PRESERVICE TEACHERS’ INTERPRETATION OF FUNCTION
INVERSES THROUGH VIDEOS AND DYNAMIC GRAPHS

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DEDICATION

This thesis is dedicated to my family. Thank you for all of your patience, love, and support throughout my education. But most of all, thank you for teaching me the importance of hard work and higher education.
ABSTRACT OF THE THESIS

Preservice Teachers’ Interpretation of Function Inverses Through Videos and Dynamic Graphs
by
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The technological advances in the past ten years have had an enormous impact on education. In particular, video technologies and websites such as YouTube have taken education to another level. However, many online educational videos do not align with what math researchers consider good math pedagogy. They show math procedures, but few instruct both mathematical procedures and conceptual knowledge. Therefore, the goal of any preservice math preparation program should be to help teacher candidates develop a more complete understanding of mathematical concepts in order to give their students that same understanding. The intent of this study is to explore the integration of effective mathematical pedagogy with the affordances of online videos with preservice teachers. In particular, an online survey was composed of a procedural video, a conceptual video, and a dynamic graph applet about function inverses. The analysis was of preservice math teachers’ responses to procedural, graphical, tabular, and open-ended questions before and after watching the videos and using the dynamic graph. Results indicated that the videos and dynamic graph enhanced the understanding of inverse functions.
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CHAPTER 1

INTRODUCTION

Over the past two decades, technology has been used to support distance learning in the form of educational videos on television and audio recordings. Now alternative forms of distance learning range from online classes to simply looking up videos on YouTube. This study attempts to explore ways in which interactive video technologies (based on effective mathematical pedagogy) can enhance traditional “talking head” videos to help preservice math teachers learn mathematical ideas in more conceptual ways.

EDUCATIONAL TECHNOLOGY

In today’s world, technological change happens so fast. Compare what people did ten years ago on computers with what they are doing today. Consider the huge changes in scale; today’s cell phones are basically pocket-sized computers that continue to advance. In a recent article in Time, Grossman (2011) points out that, “Computers are getting faster. Everybody knows that. [However], computers are getting faster faster – that is, the rate at which they’re getting faster is increasing” (p. 2). To illustrate this rapid growth, Grossman pointed out that Moore’s law reflects this increasing rate of change by describing the amount of data that fits on an electronic chip doubles approximately every two years (Grossman, 2011). This yields the exponential graph of \( y = 2^x \).

Raymond Kurzweil, an engineer who has dedicated his career to computer science, looked at similar quantities. For example, he found that the amount of computing power also doubles every two years and the amount of computing power doubles over time (Grossman, 2011). It is important to note that when \( x \) (representing time in years) is contained in the interval \((-\infty, 0)\), the graph of \( y = 2^x \) is relatively flat. However, when \( x \) is in the interval \((0, +\infty)\), the graph grows increasingly fast toward infinity as \( x \) approaches infinity. Consequently, technological growth is exponential and not linear and the rate of change of growth is also exponential. For that reason, Kurzweil predicts that within the near future, computers will become more intelligent than humans. This begs the question of how
educators can capitalize on this exponential growth to support greater learning. One way to address the issue is to reflect on earlier historical shifts in mass media.

As McLuhan, a noted media philosopher pointed out that, “We become what we behold… We shape our tools and our tools shape us” (1964, p. 19). Applying this to education, we can ask, “In what ways do education shape tools that shape us?” More specifically, in what ways does technology as a medium shape our learning? Many forms of media are used in education, one of the simplest being a pencil and paper. However, with the exponential growth in technology, graphing calculators, computer software programs, and multimedia tools are becoming more utilized in education. In particular, video technologies have grown immensely. For example, the popular website YouTube continues to grow every minute. According to YouTube’s statistics, approximately thirty-five hours of video are uploaded every minute, which is equivalent to over 150,000 or more full length movies each week. Educators have made use of sites such as YouTube and other video technologies as tools for educating preservice teachers by providing flexible ways to showing different teaching situations, by giving easy access to data, and by connecting educators to various instructional contexts (Wang & Hartley, 2003).

**YouTube Era**

Currently, many have referred to the years 2005-2011 as the “YouTube era.” Since its creation in 2005, the rate in which YouTube grows continues to increase because of its low expense, popularity among all ages, and variability in uses. The video-sharing website has had a tremendous impact on areas such as politics, entertainment and education (Tamim, Shaikh, & Bethel, 2007). In academia, YouTube is used in many ways; some educators use existing YouTube videos whereas others create videos to upload and share. Some incorporate YouTube in class and others for homework. Educators also record lectures to share with the class and the world.

Lin and Michko (2010) support the use of online educational videos to help students review material for retention with the ability to stop, rewind, and skip sections based on their understanding of the material being presented. Their research found that characteristics such as availability, comment threads, annotating, and quizzing are additional features that help improve sites such as YouTube from an educational perspective. Created in 2009, YouTube
EDU is a portion of the YouTube domain devoted to education in which only college and university video channels can upload material (YouTube, 2011). Another video sharing website, Khan Academy has a video library with over 2,100 instructional mathematics videos ranging from arithmetic, calculus, physics, and economics. In any case, videos from YouTube and other video sharing websites are a form of media that support education.

To summarize, video sharing sites cover an extensive range of topics that reach a vast number of populations. YouTube is currently available in 43 different languages and localized in 25 countries across the world. However, the majority of mathematical videos on the Internet focus on procedural algorithms instead of conceptual understanding. For example, a popular calculus video on YouTube is “Calculus in 20 Minutes” in which an actor claims to explain all of calculus in only 20 minutes. The lecturer reviews vocabulary, common equations, and procedural algorithms but fails to provide much conceptual explanations that give meaning to the vocabulary, equations, and procedures.

**PURPOSE OF THE STUDY**

Math education research continues to demonstrate that good mathematical pedagogy incorporates both conceptual and procedural understanding in order for students to have a complete understanding of topics (Grouws & Cebulla, 2000; Hiebert, 2003; Thompson, Philipp, Thompson, & Boyd, 1994). Many of the videos found on YouTube, such as “Calculus in 20 Minutes” are not aligned with this effective pedagogy. The purpose of this study is to conduct an investigation within the CSET population, prospective seventh through twelfth grade math teachers, that explores efforts to integrate research regarding effectiveness of good math pedagogy with the modern affordances of online videos and applets. The researcher hypothesized that videos and dynamic graph applets would enhance the understanding of function inverses within the subjects.

In Chapter 2, a review of relevant literature is presented, which includes summarizes relevant math pedagogy, the contributions of technology in math education, and specific research on function inverses. Chapter 3 presents the research methodology, including the phases of the research, the instruments used, the participants involved, and the data analysis process. Chapter 4 describes the results of the study in the form of student transcripts of interviews, students’ pencil and paper work, modifications in student responses after
watching the videos and working with the dynamic graph, and evidence that students
developed a deeper understanding of function inverses. Chapter 5 concludes the thesis with a
summary of the findings, local and universal implications of the findings, and suggested
follow-up research in the field.

**DEFINITION OF TERMS**

The following terms and acronyms are used throughout the thesis. *Procedural knowledge* is the understanding of how to perform algorithmic calculations. *Conceptual knowledge* is the knowledge that is abundant in relationships. And the California Subject Exam for Teachers is referred to as the CSET.
CHAPTER 2

LITERATURE REVIEW

This section addresses (a) research-based ideas for effective mathematical pedagogy, (b) the contributions of technology in math education, and (c) the specific research on inverse function. See Figure 1 for a general overview of the theoretical framework that guides this work.

![Theoretical framework](image)

**Figure 1. Theoretical framework.**

**WHAT WE KNOW ABOUT EFFECTIVE MATH PEDAGOGY**

This section will discuss (a) the *Common Core State Standards for Mathematics* and their implications for preservice teachers, (b) the importance of both conceptual and calculational understanding in mathematics, and (c) the Universal Design for Learning curriculum framework.

**The Common Core Standards for Mathematics**

Currently, forty-two states, the District of Colombia, and the Virgin Islands have adopted the *Common Core Standards for Mathematics* to help teachers ensure their students
have the proficiency and understanding needed to be successful in either a college or career track. These standards, developed by a group of teachers, administrators, and researchers from all over the United States, “provide a clear and consistent framework to prepare our children for college and the workforce” (Core Standards, 2010). The *Common Core State Standards for Mathematics* are composed of content requirements at each grade level K-12 and a set of *Standards for Mathematical Practice* for students at any level. The eight practices, based on the National Council of Teachers of Mathematics (NCTM) process standards and the National Research Council’s *Adding It Up* report (2001), are a set of practices that math educators should strive to develop in their students (Core Standards, 2010). Specifically, the standards propose the following eight practices:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

In particular, practices number three, four, and five are critical in this study. The third practice suggests that students use definitions and assumptions from the problem, develop and test conjectures, and use counterexamples to construct arguments. They are able to justify their solution and critique others’ solutions by asking clarifying questions and recognizing flaws. The fourth practice advocates that students simplify complex situations by making assumptions and approximations. Students that model with mathematics identify quantities, understand the relationships between tables, graphs and equations, and draw conclusions from the models they use. The fifth practice, which encourages the use of tools, proposes that students can use tools such as paper and pencil, concrete manipulatives, calculators, computer algebra system, or dynamic geometry software to solve a problem. Students explore with technological tools to help visualize situations by varying quantities, noticing consequences, and comparing data to ultimately develop their understanding of concepts.
Preservice teachers will be expected to teach to the *Common Core State Standards*, which include the *Standards for Mathematical Practice*. Thus, they themselves should exhibit the eight practices when working out mathematical problems. One goal of this study is to examine the ways in which various technological features can support our subjects’ propensity to exhibit practices 3, 4, and 5. Furthermore, looking at the general picture, in order for teachers to guide their students to meaningful mathematical understanding, it is essential that they have a solid procedural and conceptual mathematical understanding of the material (Even, 1990). Therefore, in addition to preparing preservice teachers for calculational CSET problems needed to pass the exam, preparation should include a conceptual understanding of topics.

**Conceptual and Calculational Understandings**

Math educators are focused on the way mathematics is taught and urge teachers to teach in a manner that emphasizes understanding and meaningful learning (Even, 1990). Grouws and Cebulla (2000) expressed the following about meaning and understanding:

> There is a long history of research, going back to the 1940s and the work of William Brownell, on the effects of teaching for meaning and understanding in mathematics. Investigations have consistently shown that an emphasis on teaching for meaning has positive effects on student learning, including better initial learning, greater retention, and an increased likelihood that the ideas will be used in new situations. (p. 13)

Hiebert (2003) claimed, “Students learn when they are given opportunities to learn” (p. 10). In other words, if students are given the opportunity to learn procedural skills such as manipulating symbols, then their knowledge will consist of those skills (Hiebert, 2003). These students are proficient in calculating, labeling, and defining more than in reasoning, communicating, conjecturing and justifying (Hiebert, 2003). On the other hand, if students are encouraged to construct multiple connections among topics then research demonstrates that students can build procedural and conceptual knowledge with an understanding of the relationships between the two types of knowledge (Hiebert, 2003). Hiebert and Lefevre (1986) define the term *conceptual knowledge* as: the knowledge that is abundant in relationships (as cited in Even, 1990). In order to provide students with an opportunity to construct conceptual knowledge, educators must provide learning conditions in which students build on their entry knowledge through classroom activities that make use of
computational skills, analyze multiple solving methods, and call for student justifications (Hiebert, 2003).

In particular, preservice teachers must have both a conceptual and procedural handle on mathematical topics in order to support their future students’ more complete understandings of subjects. Lack in either procedural or conceptual knowledge leaves teachers unable to solve problems and/or unable to explain why certain algorithms work (Even, 1990). In order to have a more complete understanding of a concept, teachers should have an understanding of the concept in various representations, an understanding of sub-concepts, and an understanding of the nature of mathematics (Even, 1990). These allow for a deep and powerful understanding of concepts.

To illustrate the difference between conceptual and procedural teaching, the main point of a study by Thompson et al. (1994) is that teachers must have a complete understanding of math subjects in order to emphasize conceptual understanding in the classroom. In the study, two teachers engaged their students in solving the same problem in their respective whole-class discussions. The problem is as follows, “At some time in the future, John will be 38 years old. At that time, he will be three times as old as Sally. Sally is now seven. How old is John now?” (Thompson et al., 1994, p. 1). The first teacher focuses the classroom discussion on procedural computations considering the problem as an opportunity to practice computational skills. This emphasis was illustrated by the calculational questions he asked such as, “Why did you divide 38 by 3?” and comments such as, “See? There are different ways to solve the same problem” (Thompson et al., 1994, p. 2). Conversely, the second teacher uses the problem to focus on quantitative relationships by allowing the students to reflect on their reasoning. He asked questions such as “What are you trying to find by dividing 38 by 3?” and comments such as “Remember to tell us what your numbers stand for” (Thompson et. al., 1994, p. 3). This latter classroom discussion involved students justifying their reasoning instead of merely explaining their procedures. Thompson et al. refer to this as a conceptual orientation in which the conception of ideas, situations, and relationships are stressed. They conclude stating, “To teach conceptually requires that one have a deep understanding of the situation. This, in turn, requires that one think beyond what is necessary merely to find ways of dealing with a situation mathematically” (Thompson et al., 1994, p.11).
In order to ensure that students are given the opportunity to learn both conceptual and procedural knowledge, the next section describes the Universal Design for Learning curriculum framework.

**Universal Design for Learning**

As illustrated in Figure 1, a third component of effective math pedagogy is to study research-based practices that focus on helping all students learn. The nonprofit research and development organization Center for Applied Special Technology (CAST) strives to increase learning opportunities for all individuals through Universal Design for Learning (UDL). UDL is a curriculum framework that offers flexible goals, methods, materials, and assessments in order to maximize learning for everyone. The term universal refers to a school curriculum that can be used and understood by all students that bring various skills, interests, and requirements to the classroom.

The term learning referred to the three types of learning that they describe as, “recognition, strategic, and affective.” Recognition, the “what” of learning, is the part of the brain responsible for gathering facts from activities such as reading, seeing, and hearing. This suggests that teachers present material in different representations with different media. The strategic part of learning, the “how” of learning, is how we organize our thoughts and ideas when solving problems. This implies that teachers should give their students multiple opportunities to express what they know by providing feedback and support. The affective component of learning, the “why” of learning, is how learners get engaged, motivated, challenged, and interested. Some students are motivated by spontaneity, novelty, and group work while other students would rather stick to a routine or work alone. Thus, teachers should provide various options for engagement to meet the needs of all students.

All three components of learning are necessary for a successful curriculum. Therefore, the term design is used to encourage a flexible curriculum that fits people both with and without disabilities. In addition, UDL promotes the use of technology in the classroom, but using technology in the classroom does not imply that your classroom follows the UDL framework. “Technology needs to be carefully planned into the curriculum as a way to achieve the goals” (National Center of Universal Design for Learning, 2011).
To summarize, the Common Core State Standards for Mathematics are newly adopted standards that support good mathematical practices. Math education research indicates that both procedural and conceptual content should be incorporated within a flexible curriculum that allows learning opportunities for all students.

**CONTRIBUTIONS OF TECHNOLOGY IN MATH EDUCATION**

This section will discuss research that illustrates (a) the characteristics and benefits of dynamic geometry software, (b) the characteristics and benefits of virtual manipulatives, and (c) the implications for the Technology, Pedagogy, and Content Knowledge (TPACK) framework.

**Dynamic Geometry**

In addition to video technologies that were discussed earlier, another form of media in mathematics that has opened up a new field of research is dynamic geometry (DG) software. This type of software was originally rooted in geometry; hence the name “dynamic geometry,” in which users can create two-dimensional and three-dimensional geometric shapes. DG software has expanded its capabilities and is now used in an array of algebraic symbol systems including graphing and computational functions. These features enable students to explore fundamental relationships between geometric and algebraic concepts. Goldenberg and Cuoco (1998) describe as DG software having objects such as points, lines, and circles; tools for working with the objects such as constructing parallel or perpendicular lines; and transformations such as rotations, dilations, reflections, and more. By manipulating objects, the observer views the varying features. A couple of popular DG software programs are Geometer’s Sketchpad (Jakicw, 2001) and Cabri (IMAG-CNRS Universite Joseph Fourier, 1998).

The major difference between using DG software and working with diagrams on paper is the plasticity that DG software offers. The term *plasticity* refers to the ability for users to manipulate parts of shapes and functions to observe consequences of that shift. As Sinclair (2008) points out, features of computer-based technologies such as DG allow for students to gain insights and intuition through solving problems, create graphical displays to
discover new patterns and relationships, test conjectures, and confirm results derived from paper and pencil algorithms.

However, any new tool brings a new set of issues. Dugdale (1993) addresses the fact that educators must be aware of common misconceptions among students when using function-plotting tools. In particular, when leaving students alone to experiment with these devices, student can be misled to interpret graphs incorrectly due, for example, to window size or scales that are not taken into consideration. Additionally, Goldenberg and Cuoco (1998) discuss issues with how students perceive plasticity, namely the moving parts of dynamic geometry software, and depending on what they perceive, how do they interpret it.

**Virtual Manipulatives**

A different type of mathematical software is virtual manipulatives. The term *virtual manipulates* encompasses two different representations of concrete manipulatives: static and dynamic. Static representations are simply pictures of concrete manipulatives whereas dynamic representations are interactive applets that can be manipulated by flipping and rotating as one could with physical concrete manipulatives (Moyer, Bolyard, & Spikell, 2002). Figure 2 shows the virtual manipulative that the subjects in this study explored by dragging the sliders to change the values of coefficients in the exponential equation.

Figure 2. Virtual manipulative used in this study.

\[ f(x) = a \cdot 10^{\frac{x-h}{b}} + k \]
Reimer and Moyer (2005) conducted a study to explore virtual manipulative computer applets for third grade instruction on fractions. In particular, the teacher questioned how the use of virtual manipulatives might enhance the understanding of fractions among students who had previous experiences with physical manipulatives. Results from the procedural and conceptual pre and posttest, student interviews, and student attitudes surveys from the 19 subjects suggested that students scored 9% higher on the conceptual questions in and 6% higher on the procedural questions in the posttest. However, perhaps the most significant data came from the 59% of subjects that gave positive feedback about the virtual manipulatives. Students indicated in the survey that the virtual manipulatives helped them learn about fractions, that they liked the immediate feedback that the applet provided, that the virtual manipulatives were easier and faster to work with than pencil and paper, and that they had fun while using the virtual manipulatives.

Bowers, Nickerson, and Kenehan (2002) found that students typically do not “come to mathematics classes with a ‘what if’ propensity or a need to explain the mathematics behind the results they see on a computer” (p. 186). Yet, when the students in the study were given the chance to explore or play with virtual manipulatives by changing parameters in linked representations, students began to develop hypotheses about how the changes in one representation would affect another. For example, when students were able to change a velocity graph, they were then provided the opportunity to predict how these changes would affect the linked position graph. Moreover, student justifications improved when they were able to refer back to the virtual manipulative.

Some critics have claimed that one disadvantage of virtual manipulatives is the lack of the hands-on kinesthetic experience. Advocates counter this by pointing out that virtual manipulatives are easily accessed on the Internet, save time by not needing to set up large and complex sets of manipulatives, and can foster mathematical ideas through dynamic images. In addition, dynamic virtual manipulatives have the capability to link dynamic visual images to abstract symbols and concepts, a disadvantage of physical manipulatives (Reimer & Moyer, 2005). The next section describes how to effectively incorporate technology into the classroom.
Technology, Pedagogy, and Content Knowledge

In this “YouTube” age, teacher educators are continually faced with the challenge of determining how to help prospective teachers learn to integrate technology into their future classrooms. Mishra and Koehler (2006) built on Schulman’s (1987) idea of Pedagogical Content Knowledge to suggest that the type of knowledge teachers need to integrate technology lies at the intersection of one’s knowledge about technology, pedagogy, and content as in Figure 3.

![Figure 3. TPACK adapted from Mishra and Koehler (2006).](image)

Content knowledge is the knowledge about the subject that is taught. Teachers must have a clear understanding of the subjects that they teach; otherwise, they can misrepresent mathematical concepts when teaching their students. Pedagogical knowledge is described as processes, practices, and methods of teaching, which includes lesson planning, implementation, and student evaluation. This type of knowledge also requires the understanding of cognitive and social learning theories and the implications for their class. Technology knowledge includes various types of technologies as well as how to use the technological tool whether it is simply chalk and a chalkboard or digital video. Because technology is constantly growing and changing, knowledge of technology continually changes with the times. Thus, the ability to learn and adapt as technologies change is important for teachers.
Mishra and Koehler (2006) describe the combination of the three types of knowledge, TPACK, as the following:

TPACK is the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (p. 1029)

Instead of simply using technology as a way to drill students, Bowers and Stephens (2011) suggest that TPACK can help teachers develop a habit of using technological tools to help students explore and understand concepts. To illustrate this point, the authors cite two contrasting case studies. In the first case, a pre-service teacher creates a technology-based lesson that engages students in the exploration of degree measures for regular polygons. This lesson demonstrates technological knowledge (the ability to use The Geometer’s Sketchpad to dilate figures), pedagogical knowledge (the activity plays on the fact that this exploration is often presented as an algorithm, but rarely explored in a conceptual way), and content knowledge (demonstrating why the sum of the angles of any regular polygon is always 360 degrees). In contrast, another student attempted to create a technologically sound activity GSP to explore the definition of the derivative. While this student was able to demonstrate some technological knowledge, she was unable to use the technology to explain why the secant of a function approaches the tangent at any point.

The goal of the study conducted by Bowers and Stephens was to identify specific characteristics that lie at the intersection of Mishra and Kohler’s theoretical diagram. Their conclusion was that educators who demonstrate TPACK strive to create activities that enable students to explore meaningful questions and figure out the causality behind what they observe on a screen by breaking down the various relationships among mathematical objects.

**Specific Research on Inverse Functions**

The algebra CSET exam covers a wide range of algebraic topics including functions. This study focuses on the topic of function inverses. The amount of research on function
inverses is limited. This section provides (a) the summary of two studies involving inverse functions and (b) how the studies informed the research.

The scant research on students’ conceptions of inverse functions generally suggests that most preservice teachers’ understanding can be categorized as either calculational or conceptual. Those who have a calculational understanding are generally able to follow the steps to compute $f^{-1}$ given $f$, but lack the conceptual underpinnings to understand why the algorithm works, or what $f^{-1}$ actually represents. Even (1992) conducted an investigation of prospective secondary math teachers’ understanding of inverse functions. She found that many students conceptualized a function inverse using the notion of ‘undoing’ (p. 557). “‘Undoing’ is an informal meaning of inverse function which captures the essence of the definition” (Even, 1992, p. 557). The study questioned 162 preservice secondary school math teachers about inverse functions through two open-ended questions.

The first question asked, “Given $f(x) = 2x - 10$ and $f^{-1}(x) = (x+10)/2$. Find $(f^{-1}(f(x)))$. Explain” (Even, 1992, p. 558). Approximately fifty percent of the subjects answered this question correctly. Of those subjects, 26 answered using calculations and properties of inverse functions (i.e., These students recognized that, for any function, $f^{-1}(f(x))=x$.) However, the remaining students completed all of the implied calculations. For example, one subject revealed that after she performed the composition calculation, she then realized that she should have known before calculating the composition that the answer would be 512.5 because of how function inverses are defined. Other subjects who got the problem correct only used a calculation to explain. Even suggests this could have been because the subjects were under the impression that a calculation was sufficient for an explanation. Even reached the conclusion that the need for calculating the composition demonstrated a lack of clear understanding of inverse functions.

In order to probe the degree to which students really understood the concept behind the algorithm, Even asked the following follow-up question: “A student said that there are two different inverse functions for the function $f(x) = 10^x$: One is the root function and the other is the log function. Is the student right? Explain” (Even, 1992, p. 560). Approximately one third of the students answered that the root of the function is the inverse because that ‘undoes’ what $10^x$ does. Even (1992) claims that the naïve conceptual understanding of the term “undoing” in this case is not enough for subjects to answer this question correctly. One
student computed the inverse of $10^x$ by taking the $x^{th}$ root resulting 10 as the answer. Then, the same student solved by taking the log and using properties of logs to result in $x$ as the answer. He answered that both the root and log answers would be correct. Even points out that only those who answered the first question correctly and conceptually (i.e. only using properties of inverse function, without writing a calculation), answered the second question correctly as well. This study suggests that many of the preservice teachers lacked an understanding of function inverses and those who viewed the inverse as ‘undoing’ had an incomplete understanding, which was explicit in question two.

Bayazit and Gray (2004) investigated student learning of function inverses from two different teachers, Ahmet and Mehmet. Ahmet focused his instruction on the idea of an inverse “undoing” operations, whereas Mehmet focused on algorithmic and procedural skills (Bayazit & Gray, 2004). Students were given pretest and posttest to assess their understanding about inverse functions before and after the classroom instruction. Results from the posttest indicated that more students from the class of Ahmet were able to answer a question regarding the domain and range of inverses correctly using verbal explanation. A second item from the posttest showed that when asked to draw the inverse graph of a given graph, students in both classes used a point-wise approach in that a point such as (1, 2) on the function would correspond to (2, 1) on the inverse. However, the most significant statistic showed that 25% of the students in Ahmet’s class chose to take a global approach in reflecting the function across the line $y = x$, yet no students in Mehmet’s class used this approach. The authors conclude that students would have difficulty constructing “meaningful understanding of inverse functions” without the exposure of conceptually focused tasks as in the class of Ahmet (Bayazit & Gray, 2004, p. 109).

**CHAPTER SUMMARY**

The findings from the significant literature about effective math pedagogy, contributions in technology, and research on inverse functions are presented in Figure 4. The newly adopted *Common Core State Standards for Mathematics* (Core Standards, 2010) describe the standards that preservice teachers will support in their future classrooms. The standards encourage both procedural and conceptual understanding, which math education
research has proven to be most effective in deepening student mathematical understandings. Dynamic geometry software and virtual manipulatives have been shown to help students make the connection between symbolic calculations and concepts. The combination of effective mathematical pedagogy with technology, such as DG and virtual manipulatives, lies at the center of TPACK. Even (1992) and Bayazit and Gray (2004) demonstrated the need for conceptual knowledge of function inverses within preservice teachers. This study builds on these findings by using video technologies and dynamic geometry software within a TPACK framework to present inverse function material to preservice teachers.
CHAPTER 3

METHODOLOGY

This chapter discusses (a) the design experiment methodology, (b) the local setting, (c) participants involved in the study, (d) each five of the phases of the design experiment, (e) the rational for the questions and videos used in phases 4 and 5, and (f) the methods used to collect and analyze data.

DESIGN EXPERIMENT

The methodology of this exploratory study is based as a design experiment. According to Cobb, Confrey, diSessa, Lehrer, and Schauble (2003), this form of design is based on iterative cycles of designing a lesson, researching its implementation, and revising it for the next round. This analysis allows researchers to sharpen the instructional focus based on student understandings. Brown (1992) states that researchers should “attempt to engineer innovative educational environments and simultaneously conduct experimental studies of those innovations” (Brown, 1992, p. 141).

This study uses a design experiment method to explore how the components of conceptual videos and dynamic graphs contribute to the development of prospective teachers’ conceptual understanding and ability to construct and interpret graphs of function inverses.

SETTING

This section describes the California Subject Examination for Teachers (CSET) and the participants involved in the study.

The California Subject Examination for Teachers

Prospective seventh through twelfth-grade mathematics teachers in California are required to pass the mathematics single-subject California Subject Examinations for Teachers
(CSET), as a requirement to obtain a mathematics single-subject teaching credential or waiver teaching credential issued in emergency situations\(^1\). The California Commission on Teacher Credentialing (CTC) has developed the CSET to assess the content knowledge of California math educators. The test is aligned with California’s K–12 Mathematics Content Standards for California Public Schools and curriculum frameworks. The single-subject math CSET is composed of three exams: algebra, geometry, statistics, and calculus. Teachers must pass the algebra and the geometry exams. Only the teachers that additionally pass the calculus exam meet the criteria to teach calculus and pre-calculus (California Subject Examinations for Teachers, 2010).

The algebra CSET exam is designed to measure future teachers’ understanding of algebra and number theory content. According to the “Mathematics Subject Matter Requirements” (CTC, 2002), the algebra exam consists of questions involving abstract algebra, polynomial equations and inequalities, functions, linear algebra, and number theory. The problems require “a deep conceptual knowledge,” and use “mathematical abstraction and symbolism” (p. 1). The importance of the CSET has led to the development of numerous preparation courses, websites, and test preparation books. However, most of these preparation options focus heavily on procedural and calculational problems that are seen on the CSET. This leaves test takers unprepared with a limited – if any – “deep conceptual knowledge” of the material. This also leaves our prospective teachers with little preparation to encourage the newly adopted Common Core Standards for Mathematical Practices (Core Standards, 2010).

**Participants**

Each subject signed a release form to allow the author to describe their background, survey responses, and interview data. The study was cleared with the Institutional Review Board of San Diego State University and each subject was given a pseudonym to preserve confidentiality.

\(^1\) Mathematics majors who enroll in the state certified single subject program may waive the CSET requirement.
Current and past students with diverse mathematics backgrounds from San Diego State University were selected to participate in the experiment. The first criterion for participants was that they had planned to take, or had recently taken, the algebra CSET and intended to teach seventh through twelfth grade math. Next, participants were chosen based on their bachelor’s degree with diversity in the ways that they have or will prepare for the CSET. A total of four subjects, three males and one female, agreed to participate in the study.

The first participant, Andy, has a Bachelor’s Degree in History and a secondary school teaching credential in history both from San Diego State University. He teaches English as a Second Language at a local adult school program and substitutes in neighboring high school districts. Andy took a summer preparation class at SDSU for the algebra CSET summer of 2009, but failed the exam the first time around. The following summer, he enrolled in the same preparation course for the geometry portion and passed the exam. He is currently studying the material from the summer preparation course for the algebra material with expectations of passing the algebra portion.

The second participant, Alex, holds a Bachelor’s and Master’s Degree in Electrical Engineering. After working in the engineering field, he was inspired to become a math teacher. Alex is currently enrolled in prerequisite classes at SDSU in order to enroll in the secondary school mathematics teaching credential program in the fall of 2011. He volunteers daily in an eighth-grade math class to gain experience with middle school students. With the aid of CSET preparation books, Alex passed both algebra and geometry portions of the exam.

Frankie has a Bachelor’s Degree in Electromechanical Engineering. Bilingual in English and Spanish, he worked in the engineering field, but due to the poor economic circumstances in the United States, he chose another career path. Frankie is taking prerequisite classes to enter the secondary school mathematics teaching credential program in the fall of 2011 at SDSU with a goal to teach math at a bilingual high school. After taking the algebra and geometry CSET summer preparation courses at SDSU, Frankie passed both sections of the exam.

Kira, the fourth subject in the study, has a Bachelor’s Degree in Applied Mathematics from San Diego State University. Kira plans to teach math at the high school level, but before she gets her teaching credential, she plans to enroll in a Master’s of Education program at
Colorado State University. After receiving her master’s degree, she will take the exam in Colorado that is equivalent to the CSET in order to be qualified to teach high school math.

In summary, three of the four subjects had little math background and three of the four took the CSET with the fourth subject expecting to take a similar exam. Of the three that took the exam, two had passed the exam on the first attempt while the third had failed on his first attempt. These subjects represent some of the populations that take the math CSET.

**EVOLUTION OF THE FIVE PHASES**

The study was composed of five phases that are described in Table 1. The design, format, and elements of subsequent phases were revised according to the feedback from previous phases.

**Phase I:** Our initial assumption guiding this experiment was that an online video data base containing short segments of mathematics instruction might support prospective teachers’ abilities to “brush up” on specific mathematical topics known to be included on the CSET exam. Therefore, during phase one, a mathematics professor was video recorded throughout three hours per day lectures during a three-week summer school CSET preparation course offered at a large southwestern university. The initial question sparking this research was, “How can we best organize and present the clips in a way that would be most educative?” The researcher began by creating PowerPoint slides that provide notes from the professor’s lecture and then linked notes to the video, creating two adjacent screens.

**Phase II:** After creating many of these PowerPoint slides and consulting with a well-known interface design researcher, the developers realized that the video and notes focused on procedural and calculational procedures and lacked conceptual information. Therefore, the researcher began to conduct a search of literature regarding aspects of video that have been found to create content retention. This research resulted in the enumeration of four possible features:

1. adding humor to the procedural videos,
2. creating conceptual videos for each concept,
3. creating Geometer’s Sketchpad dynamic graph applets, and
4. writing practice problems with solutions.
<table>
<thead>
<tr>
<th>Table 1. Cycles of Research</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Phase I</strong></td>
</tr>
<tr>
<td><strong>Design</strong></td>
</tr>
<tr>
<td><strong>Format</strong></td>
</tr>
<tr>
<td><strong>Feedback</strong></td>
</tr>
<tr>
<td><strong>Revisions</strong></td>
</tr>
</tbody>
</table>
The researcher engaged a class of pre-service secondary teachers to augment each of these features using some of the videos from the summer workshop. After completing each element, the class evaluated each component. The resulting conclusion was that the conceptual video and dynamic graphing applets were more beneficial than the humor in the procedural video and the extra problems with solutions.

**Phase III:** Once these two components were identified, the prototype for developing useful video features was modified, and the researcher decided to create an online survey to collect more in-depth information. This survey focused on one particular subject area: inverse functions. The goal of this exploratory phase was to determine what value was added by showing subjects a conceptual video after showing the calculational one. The online survey was composed of three inverse function questions to assess each subject’s initial knowledge (both conceptual and calculational), a procedural video featuring the mathematics professor demonstrating the algorithm for finding the inverse of a function given the original function, and a conceptual video demonstrating the graphical interpretation of inverse functions. The subjects were then asked the same three questions and asked if they wanted to revise any of their initial responses (the questions and videos are explained in detail in the following section). Subjects in this phase were enrolled in Math 414 class in the Spring 2011 at SDSU. Results suggested that the ability to go back and change answers was helpful and the conceptual video and applet were both enlightening from a conceptual perspective.

**Phase IV:** The results from Phase III also revealed that more in-depth interviews would enable me to probe the ways in which the subjects were viewing the videos and exactly why the subjects believed that the conceptual video enhanced their understanding. For example, did students use the stop and replay features? What about the videos enabled them to make conceptual connections? To answer these questions, the online survey was revised. The researcher conducted taped 30-to-60-minute clinical interviews (Ginsburg, 1997). During interviews, subjects had a paper copy of the tasks and were encouraged to solve tasks by explaining their answers verbally and in writing. In Phase IV, two preservice math teachers who had taken the algebra CSET answered questions through an online survey about function inverses while present with the interviewer. The interviewer noticed two issues: First, the subjects felt uncomfortable watching the videos in front of the interviewer. Second, subjects showed interest in using the function inverse dynamic graph applet used in
the conceptual video. Given that the intended purpose of this study is to inform the creation of an online website to help CSET test takers prepare for the exam through the development of conceptual videos and associated applets, the researcher concluded that the envisioned audience would most likely use the CSET preparation material at their convenience while studying for the CSET and not with an interviewer present.

Phase V: In order to address the issues of subject setting and use of applets, the interview protocol was modified again. Subjects in Phase V completed the online survey before attending the interview instead of completing the online survey with the interviewer, and they were given a link to the applet featured in the conceptual video. Since they took the online survey at home, these subjects only answered question 4 in the interview after they had seen both videos.

**RATIONAL FOR THE INSTRUMENTS**

Our learning objectives in making the online survey were to have subjects demonstrate their ability to (a) compute a function inverse given a function in the form of \( f(x) = \), (b) recognize a function inverse given a graph of the pre-image, and (c) correctly answer multiple choice questions that contain choices reflecting common misconceptions. As mentioned earlier, the survey was composed of three parts. Figure 5 gives a brief overview of what each part of the survey included.

![Figure 5. Overview of the online survey.](image)
Online Survey: Part One

Part one of the survey was a pretest consisting of four questions shown in Figure 6 through 9, which attempt to measure the subject’s understanding about function inverses. The first question (see Figure 6) was developed in order to assess the subject’s calculational knowledge: namely the ability to compute the inverse function of a given the initial function. A function was given with four choices of possible inverse functions. Calculating the inverse function would require the knowledge of ‘undoing’ in the same sense that Even (1992) uses. This was the first question because the researcher hypothesized that most subjects would remember how to do this, namely switch the $x$ and $y$ variables and solve for $y$.

Given $f(x) = -x^3 + 2$, which of the following represents $f^{-1}(x)$?

| Equation 1: | $q(x) = (-x + 2)^\frac{1}{3}$ |
| Equation 2: | $r(x) = (x - 2)^\frac{1}{3}$ |
| Equation 3: | $s(x) = (x)^\frac{1}{3} - 2$ |
| Equation 4: | $t(x) = (-x - 2)^\frac{1}{3}$ |

Figure 6. Question one.

Question two was designed to assess each subject’s conceptual knowledge (see Figure 7). Thus, it aimed to shift perspectives and give a graph of a function with four possible inverse function graphs. The subjects were to choose which graph is the corresponding inverse graph. The correct solution, Graph c, is the given graph reflected across the $y = x$ line. Subject responses to this question gave the researcher a different perspective of how preservice teachers conceive of inverse functions graphically.

Question three was designed as an open-ended question that puts the subjects in a hypothetical teaching situation (see Figure 8). Here, a student asks about the intersection of a function and its inverse. The purpose of this question was to observe how subjects conceived of the function reflecting across the $y = x$ line from the perspective of the function and inverse intersection. The researcher hypothesized two general responses for this question. First, subjects could use a distance approach, arguing that the distance from the function to
the inverse goes to zero on the line $y = x$. Second, subjects could use a point-wise approach, arguing that points on the function such as $(a, b)$ correspond to the point $(b, a)$ on the inverse; thus, the points are the same for points such as $(a, a)$ or $(b, b)$ which lie on $y = x$. The researcher was also interested whether subjects thought that a function must have an intersection or not.

The fourth question was incorporated to obtain more data on how subjects interpreted function inverses (see Figure 9). Part (a) simply asks about the definition of function inverse. If students were able to apply the definition, they would choose “d.” Alternatively, if they held the well-documented misconception that that a function inverse is synonymous with the
Figure 9. Question four, parts a, b, and c.

reciprocal (Even, 1990), they may select any of the distractors a-c. Part (b) of this question continues to probe the subjects’ conceptual understanding of the definition by providing a table of x-values and the outputs of the functions f(x) and g(x) and asking for g⁻¹(-1). Subjects must first recognize that f(x) and g(x) are not inverses of each other by looking at the table values. Since the problem is focuses on g⁻¹(-1), then subjects would need to identify the -1 in the g(x) column and the corresponding x-values, namely option b with x = 2, is the correct solution. Finally, part (c) asks another procedural question as in question 1; however, this question requires the understanding of logarithm properties.

Online Survey: Part Two

Part two of the online survey is a series of two videos that explain function inverses in differing ways. The first video is a taped classroom lecture of a math professor who is
demonstrating the algorithm to find the inverse of a function equation. First, he begins with the function \( f(x) = 2x + 1 \), graphs the line, and explains that a line is actually just made up of an infinite number of points. Then he demonstrates how to find the inverse function of \( f(x) = 2x + 1 \) by first solving for \( x \), then replacing the \( x \) and \( y \) values to graph it on the standard Cartesian plane. He concludes with \( f^{-1}(x) = \frac{1}{2}(x - 1) \).

The second video is a screen video capture with a narrator explaining a geometric interpretation of function inverses. The first part of the video explains the inverse of a function from a point-wise perspective. The narrator explains why the image of any point in a function is a reflection across the line \( y = x \) by examining the distance between the point and the line \( y = x \) and the distance between the inverse point and the line \( y = x \). Then the narrator justifies that the points are inverses of each other because their coordinates are switched, meaning the point \((3, 4)\) has an inverse point of \((4, 3)\). Then, the narrator explains why \( y = x \) is the reflection line by demonstrating the same process over the line \( y = 2x \). This shows the \( x \) and \( y \) coordinated of each point are not switched, thus implying they are not inverses. The second section of the video explains a geometric interpretation of inverse functions. Specifically, the narrator describes why one switches the \( x \) and \( y \) variables when creating inverse functions. The narrator demonstrates that the \( x \) coordinate in \( f(x) \) is the same as the \( y \) coordinate in \( f^{-1}(x) \) and that the \( y \) coordinate in \( f(x) \) is the same as the \( x \) coordinate in \( f^{-1}(x) \).

The final portion augments the geometric approach by creating a series of isosceles triangles. This portion, illustrated in Figure 10, uses a function and its inverse function and connects corresponding points on the functions to a point on the line \( y = x \) to create isosceles triangles. Recall that in isosceles triangles, the base angles are equal. The construction of this portion is represented in Figure 6. Recall that the word conceptual knowledge as the knowledge that is abundant in relationships. The researcher considered this video to be conceptual in that it provides multiple explanations of inverse functions through points, functions, and triangles and connect the underlying concepts in each explanation.

As previously noted, subjects in Phase V explored the dynamic graph applet after watching the second video. This dynamic graph allowed users to manipulate the coefficients of an exponential graph to explore the effects of the inverse function. There were no specific instructions for the dynamic graphs. Subjects were asked to move the sliders to explore the
idea of inverses as reflections, and then write a short review describing their reaction as a learner. The applet is pictured in Figure 11.

![Figure 11. Dynamic graph applet.](image)

**Online Survey: Part Three**

Part three of the online survey presented the original four questions and gave the subjects the opportunity to change their responses. Subjects had the option to keep the answer the same or change their answer. If they changed their answer, they were asked to explain why they changed their answer and if the change was based on video one, video two, or the dynamic graph applet. Allowing the subjects to repeat the same questions gave them the chance to reflect on their learning, and it gave the researcher feedback on what parts of
the videos and dynamic graph were effective in the different types of questions that were asked.

**DATA ANALYSIS**

The analysis process involved coding interview transcripts, survey responses, and pencil-and-paper work generated by the subjects. Using the order of the online survey, the researcher attempted to determine the contributions of various components: the procedural video, the conceptual video, and the dynamic graph by creating two tables to organize the data.
CHAPTER 4

RESULTS

The results are reported in three sections, (a) what specifically about the material was reported as useful, namely, the lecture video was helpful in procedural calculations and the geometrical video and applet were helpful in conceptual graphical function inverse situations, (b) how the subjects altered their responses and mathematical practices, mostly by using appropriate tools and constructing viable arguments and (c) a common misconception of thinking of the function inverse as the reciprocal of the function.

WHAT ABOUT THE INVERSE MATERIAL WAS REPORTED AS USEFUL?

The first table illustrated correct and incorrect responses for each question before and after watching the videos and using the dynamic graph. Table 2 represents the table with correct responses marked with a C, incorrect responses marked with X, and unanswered questions marked with N/A. For question three, a correct response was recorded for anything reasonable relating to function inverses. Incorrect answers were responses such as “I don’t know” or that answered with another question for the hypothetical student.

Table 2. Correct or Incorrect Responses Before and After

<table>
<thead>
<tr>
<th></th>
<th>Q1 Before</th>
<th>Q1 After</th>
<th>Q2 Before</th>
<th>Q2 After</th>
<th>Q3 Before</th>
<th>Q3 After</th>
<th>Q4a Before</th>
<th>Q4a After</th>
<th>Q4b Before</th>
<th>Q4b After</th>
<th>Q4c Before</th>
<th>Q4c After</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andy</td>
<td>C</td>
<td>C</td>
<td>X</td>
<td>C-2</td>
<td>X</td>
<td>C-2</td>
<td>N/A</td>
<td>N/A</td>
<td>X</td>
<td>C</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Alex</td>
<td>C</td>
<td>C</td>
<td>X</td>
<td>X-2</td>
<td>X</td>
<td>C-2</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C-2</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Frankie</td>
<td>X</td>
<td>C-1</td>
<td>X</td>
<td>C-2</td>
<td>X</td>
<td>C-2</td>
<td>N/A</td>
<td>X</td>
<td>N/A</td>
<td>C-2</td>
<td>N/A</td>
<td>C</td>
</tr>
<tr>
<td>Kira</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C-2</td>
<td>N/A</td>
<td>C</td>
<td>N/A</td>
<td>C</td>
<td>N/A</td>
<td>C</td>
</tr>
<tr>
<td>Total C</td>
<td>¼</td>
<td>4/4</td>
<td>¼</td>
<td>4/4</td>
<td>1/4*</td>
<td>4/4</td>
<td>1/1</td>
<td>2/3</td>
<td>½</td>
<td>4/4</td>
<td>0/2</td>
<td>2/4</td>
</tr>
</tbody>
</table>

The second table created in the analysis process organized what mathematical practices subjects demonstrated after watching the videos and using the dynamic graph. In particular, the researcher was interested in mathematical practices 3, 4 and 5 from the
Common Core State Standard for Mathematics as described in Chapter 2. Table 3, illustrates each subject and the practices exhibited in each question.

Table 3. Demonstration of 3rd, 4th, and 5th Mathematical Practices After Videos and Dynamic Graph

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4a</th>
<th>Q4b</th>
<th>Q4c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andy</td>
<td>3</td>
<td>3, 5</td>
<td>4</td>
<td>N/A</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>Alex</td>
<td>3</td>
<td>3, 5</td>
<td>3</td>
<td>3</td>
<td>3, 4</td>
<td>None</td>
</tr>
<tr>
<td>Frankie</td>
<td>3</td>
<td>3, 4</td>
<td>3, 5</td>
<td>None</td>
<td>4, 5</td>
<td>3</td>
</tr>
<tr>
<td>Kira</td>
<td>3</td>
<td>3, 4, 5</td>
<td>3, 4, 5</td>
<td>3</td>
<td>3, 4</td>
<td>3</td>
</tr>
</tbody>
</table>

In all, the researcher determined what aspects of each video and dynamic graph were reported as useful through Table 2 and student interview transcripts. Then, the researcher clarified how the mathematical practices changed after the inverse material was presented by using Table 3.

Recall that the inverse material was designed to help preservice teachers prepare for the algebra CSET. Kira’s responses, will present in which ways she reported the material, namely the videos and applet, were beneficial. Kira’s pretest responses were all correct. However, in the posttest she revised her response to the third open-ended task. In the pretest, she uses a point-wise solving approach by explaining that a point such as (3, 4) on a function will have a corresponding point of (4, 3) on the function inverse, concluding that a point such as (3, 3) will lie on both the function and its inverse, which also happens to be on the line $y = x$. After watching the videos and using the applet, she explains her alteration in the interview.

Int: You had said first that this is because you are swapping the x and y variables, so if you have a point (3, 4) on the function then (4, 3) would be the corresponding point on the inverse. And so then if you have points such as (3, 3) that would be on that line. Can you explain how video 2 helped you think about this?

Kira: The thing that I changed is that I forgot to mention that they don’t always intersect. And so um if all the examples [in the hypothetical class] happened to intersect, then um I didn’t want the student to think that always happens. And then I remember because of video two that um there was examples of… (draws Figure 12). There was the examples of the log functions and they just um, something like that, they didn’t intersect and they were still reflections over the $y = x$ line and inverses of each other.
This was the only instance in which Kira altered her initial response. It is assumed that she had previously studied function inverses in depth in her undergraduate degree program, but video two helped her realize that functions and their inverses do not necessarily need to intersect.

At the end of Kira’s interview she reflected on the function inverse study material and on a future CSET preparation website.

Int: Did the videos or applet help you think about any of these problems?
Kira: I guess I felt pretty confident in the material to begin with, but I did like that applets proof about, um, well the geometric proof. I thought it was cool to see it visually. So, um, I liked that and in the first video, that was all material that I feel confident in anyways.

Int: Do you think that a website like this having videos and applets would be helpful in preparing for the test [equivalent to the CSET but in Colorado]? Kira: Yeah I think so. I think that would be a great tool, um, a great resource, uh, in multiple categories. […] Definitely a good tool to refresh like number theory and that kind of stuff that you would have on the exam. And calculus.

Kira reports that the design of the study material would be helpful in her situation with other topics. This may also be the case with other math majors or possible math majors who have been out of the field for numerous years and need to refresh their skills.

Some may argue that the videos and applet were of little use to Kira because she was a math major and had studied math in depth during her education. Nevertheless, Kira reported the material as useful but would rather review other mathematical topics, such as advanced calculus. In addition, she reported that the inverse materials still helped her reconsider the third question. The non-math majors, on the other hand, usually take the
minimum math classes required for their degree. Science majors such as biology, chemistry, and engineering may have taken more math in college than a humanities major. However, results from Alex and Frankie, described in the next sections, illustrate benefits gained from the function inverse material.

When asked Frankie about what he found useful in the online survey, he remarked that the overall lesson design, which includes pre-assessment, watching 2 videos, an applet, and post-assessment, helped him reflect on his own learning. This comment indicates that, at least for some students, this process aids students in reinforcing function inverse concepts by reconsidering the same problems and noticing individual inconsistencies. Within the design, each video had its own constructive characteristics. The lecture video aided students in calculating the inverse of a function equation. The design of the geometrical video, audio narration over a GSP screen video capture, allows subjects to focus on the mathematical concepts being presented. Answers between the pre and posttest and subject responses provide evidence for this claim.

After the fourth phase of research, our design model of the CSET preparation website was to include a lecture video on specific content with a conceptual video explaining the topic as well as an interactive applet. However, after one the subjects explained that the design of the online survey would be a good way to set up the online CSET website, the researcher is now considering another design iteration. As the subject explained it, the answers for the questions should be given to the students after attempting the question for a third time.

Int: Is there anything else that you want to say?

Frankie: When you ask these questions on the spot, you may or may not have the answer. But then you are going to give your best guess. And then like the second opportunity gives you a better chance for you to elaborate on your answer if it was right or change your answer if it was wrong on the first time. But then the third time, it should say, ‘ok, you guessed this, then you corrected it to this, and this is the right answer.’

Frankie points out that allowing another opportunity to revise one’s original response is critical before giving the correct answer. Too many times in education, students do not persevere through a problem; instead, they give up by immediately looking at the answer. This design forces users to reflect on their learning.
Whereas the other subjects came to the interviews with the procedural knowledge about how to find the inverse of function, Frankie’s initial response to question 1 indicated that he had thought about the inverse of a function as a multiplicative inverse of an integer. He solved task one by taking the reciprocal of the function thus answering \( f^{-1}(x) = \frac{1}{x^3 + 2} \). After watching the lecture video, he was reminded of the method used to find the inverse equation of a function. Figure 13 shows how he proceeded in the posttest. The method he used was exactly the same as method the instructor in video one uses in an example. His descriptions of his steps are as follows.

\[
\begin{align*}
  y &= -x^2 + 2 \\
  y-2 &= -x^2 \\
  2-y &= x^2 \\
  x &= \sqrt{2-y} \\
  f^{-1}(x) &= \sqrt{-x+2}
\end{align*}
\]

*Figure 13. Frankie’s pretest work for task one.*

Int: Then you said that after watching the videos, you said that you changed your answer based on video 1 and you got this equation (referring to equation 1). So how did you do that? What did the video help you with?

Frankie: What I did is just um being um the same equation, then I isolate \( x \), so solve for \( x \). He (referring to teacher in video one) did exactly the same and I… and I was just like the negative of that one then, um, and I was like ‘oh yeah’ I really just have to isolate the other one and then just rename them.

In explaining the steps, Frankie refers back to the original video lecture and used similar vocabulary, such as isolating and renaming that were used in the video. In this case, the video appears to have been useful in helping Frankie calculate the inverse properly.

In the pretest, Alex also answered task 2 incorrectly. Recall that task 2 presents a graph of a function and the student is expected to select the correct inverse from a set of four
graphs. Alex chose graph d, shown in Figure 14, as the inverse, and he explains his method of solving as the following.

![Graph d](image)

**Figure 14. Alex’s pretest choice for task two.**

Alex: I rotated it 90°. I tried to make the $x$ be $y$, so it should have gone clockwise, no counterclockwise. I rotated it clockwise.

In the posttest, he corrected his original response based on the conceptual explanation presented in video 2.

Alex: Yeah so now I think it would be easy to see that it would be this one using this (points at the line $y = x$) as a mirror instead of trying to substitute $x$ and $y$ and then rotate the $x$ to the $y$. Because last time I put the plus where the minus was and I rotated one direction.

In reality, Alex still answered the question incorrectly choosing graph b, shown in Figure 15, because he did not attend to precision. However, he does use the graphical explanation supplied in video two about reflecting over the line $y = x$ to support his solution method. This case illustrates how video 2 plays a role in finding the function inverse graphically.

![Graph b](image)

**Figure 15. Alex’s posttest choice for task two.**
While some people may refute the claim that the voice-over screen video capture would be just as effective as an instructor demonstrating the same concepts on a chalkboard, I found Andy’s rational to be very compelling. When he was asked about his thoughts on the second video he stated:

Andy: It [video two] was just very helpful seeing the visuals that they had it where all you saw was the screen and you could hear them talking in the background but they were showing the different things as opposed to watching somebody write it. Or a lot of time when they are writing it, you are maybe watching them and concentrating on what they are saying as opposed to what they are doing. This one was, I mean you could concentrate on what they are saying but you have to be able to watch what they are doing. Hearing them say it and show it at the same time made it that much more effective.

In Andy’s opinion, watching the instructor can distract from the material itself. This is consistent with the proposition that the design allows for more focus on the material.

**HOW THE SUBJECTS ALTERED THEIR MATHEMATICAL PRACTICES**

The original research question focused on how research based pedagogy can be integrated with technology. However, throughout the Phase IV interviews, I identified ways in which Andy and Alex changed their mathematical behaviors, specifically by constructing viable arguments with the use of appropriate tools. In this case, tools can be used broadly to include the interactive applet along with the geometric and algebraic representations. In the following sections, I describe two examples of this change that emerged in the data.

For example, the first time that Andy was given task three, he simply stated that he could not remember if it was necessarily true that a function and its inverse necessarily intersect on the line \( y = x \). After watching the video, he immediately was able to make sense of the problem and give a thorough answer explaining Figure 16, which he constructed.

Andy: I know that there are examples where they [the function and its inverse function] won’t intersect, but \( y = x \) would be like the asymptote. I guess you could call it, um, you know, or a horizontal asymptote, I forget the terminology, oblique asymptote! You know the functions are going to be equidistant from each other, you know equidistant from \( y = x \). So maybe a point is here for \( y = x \), and a point is here. They don’t intersect at \( y = x \) but this segment and this segment are congruent to each other like what they showed in the second video where, um, I believe it was \( y = \) or \( f(x) = x \) um to the b power or something like that.
There is an important distinction between Andy’s pretest mathematical practices and posttest practices. In the pretest, he did not use any tools to help him solve the problem. A tool in this case refers to pencil and paper, concrete models, a calculator, a graph, etc. Conversely, in the posttest, he immediately used a graph to explain his explanation. In addition, his response used a practical argument of congruent distances from the line $y = x$ to analyze why a function and its inverse may not intersect.

Furthermore, Alex changes his mathematical practices in part b of task four. Originally, he does not use any tools to help solve the problem. He explains his method by explaining that if $x$ is -1, then $g(x)$ is 3 and if $g(x)$ is -1, then $x = 2$. Although he does answer correctly, his justification lacks conceptual understanding. In the posttest, he immediately solves the problem with a graphical approach. First, he plots all of the points, and then draws a line that connects the points (see Figure 17). After drawing the graph, he explains finding the answer below.
Alex: So that’s $g(x)$ so we can reflect across here (referring to $y=x$). [...] I did use that graph and it seemed like about, um, 2 or 3 and negative 1 if I were to make a reflection here. So the closest number was this one graphically (referring to 2).

Alex was able to construct a graph and use it as a tool to refer to in his explanation. This differs from the pretest in that he did not use any tool, except for language, and his original justification indicated only procedural skills.

**FUNCTION RECIPROCAL AS THE FUNCTION INVERSE MISCONCEPTION**

One common misconception that arose in three of the interviews was the idea that the function inverse is the reciprocal of the function. Interviews with the two participants in Phase IV… In part (a) of task 4, Alex answered correctly that if $f(d) = c$, then $f^{-1}$ inverts $f$ so that $f^{-1}(f(d)) = d$. Then, he states that the eighth grade students in the class that he tutors would say that $f^{-1}$ inverts $f$ so that $f^{-1}(d) = 1/f(d)$. He is right in that actually more people than just the eighth grade students would choose that as a description of $f^{-1}$. This misconception could be because of the notation $f^{-1}$, namely the function to the negative 1 power or because of the multiplicative inverse of an integer is simply its reciprocal. Students with these misunderstandings seem to not have a complete understanding of what a function inverse is. This demonstrates the why a conceptual understanding is essential.

Without delay, Andy solved task one in the pretest by switching the $x$ and $y$ variables, and then solving for $y$. One would then assume that he understood procedurally how to find a function inverse. However, in the pretest of part c in task four, he was unable to find the correct function inverse.

Andy: I just figured on the inverse of, you know, like I was just thinking of integers and the inverse of 3 would be 1/3 so I just figured that the inverse of 100$^t$ would be 1/100$^t$.

Still, after watching the videos, he was incapable of completely realizing his error.

Andy: I know there’s gotta be some sort of logarithm in there, especially if it’s in the exponent. Um although it could be a simple variable, for this one I’m going to stick with (a) and then it’s just the reciprocal when you are using the inverse.

Even though he knew that in task one the correct procedures in finding the inverse of an equation, he resorted to choosing the answer that was the reciprocal of the function as his
answer. He expressed doubt in his answer by stating, “There’s gotta be some sort of logarithm,” but did not bother to persevere to find an alternative solution.

Researchers may disagree with the claim that Andy thinking is encumbered by this misconception arguing that Andy answered incorrectly because he simply forgot his exponential and logarithm properties. However, Andy’s forgetting the properties is not the main issue. The problem is that he resorts to thinking of the function inverse as a reciprocal. Instead of using the same solution methods as in task one, Andy’s explanation for task four implies that he has not fully grasped the idea of that a function inverse undoes an operation.
CHAPTER 5

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

This chapter will include (a) a summary of the background, purpose, methodology, and results of the study, (b) the significance of the study, and (c) recommendations for future research.

SUMMARY

The results of several math education studies (Grouws & Cebulla, 2000; Hiebert, 2003; National Center of Universal Design for Learning, 2011; Thompson et al., 1994) demonstrate what they consider to be effective math pedagogy, giving students the opportunity to learn through a “UDL” approach that emphasized both procedural and conceptual knowledge. With the advances in technology, math educators such as Goldenberg and Cuoco (1998); Reimer & Moyer (2005), and Bowers and Stephens (2011) advocate the use of technology in the classroom through dynamic geometry, virtual manipulatives, or other technology that facilitates students’ explorations of mathematical relations. As for inverse functions in particular, Even (1992) concluded from her study that many preservice math teachers lacked a complete understanding of inverse functions. Bayazit and Gray (2004) suggest that when in-class teachers lack conceptual understandings of inverse functions, their students suffer as well.

The purpose of this study was to integrate this previous research in math pedagogy, technology, and inverse functions to explore the application of research based theories of best practices with the modern affordances of online videos and applets within the CSET population of preservice teachers.

Data collection and analysis were based in design research that consisted of iterative cycles of (1) designing a lesson, (2) researching its implementation, and (3) revising it for the next round. During Phase II, 13 students created initial CSET video lessons. During Phase III, 14 subjects completed an initial online survey. During Phases IV and V, four subjects
with diverse educational backgrounds (all of whom were particularly interested in passing the CSET) participated in clinical interviews (Ginsburg, 1997) and responded to four questions relating to inverse functions before and after watching a procedural video, conceptual video, and dynamic applet consisting of information about inverse functions.

Of the phases of research, only the results of phase IV and V were elaborated in this thesis. To sum up the data, students improved their responses after watching the videos and using the applet by either correcting their responses and/or exhibiting one or more of the Standards for Mathematical Practice. Two of the four subjects reported that the procedural video reminded them of the algorithm for determining a function inverse. In contrast, all of the subjects reported that the visual nature of the conceptual video helped them to understand the underlying imagery of function inverses. Some of the respondents also reported that the imagery on the screen, focusing on graphs rather than a professor, focused their attention on the mathematics instead of the lecturer. Interview transcripts revealed that subjects used language from the conceptual video such as “mirror” or “reflect” to explain and justify their responses. Of the two subjects that used the inverse function applet, one referred to dynamic applets as “great tools” that can “help students refresh their skills in all areas of math.” Data from the interviews and survey responses also revealed a common misconception in which subjects viewed the function inverse as the multiplicative inverse of an integer.

**CONCLUSION**

There were three major conclusions drawn from the study: (a) several of the features of the conceptual video were reported as beneficial by the subjects, (b) the dynamic applet provided an opportunity for students to explore the “what if” propensity, and (c) designers of online videos should anticipate documented misconceptions.

Conclusion (a): Several features of the conceptual video were reported as beneficial. One subject found that the voice-over screen capture video allows viewers to focus their attention on the mathematics instead of a lecturing teacher. He explained that lecturing teachers can distract viewers whereas, with a narrator, this is not an issue. The full-screen visuals in the conceptual video, precisely a screen capture over the Geometer’s Sketch Pad applet, may have led to deeper connections among multiple representations of function inverses. These visuals were reported as better than paper-and-pencil graphs because of the
plasticity, accuracy, and color capabilities. Researchers Mishra and Koehler (2006) described that the stop and rewind capability of videos benefits learners. However, the subjects that watched the conceptual video in the interview did not make use of that function which could have been due to the interviewer’s presence.

Conclusion (b): The two students that used the applet reported positive feedback. In particular, Frankie mentioned that advantages of applets over pencil and paper methods are the ability to change parameters easily and to take note of the effects of single or multiple parameters. This is consistent with the conclusion that Bowers et al. (2002) described wherein student exploration of virtual manipulatives allow students to pose “what if” questions that spark an interest in students to develop hypotheses and test them by changing parameters. Although Alex (who was interviewed during phase IV) was not given the opportunity to use the applet, when asked if he would like to use an inverse applet, he replied that he would use the applet to look at the extremes of when $x$ approaches positive and negative infinity. This demonstrates the “what if” curiosity that students bring when working with applets.

Conclusion (c): Of the two articles relating to function inverses summarized in this thesis, (Bayazit & Gray, 2004; Even, 1992) neither referred to perhaps the most unanticipated result from the data, which should have been expected, “the function inverse as a reciprocal” misconception. Thus, effective online videos should anticipate documented misconceptions to help students avoid making common mistakes.

**RECOMMENDATIONS**

Based on the three conclusions, I provide three recommendations for further studies: (a) what features make a good conceptual video, (b) what features make an effective applet, and (c) how developers can integrate student misconception in mathematical videos and applets.

The first recommendation is to explore the specific features that make a good conceptual video. The results of this study indicate that when lesson is augmented with multiple visual representations, such as the *Geometer’s Sketchpad* dynamic graphs, students focus their attention on the material instead of a lecturer. Further research should explore the
value of stop and think buttons, captions, and other video enhancements to online videos to acquisition of conceptual understanding.

The second recommendation is to look at other aspects of dynamic applets that allow students to explore concepts more deeply. This may include several applets depicting the same mathematical concept but portrayed in various ways. Subjects would report which applet and what features of the applet they found as most enlightening.

The final recommendation for further research relates to the notion of integrating mathematics education results into development of online video instruction. If developers can use current knowledge relating to students misconceptions to anticipate potential areas of student confusion, they can produce videos that target specific challenges that students might encounter. This study demonstrated that students might interpret the function inverse as the reciprocal of the function. Online videos and applets can be designed in such a way that potential misconceptions are anticipated and explicitly contrasted. For example, had the applet included a graph of \( h(x) = \frac{1}{f(x)} \) students may have clearly seen the difference between \( h(x) \) and the correct function inverse \( f^{-1}(x) \). Further research needs to be conducted to determine how developers could create questions and activities that would allow students to discover the inconsistencies between their view of inverses as reciprocals and the geometric view of functions as reflections.
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