figure 4.5a), the extra region of stability in the 2nd quadrant between the degeneracy line and the MRS is clearly present in Figure 4.5a, and exists but is not visible in Figure 4.8a. As \(R\) increases through the interval \([0.20, 0.249]\) with \(A\) fixed at \(A_{3}^*\), the perpendicular distance between the degeneracy line the 2nd quadrant face of the MRS shrinks through the approximate interval \([3.51, 0.074]\). Additionally, the point of intersection of Curves 1 and 3, previously located in the 1st quadrant when \(R = 0.20\), appears much closer to the corner point of the MRS at \((B, C) = (A_{3}^*, 0)\) as \(R \to 0.249^−\). This suggests the surface for \(R = 0.25\) may have this intersection point at \((A_{3}^*, 0)\). Curves 3 and 4 intersect with Eqn. (3.8) at the Point of Transition when \(A = A_{3}^*\), but instantly following the transition, Curve 4 enters the boundary of stability. For \(R = 0.249\), Curve 4 takes on a portion of the stability boundary in the 1st and 2nd quadrants, in the same fashion as the red curve in Figure 4.5a, with the exception that the gap between Curve 4 and the 2nd quadrant face of the MRS would not be visible in the \(BC\) cross section without zooming in.
Figure 4.8. Region of stability for $R = 0.249$ at and after the third transition.

Figures 4.7e and 4.7f show Curves 3, 9 and 15 (black) and 6, 12 and 18 (orange) on the boundary and Curves 4, 10 and 16 (red) and 7 and 13 (blue) outside of the boundary of stability. Prior to $A^*_3$, several transitions, $A^*_{3n}$, occur outside of the stability boundary, rendering Curves $3n + 1$, previously having end behavior keeping each curve away from the boundary, near the stability boundary. Following $A^*_3$, Table 4.1 and Figure 4.6 indicate a second transferral occurs, $A^*_{1,7}$, marking the subsequent entry of Curve 7 onto the boundary, via an intersection with the plane and Curve 1 at the zero eigenvalue, Eqn. (3.5).

As $A$ is incremented past $A^*_{1,7}$, higher-ordered bifurcation curves, indexed by $3n + 1$, rapidly enter the stability boundary in an orderly fashion, by way of tangencies. First,
Curve 10 enters the stability boundary in the 1\textsuperscript{st} quadrant after becoming tangent to Curve 4 at $A_{4,10}^t \approx 749.9537$, followed by Curve 13 which becomes tangent to Curve 7 in the 4\textsuperscript{th} quadrant four when $A_{7,13}^t \approx 750.044$. Between $A_{4,10}^t$ and $A = 1000$, Figure 4.6 indicates Curves 16, 19, 21, \ldots, 121 come onto the stability surface, via tangencies $A_{10,16}^t$, $A_{13,19}^t$, $A_{16,22}^t$, \ldots, $A_{115,121}^t$, which alternate in location by quadrant.

A picture of the two-dimensional boundary of stability with $A$ fixed at 1000 is given in Figure 4.8b. Forty-three curves are found on the boundary: $A + B + C = 0$, Curve 2 (green) and Curves 4, 10, 16, \ldots, 118 (red) and Curves 1, 7, 13, \ldots, 121 (blue). The near symmetry of Curves 4, 10, 16, \ldots, 118 and Curves 1, 7, 13, \ldots, 121 is visible. Additionally, one can see from comparison of Figures 4.8a and 4.8b that the ratio of the area of the stability region to the area of the MRS is smaller in Figure 4.8b. If one zooms in on Curves 1, 7, 13, \ldots, 121 in Figure 4.8b near the zero eigenvalue intersection of Curve 1 and the real root-crossing plane, Figure 4.8c would be seen. The boundary there is clearly complex, but the arrangement, ordering and near symmetry of the curves along the boundary is similar to the way it was before the third transition.

Figure 4.8d shows the stability boundary for $A = 2000$. After inspecting the sequence of pictures given in Figures 4.7 and 4.8, one can visually see that the ratio of the area of the stability region to the area of the MRS has peaked at $A = A_3^*$. The stability boundary for $R = 0.249$ can be seen converging toward the MRS for rising $A$ beyond 750, but we conjecture the ratio of the area of the stability region to the area of the MRS asymptotically approaches a value slightly larger than one.

### 4.2.2 Case Study $R = 0.2495$

Between the dense band of tangency curves (black and green) on Figure 4.6 lies the third transition curve, $A_3^*$ (red), which is effectively at infinity when $R = 0.25$. Below and to the right of $A_3^*$ lies transferral curve $A_{1,6}^t$. Below and to the right of $A_{1,6}^t$ in Figure 4.6 lies tangency curves $A_{3n,3n+6}^t$ and $A_{3n+6,3n}^t$. Above transition curve $A_3^*$ lies transferral curve $A_{1,7}^t$ and above it tangency curves $A_{3n+1,3n+7}^t$ were found. Upon inspection of Figure 4.6, if we let the value of $R$ increase from 0.249 towards 0.25 and examine (finite sections of) the corresponding stability surfaces, we see basically that a series of bifurcation surfaces, whose indices differ by six, will come on and off of the stability surface (between $A_3^*$ and $A_3^*$) in a manner similar to the case when $R = 0.249$. The family structure of $R = 0.25$ can be seen presenting itself on the stability surface for a wider range of $A$ (underneath the $A_3^*$ curve in Figure 4.6) as $R \to 0.25^-$. For example, consider Figure 4.6 and fix the value of $R$ to be 0.2495. The stability surface begins at $A_0$, where Surface 1 and $A + B + C = 0$ form the earliest stage of the
surface. Subsequently, Surfaces 2 and 3 self-intersect forming stability spurs that connect to the larger surface at $A_1^*$ and $A_2^*$, followed by the addition of Surface 6 to the stability surface at $A_{1,6}^*$. Afterwards, we see tangencies arise. When $A = 150$, the two-dimensional boundary of stability looks similar to Figure 4.7a, and 8 curves are found on the boundary. When $A$ is increased from 150 to $A_{69,75}^* \approx 930$, Figure 4.6 shows 21 tangencies have occurred, so that 26 curves can be found on the boundary of stability. It is also clear from Figure 4.6 that Curve 75 is not on the stability surface long since reverse tangency $\hat{A}_{75}^* \approx 1000$.

On the interval [1000, 1500] the sequence of bifurcation surfaces $\{3n\}$ from $n = 25, 21, \ldots, 3$ peel off of the stability surface, leaving the boundary as reverse tangencies. Then Surface 6 leaves the boundary via reverse transferral, $\hat{A}_{6,1}^*$. Shortly thereafter, $A_3^* \approx 1500$ marks the connection of a third stability spur, given by Curve 4. At $A_3^*$, we find the boundary has the same simple, near symmetric shape as the one depicted in Figure 4.8a and is constructed by the the same five curves. The perpendicular distance between the 2nd quadrant face of the MRS and Eqn. (3.8) decreases to roughly 0.035, and the 1st quadrant intersection of Curves 1 and 3 seems to converge to the right corner point of the MRS at $(B, C) = (A_3^*, 0)$. After $A$ is increased beyond $A_3^*$, Curve 3 leaves the boundary, then Surface 7 is added at $A_{1,7}^*$. Finally, the stability surface associated with $R = 0.2495$ rapidly takes on the sequence of bifurcation surfaces indexed by $\{3n + 1\}$, which enter the two-dimensional boundary by way of tangency. In addition, the bulge in the stability region away from the MRS (in the 1st and 4th quadrants) becomes less pronounced, as the ratio of the area of the stability surface to the MRS Area peaks at $A_3^*$ and then appears to asymptotically converge to the MRS. Although, for large values of $A$ we conjecture that one can zoom in near the MRS and still detect a slight bulge.

### 4.2.3 Asymptotic Region of Stability for $R = 0.25$

Case studies detailing the two-dimensional boundary and region of stability for Eqn. (3.1) with fixed values of $R$ slightly less than 0.25 have been presented in the previous subsections. Those case studies have described the stability surface before and after the connection of a third stability spur at $A_3^*$. From Eqn. (3.6), the equation defining $A_3^*$, we see that as $R \to 0.25^-$, $A_3^* \to \infty$. Using the continuity of the characteristic equation, (3.2), Figure 4.6, the family structure of $R = 0.25$ and the preceding case studies, we are able to characterize the complete, unbounded stability surface associated with the two-delay ratio, $R = 0.25$.

Increasing $A$ past $A_2^*$, the second spur-connecting transition, we see members of families three and six, bifurcation surfaces whose indexes differ by six, enter the boundary of stability and come onto the stability surface. More and more bifurcation surfaces intersect and
fabricate the stability surface. Extra regions of stability outside of the MRS pyramid become predominant and visible on the surface. When \( A \) is sufficiently large enough, those same bifurcation curves start shedding off of the stability surface one by one (with rising \( A \)) and leave the two-dimensional boundary of stability via a reverse tangency. A smaller and smaller number of bifurcation surfaces would be seen constructing the stability boundary as \( A \) gets close to \( A_3^* \). At \( A = \infty \), the extra large bulges in the stability surface outside of the MRS persist and are symmetric. The two-dimensional stability region has the same shape as the region depicted in Figure 4.8a, and is bounded by the same five curves.

4.2.4 Stability Region Area at \( A_3^* \)

Because of the high number of curves along the boundary for some values of \( A \), we recognize that applying quadrature algorithms approximating volumes of stability surfaces would be a daunting task. However, in order to quantify how much larger the stability surface is than the MRS for large \( A \), the ratio, \( \rho(R) \), of the stability region area to the MRS area was measured over the interval \([0.249, 0.25)\], with \( A \) fixed at \( A_3^* \). Figure 4.9 shows the stability boundary at the transition in both the \( BC \) plane and in the \( XY \) plane. One can tell from inspecting Figure 4.9 that applying quadrature algorithms to the stability region at \( A_3^* \) in the rotated coordinate system is a simple calculation. In order to preserve the area when the rotation is applied to the stability region, Eqns. (3.9) and (3.10) had to be scaled by \( \frac{1}{\sqrt{2}} \). Both Composite Simpson’s and Trapezoid quadrature algorithms were used, with \( N = 10^7 \). The results suggest that the stability region for \( R = 0.25 \) is 26.86% larger than the MRS for large values of \( A \).

Figure 4.9. Asymptotic Region of Stability for \( R = 0.25 \).
In addition, the following two measurements were taken with $A$ fixed at $A^*_3$: the perpendicular distance between the degeneracy line, Eqn. (3.8), and the 2$^{nd}$ quadrant face of the MRS, and the distance between the intersection point of Curve 1 with Curve 3 (located in the 1$^{st}$ quadrant) and the corner point, $(B, C) = (A^*_3, 0)$. Both distances converged to zero as $R$ was swept through $[0.249, 0.25)$. The numerical evidence presented in this chapter provides the fundamental motivation for seeking an analytic proof that the region of stability for $R = 0.25$ remains larger than the MRS.
CHAPTER 5
CONCLUSIONS AND FUTURE WORK

Delay differential equations (DDEs) arise in a wide range of applications. Understanding the stability properties of DDEs is important for analysis of models with delays. DDEs are infinite dimensional and as such, present analytical challenges. Multiple delays further complicate this analysis. We examined one of the simplest two-delay differential equations, given by (3.1). This thesis provided valuable tools for performing the analysis.

The work concentrated on two aspects of the complicated analysis. The first topic was a detailed study of stability spurs. These create disconnected regions of stability in two-dimensional parameter space though connect through transitions in three-dimensional parameter space. Our second topic was to explore the idea that rationally-dependant delays lead to larger regions of stability. Coincidently, these two topics were shown to be connected. Due to the complexity of the problem, our work was largely limited to numerical studies of special cases.

A complete characterization of the unbounded stability surface in the ABC parameter space of Eqn. (3.1) for delay ratio $R = 0.25$ was provided. Numerical evidence from Section 4.2 provides the basis for seeking a proof that the region of stability for $R = 0.25$ is substantially larger than the Minimum Region of Stability (Theorem 3.1). Evidence presented in Section 4.1 supports the conjecture that for $R = 0.25$ there are three stability spurs with the third stability spur being residual and joining the stability surface at $A_{3}^{*} = \infty$. Our numerical studies show that the stability spurs draw bifurcation surfaces away from the MRS and limit the number of bifurcation surfaces playing a role on the boundary of the region of stability. This creates a significant bulge in the stability surface, which gives the increased region of stability, asymptotically, for $R = 0.25$.

Computational and visualization tools built in Matlab during the research phase of this thesis, one of which is given in the Appendix, have allowed us to quickly examine changes along the two-dimensional stability boundary, for other delay ratios, $R$, and try to inspect their properties for large $A$. Figure 5.1 gives three examples showing the first 100 bifurcation curves for $A = 1000$ and delays of $R = \frac{1}{3}$, $R = 0.45$, and $R = \frac{1}{2}$. Assuming that 100 bifurcation curves give a good representation of the stability region, this figure shows how different the regions of stability are for the different delays with the stability region for $R = \frac{1}{2}$ significantly greater than the others and the stability region of $R = 0.45$ being very close to
the MRS. As seen in Figure 5.1, many bifurcation curves can intersect often along the boundary of the region of stability, which creates significant challenges in describing the evolution of the complete stability surface for Eqn. (3.1) in the ABC-parameter space. Future work aims to rigorously prove a number of results, which have been observed numerically. Below are several specific aims that might be addressed.

**Aim 1: Stability for Rational \( R \) -** The primary goal is an analytic proof of larger regions of stability for rational \( R \) at least for delays \( R = \frac{1}{n} \). There are folk theorems suggesting that rationally dependent delays lead to regions of larger stability, and this thesis provides numerical evidence to suggests this. Using analytic tools, it may be possible to prove these larger regions of stability and show there is an ordering in the magnitude of the stability regions for \( R = \frac{1}{n} \) with \( R = \frac{1}{2} \) having the largest asymptotic region of stability.

**Aim 2: Enumerating Stability Spurs -** Some bifurcation surfaces generated in the ABC-parameter space for fixed \( R \) self-intersect, creating protrusions on the stability surface, and we label these objects as "stability spurs."[35] These stability spurs create distortions on the stability surface, which enlarge the region of stability. We conjecture the existence of \( j \) stability spurs for \( R \in \left[ \frac{1}{j+2}, \frac{1}{j+1} \right) \). Determining conditions for the self-intersection and existence of stability spurs are required to fulfill this goal.

**Aim 3: Bifurcation Surface Families -** For rational \( R \), we demonstrated that bifurcation surfaces for finding the stability surface organize into a finite set of families. This creates a type of resonance, which prevents the surfaces from approaching the asymptotic minimum region of stability. One may be able to prove these families organize into groups with monotonic properties, which result in very few bifurcation surfaces playing a role asymptotically (large \( A \)) on the boundary of the region of stability, which forces enlargement of the stability region.
Aim 4: Other Rational $R$ - We provided details for $R = \frac{1}{4}$, but more numerical studies centered around rational delays of the form $R = \frac{1}{n}$ are needed. These numerical studies could aid Aim 1. Also, other rational $R$, such as $R = \frac{2}{5}$ or $R > \frac{1}{2}$, need more investigations.

Aim 5: Asymptotics for small $R$ - Our numerical studies showed the stability spurs become larger and increase in number as $R \to 0$, and at $R = 0$, we have the one delay case, which creates a cylinder of stability. (These numerical studies were not included in the thesis, as too little was done.) How does this multi-spurred region evolve into the cylinder as $R \to 0$? This problem requires extensive examination numerically and analytically.

Using the continuity of Eqn. (3.2), the definitions for Transitions, Transferrals and Tangencies, Newton’s Method of Root Finding, Eqns. (3.3) and (3.4) and our Matlab tools, we were able to extend Figure 4.1 and numerically track these changes in the $RA$ plane for $R \in [0.20, 0.40]$ and $A \in [A_0, 200]$. As a conclusion to this thesis, we leave the reader to ponder the structure of Eqn. (3.1), as given in Figure 5.2.

Figure 5.2. A summary of boundary changes.
BIBLIOGRAPHY


[48] E. Zaron, The delay differential equation: \( x'(t) = -ax(t) + bx(t - \tau_1) + cx(t - \tau_2) \), technical report, Harvey Mudd College, Claremont, CA, 1987.
APPENDIX
TWO-DIMENSIONAL BOUNDARY VIEWING ALGORITHM
function two_delay_bdy(A,R,window,j,rotate_frame)

% two_delay_bdy.m produces a static plot of the first j bifurcation curves of the two delay system in either the BC or XY plane, for fixed values of A and R.
% Dynamic two-dimensional boundary plot and Matlab GUI available upon request. Send inquiries to timbusken@gmail.com

% INPUT VARIABLES

% A - a scalar
% R - a scalar in the interval [0,1]
% j - the number of bifurcation curves

% window - a positive scalar or a 1 by 4 vector that sets axis window
% if window is a vector, then that is passed to the 'axis' command, i.e. set axis(window)

% rotate_frame - Enter 0 to view the 2D boundary on the BC plane;
Enter 1 to view the 2D boundary on the XY plane;

% COLOR SCHEME - the color of the jth curve is the same as the color of the mod(j,6)'th curve, e.g. curve 7 has the same color as curve 1, curve 9 has the same color of curve 3, etc. This color scheme follows the six family structure of R=0.25. Change internal variables 'mod_number' and 'color_code' to change this color scheme to follow another family structure.
'color_code' is a vector of RGB triplets. Calling 'uisetcolor' in the command window can help you determine an RGB triplet corresponding to a particular color.
% BIFURCATION CURVE COLOR

% 0 = A+B+C  purple
% 1 blue  4 red
% 2 green  5 dark green
% 3 black  6 orange

clf

% Eqns. 3.3 and 3.4
B = @(w,A,R)( A.*sin(w*R)+w.*cos(w*R))./sin(w*(1-R));
C = @(w,A,R)(-A.*sin(w) +w.*cos(w) )./sin(w*(1-R));

% the following is the rotated coor coordsys (B,C)--> (x,y) with area preservation
XX = @(w,A,R)((B(w,A,R)+C(w,A,R))/sqrt(2) );
YY = @(w,A,R)((B(w,A,R)-C(w,A,R))/sqrt(2) );

% deg line - Eqn.3.8
Btrans = @(jj,R)((-1)^jj*( (1-R).*cos(jj*pi*R./(1-R)) ...
   -jj*pi*R.*csc(jj*pi*./((1-R).*2));
Ctrans = @(jj,R)((-1)^jj+1*( (1-R).*cos(jj*pi./(1-R)) ...
   -jj*pi*csc(jj*pi./(1-R)) ) ./ ((1-R).*2));
CC = @(Bt,B,Ct,jj)((Bt-B+(-1)^jj*Ct)/(-1)^jj);

syms W X Y c b p
S = solve('X+(-1)^p*b +(-1)^p*(Y-c)=0','W+X+Y=0','X','Y');
% S is an array which gives the point of intersection (X,Y) = (B,C) in BC
% space where the degeneracy line intersects the plane. The point is stored
% in analytic form with the above declared symbols. Use the subs command
% repeatedly from the symbolic toolbox to evaluate this expression. To
% access the soln in S type "S.X" or "S,Y".

color_code = [0 0 1;0 1 0; 0 0 0;1 0 0;0.2314 0.4431 0.3373; 1 .50196 0;];
mod_number=6;

for q = 1
   h = get_trans_points(R);
for i = 1:j %number of bif curves determined by input var j
    w = linspace((i-1)*3.1416/(1-R),i*3.14158/(1-R),11000);
    if rotate_frame
        if isequal(mod(i,mod_number),0)
            k = mod_number;
        else
            k = mod(i,mod_number);
        end
        plot(XX(w,A,R),YY(w,A,R), 'color',color_code(k,:))
        hold on
        clear w;
    else
        if isequal(mod(i,mod_number),0)
            k = mod_number;
        else
            k = mod(i,mod_number);
        end
        plot(B(w,A,R),C(w,A,R), 'color',color_code(k,:))
        hold on
        clear w;
    end

% PLOT THE COORDINATE AXES
    ttt=max(window);
    b=linspace(-ttt,ttt,1100);
    plot(0,b,'black',b,0,'black')

% PLOT DEGENERACY LINE — IF NEAR THE TRANSITION
% PLOT PLANE
% PLOT MRS

if rotate_frame

% PLOT DEGENERACY LINE
    if (~isequal(h,0))
        for k = 1:length(h(:,1))
            if(abs(A-h(k,1))<10^(-1)) %then A is near Atrans
                c=CC(Btrans(k,R),b,Ctrans(k,R),k);
                plot((b+c)/sqrt(2),(b-c)/sqrt(2), 'LineStyle','-','Color','black','LineWidth',1)
            clear c
        end
    end
% PLOT rotated A+B+C=0 IN VIOLET
b; c=-(A+b);
plot((b+c)/sqrt(2),(b-c)/sqrt(2),'color',[0.6 0.6],'LineWidth',1);

% NOW PLOT THE MRS
if (A > -(R+1)/R) && (A > 0)
    line([[0+A/sqrt(2)], [(A+0)/sqrt(2)]],...
         [((0-A)/sqrt(2)), ((A-0)/sqrt(2))],...
         'Marker','.','LineStyle','-','Color','black')
    line([A/sqrt(2) (-A)/sqrt(2)],...
         [(A)/sqrt(2) (A)/sqrt(2)],...
         'Marker','.','LineStyle','-','Color','black')
    line([-A/sqrt(2) A/sqrt(2)],...
         [-A/sqrt(2) -A/sqrt(2)],...
         'Marker','.','LineStyle','-','Color','black')
end

% SET AXIS LABELS
xlabel('X axis','FontSize',18)
ylabel('Y axis','FontSize',18)

else
    % plot the degeneracy line, eqn.3.9
    if(~isequal(h,0))
        for k = 1:length(h(:,1))
            if(abs(A-h(k,1))<10^(-1)) %then A is near Atrans
                c=CC(Btrans(k,R),b,Ctrans(k,R),k);
                plot(b,c,'LineStyle','-','Color','black','LineWidth',2)
                clear c
            end
        end
    end
end

% PLOT A+B+C=0 IN VIOLET
plot(b, -(A+b),'color',[0.6 0.6]);

% NOW PLOT THE MRS
if (A > -(R+1)/R) && (A > 0)
    line([0 A],[0 0], 'Marker','.','LineStyle','-','Color','black')
    line([A 0],[ 0 -A], 'Marker','.','LineStyle','-','Color','black')
    line([-A 0],[0 A], 'Marker','.','LineStyle','-','Color','black')
% SET AXIS LABELS
xlabel('B axis','FontSize',18)
ylabel('C axis','FontSize',18)
end

% PRINT PARAMETER VALUES TO THE TITLE
s = sprintf('R = %1.8f and A = %16.7f',R,A);
title(s,'FontSize',18)

% SET AXIS WINDOW — FOR A SCALAR OR AXIS VECTOR
if isequal(length(window),1)
    axis(window * [−1 1 −1 1])
else
    axis(window)
end

grid off;
axis square

hold off
set(gca,'FontSize',18,'FontWeight','bold','FontName','Times New Roman')
end
return

function h = get_trans_points(R)
% Identifies transition points; so dynamicBif.m can detect and plot
% the degeneracy line, Eqn. 3.8, when appropriate.
% Returns an array whose rows are (A,B,C) triplets (Eqns. 3.7)
% at which there exists a transition, for the
% given value of R
j = 1;
 hh=zeros(20,3);
Atrans =0;
flag =0;
h=[];
while (abs(Atrans) < 10^5)
    Atrans = −j*pi/(1−R)*cot(j*R*pi/(1−R));

    Btrans = (−1)^j∗( (1−R)*cos(j*pi*R/(1−R)) ... 
             −j*pi*R*csc(j*R*pi/(1−R)) ) / ((1−R)^2);
\[
C_{\text{trans}} = (-1)^{(j+1)} \cdot \left( (1-R) \cdot \cos\left(j \cdot \pi / (1-R) \right) \right) \ldots
- j \cdot \pi \cdot \csc\left(j \cdot \pi / (1-R) \right) / ((1-R)^2);
\]

\[
\text{hh}(j,:) = [A_{\text{trans}} \ B_{\text{trans}} \ C_{\text{trans}}];
\]

\[
\text{temp} = \text{sign}(A_{\text{trans}});
\]

\[
\text{if } \text{isequal(temp,1)}
\]

\[
\text{flag} = 1;
\]

\[
\text{if } \text{isequal(temp, -1)} \text{ && isequal(flag,1)}
\]

\[
\text{return}
\]

\[
\text{if } (\text{abs}(A_{\text{trans}}) < 10^5)
\]

\[
\text{h}(j,:) = \text{hh}(j,:);
\]

\[
\text{end}
\]

\[
\text{j} = \text{j}+1;
\]

\[
\text{end}
\]

\[
\text{if } \text{isempty(h)}
\]

\[
\text{h} = 0;
\]

\[
\text{end}
\]

\[
\text{return}
\]