Backward Transfer: How Mathematical Understanding Changes as One Builds Upon It

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by

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DEDICATION

This dissertation is dedicated to the incredibly supportive faculty and staff associated with the Mathematics and Science Education Doctoral Program (MSED) run jointly by UCSD and SDSU and the Center for Research in Mathematics and Science Education (CRMSE).

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The use of prior knowledge has been a much-researched topic. For example, it has been well established that prior knowledge can act as a foundation on which new knowledge is constructed (Smith, diSessa, & Rochelle, 1993). Other studies have looked at the forgetting of prior knowledge (Ebbinghaus, 1885) and at the inhibition of retrieval of prior knowledge (e.g., Estes, 1987). What has remained understudied in mathematics education research is how prior knowledge changes as new knowledge is
built upon it. However, a limited amount of transdisciplinary research (particularly in linguistics) has examined changing prior knowledge as a case of *backward transfer* (i.e., as the generalization of recently-constructed knowledge “backwards” onto longer-held knowledge). This study extends that work by examining backward transfer in a mathematics classroom environment. Specifically, this study examines how previously-held linear functions knowledge changes as middle school students construct beginning conceptions of quadratics functions. Using a design-based instructional intervention on quadratic functions, this study aimed to create backward transfer that productively influenced students’ linear functions knowledge.

Using a mixed methods approach (i.e., a priori and inductive codes; Miles & Huberman, 1994), qualitative analysis of pre- and post-interviews and the intervention revealed three major findings. First, productive backward transfer was produced. In particular, evidence was found that students’ understanding of linearity was deepened as a result of the quadratics instructional intervention (e.g., students reasoned proportionally with the changes in the independent and dependent variable). Second, evidence was found supporting the claim that backward transfer is a particular kind of transfer (i.e., three relationships between backward and forward transfer were found). Third, the process of noticing was shown to provide explanatory power for the occurrence of backward transfer.

The potential significance of this study is that it (a) addresses the relationship between prior knowledge and new knowledge from a new and potentially informative angle; (b) contributes to the emergence and theoretical conceptualization of alternative transfer perspectives; and (c) provides general principles by which to address
backward transfer and informs the teaching of linearity and quadratics within Algebra 1 courses.
CHAPTER 1:
RATIONALE

According to the tenets of constructivism, new understanding is created from the building blocks of prior knowledge (Smith, et al., 1993). This raises a question: what happens to a learner’s prior knowledge as a result of learning about a new concept for which the prior knowledge serves as a foundation? An analysis of pilot data collected as part of a larger design-based research study on the learning of quadratic functions revealed that some participating 8th grade subjects’ prior knowledge of linear functions was unproductively influenced by the instruction on quadratic functions\(^1\). This finding was surprising because the students displayed a fairly strong understanding of linear functions prior to the instruction and because the instructional designers for the study from which the pilot data came considered linear functions to be foundational for the development of quadratic functions.

Consequently, I became interested in the phenomenon of how mathematical understanding changes as one builds upon it. To give the reader a sense for the kinds of changes that were observed in these students’ prior knowledge and why these changes were attributed to the instruction, I begin this chapter by presenting vignettes from the pilot data. First, two vignettes from pre- and post-interviews with a particular student will be presented and compared to show how this student’s prior knowledge changed. Then, his changed prior knowledge will be compared to his reasoning about the new concept in a third vignette, from the post-interview, which
shows that there were common patterns in reasoning. It was these common patterns that led me to conclude that the instruction about the new concept influenced the changes in the student’s prior knowledge.

Vignettes Introducing the Phenomenon

The vignettes presented here derive from two interviews conducted in the spring of 2007 with Matt, an 8th grade student. The interviews occurred just before and immediately after a two-week instructional unit on quadratic functions conducted with Matt and seven other students.

Linear Task from Pre-Interview

Matt was presented with tabular data that represented a linear relationship between time and altitude for a remote-control plane (see Figure 1.1). The task was complicated by the fact that the time intervals in the table were not uniform. In the following vignette, the interviewer (I) asked Matt (M) to consider whether or not the plane was climbing at a steady pace. Except for a brief moment at the beginning, Matt reasoned in such a way as to preserve the linear relationship between the two variables time and altitude and he correctly concluded that the plane was climbing steadily:

1 I: And as this airplane takes off, is it climbing steadily, or do you think it might be climbing really fast and then tapering off in its climb, or is it climbing slowly and then faster, faster, faster?
2 M: At first, I think it’s climbing steadily at first, up until 10. And then it looks like it starts going steeper.

1 All research subject names are pseudonyms, although subject gender has been preserved.
I: What were you looking at that made you think it was going steeper after the 10th second?

M: I thought that time, since in 2 seconds it climbed to 7 feet, and 2 goes into 8 seconds for the next one 4 times, the altitude times 4 is 28. And then another 2 seconds it will climb another 7 feet, so that would have been 35. And then in 10 seconds, oh, then it does climb steadily.

I: What made you just change your mind, what did you see?

M: 10, 20, 2 goes into 20, 10 times. And then 7 times 10 is 70, so.

I: Can you tell me one more time, which things you were multiplying?

M: I was getting the seconds [points at 2 seconds in table].

I: The 2 times 10 is 20 [points at 2 s in table] is that what you were doing?

M: No, I was doing the seconds. In 2 seconds, it climbs 7 feet [points at 2 s and 7 ft in table], so 2 goes into 20 [points at 20 s in table], 10 times. And then I multiply the 7... [points at 7 ft and 70 ft in table].

I: Times 10 and you got the 70, very good. Your thinking is very clear to me, you’re doing a great job articulating it. How about here, do you think that is also climbing steadily, between this point and this point? [points between 20 s and 24 s in table].

M: Yeah.

I: How did you decide?

M: I got 20, and there’s a difference of 4 seconds, so 4 seconds. And then say it’s 2 goes into 4, 2 times, 7 times 2 is 14, and then I added 14 to 70.

I: So you think it’s climbing steadily?

M: Um-hm.
I: What is the elevation after 1 second?
M: After 1 feet, it should be . . .
I: 1 second.
M: 1 second, it should be 3½.
I: How do you know that, what did you do?
M: Because 1 is half of 2, so then I’d have to get half of 7, which is 3½.
I: Can you use that to figure out what the elevation is after 4 seconds?
M: Yeah. 3.5 times 4 equals 14.

The following evidence suggests that Matt recognized the linear relationship between the variables: (a) he said that the plane was climbing at a steady rate (lines 4 and 16); (b) he reasoned proportionally with the independent and dependent variables (lines 6, 10, 14, and 22), and (c) he reported the constant rate of climbing as 3½ ft per s (lines 20, 22, and 24).

**Linear Task Revisited in Post-Interview**

In the following vignette from the interview that occurred after the instructional sessions, the interviewer presented Matt with the identical table of time-altitude data for the remote-control plane. Matt was again asked to consider whether or not the plane was climbing at a constant rate. However, in this interview Matt did not reason in such a way as to preserve the linear relationship between the two variables, despite doing so prior to instruction:

I: We’ll start with the first plane. So do you think that he is climbing at a constant rate? Or do you think he’s climbing really fast and then he slows down climbing? Or is he climbing faster, faster, faster?
M: [mumbles to self as he calculates and writes down first differences to the right of the altitude column in Figure 1.2] Let’s see, over here he went 20, over here he went 27, wait. From 20 he went to 28, from 7, he went up to, and from 20 to 35 to, he went 7. From here, that’s less than 40, 35, so then 40, so then he went . . .
I: So what does that tell you?
M: 28, hold on, 29, 30, 31, 32, 33, 34, 35, 7 looks weird.
I: 7 looks weird? Is that what you said?
M: Yeah. The second 7 [points to the first difference of 7 from an altitude of 28 to 37 ft].
I: How come? Why does that look weird?
M: Just because it looks out like the plane went up and then kind of slowed down its climb, and then went up even steeper than this climb [points to 21 in first difference column in Figure 1.2], and then again slow.
I: Where do you think it climbed the fastest?
M: Probably from 35 to 70.
I: How come?
M: Because that’s the biggest interval of feet.
I: Ah! Ok. So, show me with your hand how you think he’s climbing.
M: I think he’s going, like he’s going like this [hand moves up slowly] and then increasing [hand moves up faster]. And then slowing down again [hand moves up slowly]. And then going steeper and then going down again [hand moves up faster].
I: You said he climbed 14 feet in 4 seconds. Is that what you said? Between 20 and 24? [M nods] How many feet do you think he climbed in one second?
M: How much do I think he’s climbed in one second? In one second, he’s probably climbed like. In one second, I don’t know. It would be . . . The 14 divided by 4.
I: Ok, so you think it’s climbed how many feet in one second?
M: 3.5.

Figure 1.2. Matt finds first and second differences between altitudes during second interview.
43 I: What are you concluding?
44 M: That 3.5 is probably the acceleration of it.
45 I: The acceleration?
46 M: The, yeah, the rate of climb.
47 I: The rate of climb?
48 M: Or, yeah, I think. The rate at which he is climbing, I think, yeah.

49 M: Well, let’s see, from 21 minus 7, that’s- I don’t know [mumbles to self as he calculates and writes down second differences to the right of the first differences in Figure 1.2]. Would equal 14. I think. It crosses now? So it’d be 14, yeah. And then 21 over there, and then 14, and then 14, and then 35 minus 7 would equal 28 and then here it’d be 35 minus 14, that equals 21. So it looks like it’s kind of going like this [gestures off camera] and then like it goes like this and then gets steeper.
50 I: OK. Now, I saw you working with a lot of numbers over here [points at altitude column]. Do you work with these numbers at all, the seconds? [points at time column in the table]
51 M: No, not really.
52 I: Not really?
53 M: Well, no, not really.
54 I: These are the ones that matter?
55 M: Yeah.

Surprisingly, Matt’s understanding of linear functions seemed to worsen as a result of his participation in the quadratic functions unit. There were three ways in which his thinking changed by the second interview. First, Matt went from engaging in covariate reasoning to engaging in univariate reasoning. In the pre-interview, he attended to how the dependent and independent variables were changing simultaneously (lines 4, 6, 10, 14, and 22). In the post-interview, he attended primarily to the changes in the dependent variable (lines 26, 28, and 34). In fact, when the interviewer asked him if he needed to consider the independent variable, he said no (lines 51 and 53).

Second, Matt went from reasoning proportionally to reasoning exclusively
with differences. In the pre-interview, Matt reasoned proportionally to conclude that if, in 2 s the plane was at an altitude of 7 ft, then in 20 s it would be at an altitude of 70 ft (lines 6 and 10). He was also able to reason proportionally to find that between 20 and 24 s the plane would climb 14 ft (line 14). In the post-interview, Matt reasoned with the differences in the dependent variable (lines 26, 28, and 34). Later in the post-interview (not captured in the vignette), Matt calculated the second differences and came to the same conclusion that the plane was climbing at variable rates. He also made bracket inscriptions on the table to highlight the first and second differences in altitude (see Figure 1.2). By reasoning with only the altitude quantity, Matt showed that he was not reasoning proportionally.

Finally, in the pre-interview Matt appeared to conceive of the altitude climbed in 1 s as a rate when he used it to calculate the altitude after 4 s (line 24). In the post-interview, he referred to the altitude climbed in 1 s as both “the acceleration” (line 44) and as “the rate of climb” (lines 46 and 48) and he did not use it to find other altitudes. Thus, there appeared to be some uncertainty in the second interview about what the altitude climbed in 1 s represented.

These changes were unanticipated and undesirable. However, closer inspection of the post-interview data revealed similarities between Matt’s reasoning with linear functions in the airplane task and the quadratic functions tasks. To highlight these similarities, a third vignette of Matt engaging with a quadratic function task is presented next.

**Quadratic Task in Post-Interview**

In the following vignette, Matt reasoned about a quadratic relationship between
distance and time for a swimming fish. He was given a table and asked to find the
next entry (see Figure 1.3). Matt’s reasoning about this quadratic function data was
for the most part productive. More importantly, it shared features with his thinking on
the linear task from the post-instruction interview, particularly the reliance on
reasoning with a single quantity and the use of first and second differences:

56 I: What do you think the next entry in the table will be?
57 M: It’s harder when it’s speeding up.
58 I: [laugh] Well, that’s an understatement! It is hard when it’s
speeding up!
59 M: Ok, then, well, let’s go 20 and that’s 40 and that’s 100 [writes 20,
40 and 100 as the first three 1st differences to the right of
distance column]. And then 320 minus 180 equals 150 [writes
140 as the fourth 1st difference]. So from here to here, so right
here it goes 60 [writes 60 as first 2nd difference between 20 and
40], no, not 60, 20 [changes 60 to 20]. It’d be 60 here [writes 60
as the second 2nd difference]. 40 here [writes 40 as the third 2nd
difference]. Right? 10 to 20, 40, 20 [points at first two 1st
differences], oh, no wonder, I thought something was wrong.
That’s 60 [changes second 1st difference from 40 to 60], that’s
right, it’s 60, that means that’s 40 [changes second 2nd difference
from 60 to 40]. From there to there would be 100 [points to third
1st difference]. And that said would be [uses calculator] let’s see
320 minus 180 would equal, yeah, 140. So then, it looks like it’s
going 20 seconds, or from here to here, it’s going 40 [changes

![Figure 1.3. Matt finds first differences for quadratic data.](image)
And from here to here, it’s going 40 points at second 2nd difference, and it’s going 40 [points at third 2nd difference], yeah, so it’s accelerating by 40. So that means, and I think it’d be 320 plus 180, isn’t that 500? Woops, hold it, 320 plus 180 would equal, wait, 320 plus 180 would be [uses calculator], there, 500 [writes in last entry of table].

60 I: Ok, tell me how you figured that out. I think I followed you to about here.

61 M: From the, well, I found out that the acceleration would be 40 feet per second per second. So that means the increase from here to here would be 40, and then 40, and then 40 [points at second differences]. Like 20 is added. And then you add 40 more to that and get 60.

While Matt made productive use of what he learned in the quadratics unit to reason that the missing entry in the data table was 500, his similar reasoning with the linear task had a less productive outcome. Specifically, there are three features of Matt’s reasoning that were similar across the post-interview quadratic and linear functions tasks. First, Matt reasoned primarily with the dependent variable for both linear and quadratic tasks in the post-interview (e.g., lines 26 for the linear task and line 59 for the quadratic task). On the quadratic task, reasoning with the distance, while ignoring the time, was productive because the times were equally spaced (i.e., 2 s intervals). Thus, to find the next entry in the table, Matt could ignore the time and look for a pattern in distances. However, on the linear function plane task, reasoning with the time alone was unproductive because the time values were unequally spaced.

Second, Matt reasoned additively instead of proportionally on both the quadratic and linear function post-interview tasks. On the quadratic function task, Matt found the first and second differences between successive distances of the given data table (see Figure 1.3) and reasoned with those differences. He engaged in similar reasoning on the linear task. Reasoning additively instead of proportionally on the
quadratic function task was again productive because the times were presented equally spaced and was unproductive on the linear function task because the times were unequally spaced.

All the quadratic data that the students explored in the instructional intervention were presented with equally spaced independent variable values (e.g., see Figure 1.3). In fact, for 8th graders, providing data in which the independent variable of a quadratic function increases in constant intervals is virtually a necessity because, with unequally-spaced independent variable intervals, quadratic function data are difficult to interpret. However, by always providing data with equally spaced independent variable values, an unintended consequence may have been that it influenced students to reason exclusively with the dependent variable in other contexts like, for example, the linear function plane task.

Third, in the post-interview Matt appeared to think that acceleration was involved in both the linear task (e.g., line 44) and the quadratic task (e.g., lines 59, and 61). Specifically, he referred to the second difference of 40 ft/s/s as the acceleration of the swimming fish in the quadratic task, and he also referred to the climbing rate of the plane in the linear task (i.e., 3.5 ft/s) as the acceleration. Acceleration is a useful context in which to introduce rate of change of a rate of change, and acceleration is an important aspect of quadratic distance-time functions because the acceleration is always constant. Because students have experience with acceleration, and because

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2 Here and throughout the dissertation, the distances in distance-time functions are taken as the total distances traveled from some starting 0 point. When contrasting distances and changes in distances, the term accumulated distances will also be used to refer to the total distance travelled from 0.
acceleration can be easily demonstrated with computer simulations of moving objects, a beginning conception of acceleration should be accessible to 8th graders (Lobato, Hohensee, Rhodehamel, & Diamond, in press). Therefore, acceleration was a concept that was given prominence in the instructional sessions. However, Matt appeared to conflate the rate of change in a linear distance-time context with the acceleration in a quadratic distance-time context.

In summary, analysis of pilot data revealed that there were several surprising changes in Matt’s prior knowledge about linear functions. More importantly, these changes appeared to be related to what he learned about quadratic functions. Furthermore, Matt was not unique among the study participants. This suggests that the instruction of the new quadratic function concepts, for which students’ prior linear function knowledge was to serve as a foundation, influenced their thinking about linear functions. It is these kinds of changes to one’s prior knowledge that are examined in this dissertation study. Next, I define the phenomenon of interest more specifically.

**A Case for Investigating the Phenomenon**

The phenomenon investigated in this study is the influence of learning experiences on prior knowledge, in those cases when the prior knowledge is conceived of as a fundamental building block for the targeted content in the learning experience. In the remainder of this chapter, I build a case for studying this phenomenon. In the first section, I make the topic of investigation explicit (a) by naming it backward transfer, (b) by providing a definition for backward transfer, and (c) by further
characterizing the phenomenon. In the second section, I present an argument for why the phenomenon that was studied should be situated within the more general concept of transfer and I situate the dissertation study within other findings about backward transfer from prior research. In the third section, I offer (a) an additional vignette from the pilot data which shows a more productive example of the phenomenon, (b) several mechanisms as potential ways to account for both productive and unproductive backward transfer effects, and (c) a conjecture about instructional supports that might promote productive backward transfer. In the fourth section, I provide the research questions and the significance that answering these questions have for the field of mathematics education.

Defining the Phenomenon

While very little mathematics education research has been focused on how new knowledge influences prior knowledge, other fields of research do have a history of investigating this kind of phenomenon. In those studies, the phenomenon is often called *backward transfer*. This term has been adopted for the dissertation study. Using *backward* as part of the label is appropriate because the directionality of the influence being investigated is from more recent knowledge back onto prior knowledge.³ Later in this chapter, I situate the phenomenon within the larger construct of transfer and thus explain the other half of the *backward transfer* label.

In Chapter 2, I discuss how the definitions of *backward transfer* found in

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³ The term *forward transfer* will be used to distinguish other kinds of transfer from backward transfer and the term *transfer* will be used to refer to both forward and backward transfer.
transdisciplinary studies are inadequate for the kinds of observations that were made in the pilot data. Therefore, the definition used in this study is an adaptation of a definition that comes from a particular approach to transfer called the *actor-oriented approach* (AOT; Lobato, 2008a). Specifically, the definition used for the dissertation is the following:

Backward transfer is the influence on prior knowledge by the acquisition and subsequent generalization of new knowledge.

This definition is the backward correlate of the definition of transfer that is used by the AOT approach, which is that transfer is “the influence of a learner’s prior activities on his or her activity in a novel situation” (Lobato, 2008a, p. 169). More will be said later about this approach.

Backward transfer involves an influence *backward*. Transfer has also been characterized using the directional adjectives *vertical* and *lateral* (Gagné, 1970; Singley & Anderson, 1989; Barnett & Ceci, 2002). Vertical transfer is defined as transfer from lower-level to higher-level skills, where the lower-level skills are prerequisite. Lateral transfer is defined as the generalizations that are made over contexts that are at a similar level of complexity (Gagné, 1970; Singley & Anderson, 1989). Lateral transfer has received more research attention and is the more contentiously debated form of transfer. Vertical transfer is less controversial and is accounted for by most learning theories (Lobato, 1996).

The primary interest in this study was to investigate backward transfer along the vertical dimension as it might occur in the course of regular instruction, where one concept builds on another. In other words, the goal was to examine a mathematical
context in which particular prior knowledge acted as a foundation on which new knowledge was built. Together, linear and quadratic functions form such a context. This does not mean that possessing linear functions knowledge is a necessary prerequisite for learning about quadratic functions. However, several researchers have shown that linear functions knowledge can be used to help build understanding of quadratic functions (e.g., Movshovitz-Hadar, 1993).

In describing the vignettes presented earlier, the terms productive and unproductive influences were used. The way that productive is being used aligns with Wertheimer’s (1945/1959, cited in Greeno, 1989) definition of productive thinking, which he defined as possessing the understanding of the “essential properties and relations of the ideas in a problem” (p. 138). By productive or unproductive influences, I mean influences that move the student either toward or away from productive thinking. For example, Matt seemed to have a diminished understanding of the essential properties of the linear function data in the post-interview compared to the pre-interview, which suggests that there was an unproductive influence. Later, I provide evidence of a student who exhibited greater understanding of a linear function in the post-interview compared to the pre-interview. This evidence suggests that she experienced a productive influence. Next, I turn to how the construct of backward transfer is situated within the more general construct of transfer and within the transdisciplinary research on backward transfer.

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4 Productive thinking is distinct from mechanical knowledge, which is the ability to operate with symbols without the appropriate understanding (Greeno, 1989).
Situating Backward Transfer

Despite a vast body of work dedicated to theorizing about and studying transfer (Haskell, 2001), there is little agreement about the nature of transfer among the research community (Barnett & Ceci, 2002; De Corte, 1999). In particular, many researchers have identified shortcomings with the traditional approach to transfer (e.g., Lave, 1988; Green, Moore, & Smith, 1993) and have developed alternate approaches (e.g., Beach, 1999; Lobato, 2008a; Marton, 2006). The reader may wonder if it was prudent to associate the phenomenon of interest with transfer when the general construct of transfer is so controversial. However, there was an important reason for doing so, namely that one of the alternate approaches, the actor-oriented approach to transfer (Lobato, 2006, 2008a; Lobato & Siebert, 2002), provided a fruitful perspective for investigating the phenomenon observed in the pilot data.

Next, I briefly describe and contrast the AOT approach to transfer with the traditional approach. I will highlight the key features that make the AOT approach suitable for studying the phenomenon of interest. In the second part of this section, I give examples of empirical findings from linguistics research, which has a history of investigating backward transfer. The purpose of these examples is to situate the dissertation study within other research on backward transfer. I also present similarities and differences between the findings from linguistics research and the preliminary findings from the student interviews presented above.

Theoretically situating the phenomenon as transfer. The actor-oriented transfer (AOT) perspective is an alternate approach that addresses various concerns with the traditional transfer paradigm. One concern is that the traditional approach to
transfer underestimates the amount of transfer in a given situation (Lobato, 2008a; 2008b). This is because, from the traditional approach, only expert-like performance on novel tasks is counted as transfer (Singley & Anderson, 1989). In fact, Lave (1988) contends that traditional transfer studies often become an “unnatural, laboratory game in which the task becomes to get the subject to match the experimenter’s expectations,” rather than an investigation of the “processes employed as people naturally bring their knowledge to bear on novel problems” (p. 20). The AOT approach addresses this concern by counting as transfer any influence of prior experiences on a learner’s thinking about a novel situation (Lobato, 2006). As indicated by the name, AOT approach means investigating the actor’s (or learner’s) personal ways of connecting initial learning and transfer situations.

A second concern with the traditional approach to transfer is that knowledge is conceived of as a static (conceptual) tool that can be applied to a situation without change, much in the same way that a physical tool can be pulled out of a tool box and used (Lave, 1988). The AOT approach addresses this concern by conceiving of transfer as a dynamic rather than a static process (Lobato, 2006). Researchers who treat transfer as a dynamic process account for the ways in which people change transfer situations until they become similar to something they know and the ways in which people reconstruct their understanding of initial learning situations in order to make connections to the transfer situation (Bransford & Schwartz, 1999; Carraher & Schliemann, 2002; Lobato & Siebert, 2002). A third concern with the traditional approach is that transfer is treated as occurring entirely within the mind, without accounting for the ways in which other people, discourse practices, artifacts, and other
socio-cultural features of the environment contribute to the generalization of learning (Greeno et al., 1993). In response, the AOT approach conceives of transfer as being “distributed across mental, material, social, and cultural planes” (Lobato, 2008a, p.174).

These features suggest that transfer, as defined by the AOT approach, is related to the observed phenomenon of interest in my study. The first feature, that transfer is not just about the correct application of knowledge but about any influence from a previous learning experience, applies (in reverse) to the phenomenon of interest because, as seen in the vignettes, the quadratic functions instruction appeared to influence Matt’s prior knowledge of linear functions, but not in ways that led to expert-like performance. Taking an AOT approach was particularly important for this dissertation study because the understanding of linear functions of 8th graders (like those that participated in this study) is normally still developing and not going be expert-like for some time (Harel, Behr, Lesh, & Post, 1994; Kaput & Maxwell-West, 1994; Lamon, 2007; Lesh, Post, & Behr, 1988).

The second feature of the actor-oriented approach, namely that transfer is conceived of as a dynamic process (i.e., learners may reconstruct their thinking), also applies to the observed phenomenon because the interesting observation that emerged from the pilot data was that prior knowledge changed. The third feature, that transfer is distributed, also applies to the phenomenon of interest, because noticing, which was shown to be a forward transfer process and which comes about through the interplay between mental, material, social and cultural factors (Lobato, Rhodehamel, & Hohensee, 2011a), also offers promise as a backward transfer mechanism. Therefore,
this study examined whether noticing could provide explanatory power for backward transfer.

In summary, the reason for situating the phenomenon of interest within transfer was that it aligned well with an alternate approach to transfer—the AOT approach. The preceding comparison was meant as a general overview of this approach. Comparisons between the AOT approach and other approaches to transfer, as well as methodological implications, are presented in greater detail in Chapter 2.

**Transdisciplinary research on backward transfer.** As stated above, research that explicitly focuses on backward transfer in mathematics education is virtually non-existent. However, there is a small but established body of research originating primarily in the field of linguistics that addresses backward transfer directly. Examples from linguistics research will be used to situate the dissertation study. In Chapter 2, a more complete review of existing research on this topic is presented.

Ulrich Weinreich (1953, cited in Cook, 2003), a well-known linguist, was one of the first to suggest that there could be an influence by one language on the other for those who spoke two (or more) languages and that the influence could occur from the second language (L2) to the first language (L1). This conclusion was made after listening to people’s accents. Since then, many linguistics researchers have found evidence of backward transfer from L2 to L1. One of the influences involves language comprehension strategies: bilinguals use language strategies from L2 to comprehend L1. For example, Su (2001) compared monolingual Chinese with bilingual L1-Chinese speakers (English was L2) and monolingual English with bilingual L1-English speakers (Chinese was L2) to see how they determined the agent-
patient relation in a sentence. The study found that those who had attained intermediate and advanced levels of L2 (Chinese or English) employed L1 sentence processing strategies (word-order cues or verb-animacy cues) differently than monolinguals. This was interpreted as evidence of backward transfer.

A second influence by L2 on L1 involves language production. For example, Romero (2008) found that young L2-Turkish speakers often filled in the lexical vacuum created by forgotten words in L1-Spanish with Turkish words. In fact, the production of Spanish by younger generations of L1-Spanish speakers in Istanbul was strongly impacted at the lexical, phonological, morphological, and syntactic levels, by their L2-Turkish.

There are at least two ways that the backward transfer observed in the pilot data was similar to backward transfer effects from L2 to L1. First, in both cases, the effects appeared to be predominantly unproductive. In particular, Matt’s ability to reason about linearity appeared to diminish, as did bilinguals’ ability to comprehend or produce L1. Second, both effects appeared to occur without conscious effort. For example, the student in the mathematics interview did not appear to realize that his thinking had changed.

There are also two ways that the backward transfer in the pilot data was different than the backward transfer in the linguistics studies. First, in the linguistic

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5 The agent is the doer of an action, while the patient is the recipient of or participant in the action. For example, in the sentence, “the boy hit the balls” (Su, 2001, p. 84), the boy is the agent and the balls are the patient. In English, animate objects are more often agents than inanimate objects. So in the example, “the kite chased the mice” (p. 84), a native speaker of English might use animacy cues to conclude that the mice are doing the action.
studies, L2 was not necessarily more complex than L1, and L1 was not foundational for acquiring L2. In contrast, quadratic function knowledge is more complex than linear functions knowledge and it has been shown that the former can be built on the latter (e.g., Movshovitz-Hadar, 1993). Therefore, the linguistics example of backward transfer was along the lateral dimension—described earlier as transfer across contexts with similar levels of complexity—while the mathematics example from the pilot data was along the vertical dimension. Second, the linguistics research was about non-school-based learning, while the mathematics education research was about schooled-based learning.

In summary, prior research has shown that backward transfer can be an unproductive and unconscious influence, which can occur in the lateral dimension in non-schooled contexts. This dissertation study expands on this research because backward transfer in the vertical dimension for school-based learning was examined. Additionally, this study expands on prior research because a goal of the study was to produce a productive backward transfer influence, as will be explained in the next section.

The Productive Side of Backward Transfer

In this section, I present a second student from the pilot data. This student exhibited a productive backward transfer effect. I also discuss several potential mechanisms for productive and/or unproductive backward transfer, including the process of noticing, which offered promise as an explanatory frame. I conclude this section with an elaboration of a conjecture regarding instructional supports that may promote productive backward transfer and limit unproductive backward transfer. This
conjecture served an important function in the study because it guided the design of the instructional intervention.

**Illustrative example of productive backward transfer.** This example comes from interviews with Kala, who was also an 8th grader. Her interviews were different from Matt’s interviews for three reasons. First, Kala participated in the second iteration of this design-based research study, one year after the iteration that Matt participated in. Therefore, the design of the quadratic instructional intervention she received was a refinement of the previous year’s instructional design. Second, in the first year, students were interviewed before and after the intervention, whereas in the second year students were only interviewed once, after the intervention. Instead, they were given a short written pre-test as a substitute for the pre-instruction interview. Third, and most important, this student’s prior knowledge appeared to benefit rather than suffer from an influence of newly acquired knowledge. Kala’s written responses prior to the instruction and a vignette of her post-interview will be presented first. Then, four reasons will be given supporting the conclusion that her understanding of linearity had deepened from pre- to post-interview.

On the short written pre-test, Kala wrote that she thought that the remote control plane, as represented by the altitude-time data (see Figure 1.4), was getting slower:

> It is climbing slower because at 5 ft it is really supposed to be at 40 ft but instead, it is at 17.5. I found this out by multiplying 7 • 5 which equals 40. I did this because if it was moving equally it would climb 7 ft for each sec.

In this excerpt, Kala appeared to be attempting to reason proportionally by
trying to find the altitude after 5 s if the plane climbed at a steady rate of 7 ft per s.

However, she appeared not to notice that the plane’s altitude after 2 s was 7 ft, which would translate to a climbing rate of 3.5 ft per s. Thus, she concluded that the plane’s climbing rate was slowing down. Kala’s conclusion could be the result of her not noticing an important feature of this linear function, namely that 7 ft was paired with 2 s. Alternately, it could mean that Kala was not entirely clear how to determine the rate of a linear function.

During the post-interview, Kala (K) was shown the same altitude-time data by the interviewer (I). She made the following comments:

62 I: What did you conclude?
63 K: That it goes, every second it goes 3.5 ft.
64 I: How did you find that out?
65 K: I did 2 divided by 7, and I found out that it’s 3.5
66 I: And then you did some more things, can you tell me about what’s in there? [points at data table; see Figure 1.5]
67 K: Since the other seconds aren’t there, I just put 3 [points at the 3 she had written in time column], and then I did 7 plus 3.5, and I got 10.5. And then I did 10.5 plus 3.5 and I got 14. Then I did, so I just wanted to check it, so I did 14 plus 3.5 and I got 17.5. And then I did that [points at 17.5 in altitude column] plus 3.5 and got 21, and that plus 3.5 [points at 21 in altitude column], and got 24.
68 I: So what did you finally conclude about whether it was climbing equally fast or . . . ?
69 K: It was climbing equally fast.
70 I: When you first did this, did you think that it was faster in places?
71 K: Yes.
72 I: Where did you think it was the fastest?
73 K: Right here. [points at first difference 17.5]
74 I: What changed your mind?
75 K: When I saw that the seconds were missing.
This vignette shows that Kala’s thinking had changed. Specifically, it demonstrates a deepening in her understanding of linearity. There are four pieces of evidence supporting this conclusion. First, in the post-interview she calculated the amount of altitude the plane would climb in 1 s by dividing, 7, the altitude the plane attained after 2 s, by 2 (lines 63 and 65). Second, Kala repeatedly added the 3.5 to find other values of altitude for which the time was not represented in the table (see Figure 1.5 and line 67). Third, Kala concluded that the plane was “climbing equally fast” (line 69). Fourth, she noticed that some of the time values were missing (line 75).

These changes in Kala’s thinking appeared to be closely linked with noticing that some of the time values were missing. This was followed by finding the altitude after 1 s which, in turn, was followed by filling in absent rows in the table so that all integer values of time were represented. The result of this filling in of rows led Kala to conclude that the plane was climbing at a constant rate. Therefore, noticing appeared to play an important role in her changed thinking. This suggests that noticing holds promise for helping explain the occurrence of backward transfer. More
will be said about this process in the next section.

The changes in Kala’s thinking involved her reasoning with both altitude and time. As her inscriptions show (see Figure 1.5), Kala coordinated changes in time and altitude to fill in the missing rows. Coordinating two quantities in this way is a feature of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Therefore, this example suggests that promoting covariational reasoning in a unit on quadratics may lead to productive changes in students’ prior knowledge of linear functions. As will be explained later, this conjecture played a prominent role in the design of the instructional intervention. In the next section, several mechanisms that may account for such changes in prior knowledge will be discussed.

**Potential backward transfer processes.** A major goal of this study was to investigate the processes by which backward transfer occurs during an instructional intervention. There are a variety of mechanisms that might account for aspects of backward transfer (Glenberg & Swanson, 1986; Ebbinghaus 1885/1913, cited in Estes, 1987; Isurin & McDonald, 2001; Karmiloff-Smith, 1992; Kristjánsson, Wang, &

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**Figure 1.5. Kala’s inscriptions during post-interview remote-control car task.**

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>Altitude (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3.5</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>17.5</td>
</tr>
<tr>
<td>10</td>
<td>35</td>
</tr>
<tr>
<td>12</td>
<td>42</td>
</tr>
</tbody>
</table>

First differences

\[
3.5 + 8 = 9
\]
Nakayama, 2002; Lobato et al., 2011a; Luchins & Luchins, 1991; Pirie, Martin, &
Kieren, 1996; Schwartz & Bransford, 1998; Shuell, 1986). The following potential
mechanisms will be briefly reviewed (greater detail will be provided in Chapter 2): (a)
the recency effect, (b) retroactive inhibition, (d) Einstellung, (e)
discrimination/differentiation, and (f) noticing. My review will show that noticing
offers the most promise for providing an explanatory framework for the backward
transfer that might be produced in the dissertation study.

The recency effect is defined as learners having the best recall for information
that is presented most recently (Bjork & Whitten, 1974; Glenberg et al., 1983;
Glenberg & Swanson, 1986) and retroactive inhibition is defined as new information
interfering with the retrieval of older information stored in memory (Isurin &
McDonald, 2001). These mechanisms were considered because, for example, perhaps
Matt’s (from the pilot data) new knowledge of quadratic functions was more readily
available for recall and was acting as interference. However, these mechanisms offer
limited explanatory power for the kinds of improvements that Kala showed because
neither mechanism would account for a positive backward transfer effect.

Einstellung (Luchins & Luchins, 1991) is a mechanism that is described as
learners using complex methods to solve problems even though using simpler methods
would be more efficient. This mechanism was considered because Matt may have
been using the more-complex problem-solving methods from the quadratics
instruction to try to interpret the remote-control plane data instead of using the less-
complex methods associated with linear functions. However, a key feature of
Einstellung is that it occurs when learners learn the more-complex methods prior to
learning the less-complex methods. In Matt’s case, he learned about problem solving methods associated with linear functions (i.e., the less-complex methods) prior to learning about problem solving methods for quadratic functions (i.e., the more-complex methods). Therefore, this mechanism offers limited explanatory power for the effects that Matt exhibited.

*Discrimination* is defined as becoming attuned to distinguishing features (Marton, 2006). By making finer and finer discriminations between contexts, students can come to understand which features are critical and which are superficial. For example, Seguin (1907, cited in Marton, 2006) taught retarded children to discriminate between sensations by first presenting sensations at opposite ends of a continuum (e.g., hot/cold) and gradually presenting sensations that were more and more similar. Schwartz and Bransford (1998) identified a similar mechanism which they called *discerning* or *discovering distinctions*. They explained that through analyzing contrasting cases, this mechanism prepares learners for understanding text. “Contrasting cases help people notice specific features and dimensions that make the cases distinctive” (Schwartz and Bransford, 1998, p. 479). It is possible that these mechanisms played a role in the productive changes in Kala’s understanding about linear functions. For example, the instruction on quadratics may have functioned for Kala as a contrasting case for linear functions and thus enabled her to discover new features that distinguish linear functions. However, neither of these mechanisms account for negative effects and thus offer limited explanatory power for unproductive backward transfer.

Thus far, I have presented a sampling of mechanisms that, for various reasons,
appear to be limited in providing an explanatory framework for the productive and/or the unproductive backward transfer that was produced in pilot data. Next, I elaborate on the process of noticing.

Lobato et al. (2011a) have recently offered noticing as a transfer process. Noticing refers to selecting, interpreting and working with particular mathematical features or regularities when multiple sources of information compete for students’ attention (Lobato, Rhodehamel, & Hohensee, 2011b). However, Lobato and colleagues follow Goodwin (1994) by acknowledging that “the ability to see a meaningful event is not a transparent, psychological process, but is instead a socially situated activity” (p. 606). Consequently, they have developed the focusing framework as a way to coordinate what individuals notice with the social organization of noticing. The focusing framework is composed of the following four parts: (a) the centers of focus, (b) the focusing interactions, (c) the features of the mathematical tasks, and (d) the nature of the mathematical activity. The centers of focus are the features of a domain of scrutiny that individuals notice. In the context of mathematics education, centers of focus would be mathematical features, regularities, or conceptual objects. Focusing interactions are the discursive practices (e.g., gestures, diagrams, talk) that direct attention to some features of the domain of scrutiny and not to others. The features of the mathematical tasks refer to the features of particular problems and activities, which influence what students’ notice mathematically. The nature of the mathematical activity refers to the ways that classroom participation is organized, which contribute to the emergence of particular centers of focus (e.g., teacher and student roles, expectations for participation). Furthermore, according to the
framework, the centers of focus that emerge are the result of interplay between the latter three components of the focusing framework.

Noticing offers promise as a backward transfer process because it could potentially explain the occurrence of both productive and unproductive effects. For example, perhaps during the quadratic function instruction that Matt participated in, his center of focus became the dependent variable. That may explain why he reasoned primarily with the dependent variable on the linear function post-interview task. In contrast, perhaps during the quadratic function instruction that Kala participated in, her center of focus became the coordination of two variables. That may explain her more-productive reasoning with both the dependent and independent variables on the linear function post-interview task.

A more complete explanation of noticing, the focusing framework and the AOT approach to transfer that was used to develop these constructs will be provided in Chapter 2. Next, I present two instructional supports that could promote productive noticing, as well as the conjecture on which these supports are based.

**Conjecture about instructional supports.** One of the research questions for this dissertation study asks about the ways to promote productive backward transfer in mathematics classrooms. As shown above, the pilot data provided evidence of a deepening, from pre- to post-interview, in Kala’s understanding of linearity. The reason why the pilot data were not used to address the present dissertation questions is because the two previous iterations of the study from which the pilot data came were not designed to investigate backward transfer. It was serendipitous that a brief linear function task was included in the pre- and post-interview protocols for those studies.
However, the data from those tasks were too sparse to facilitate a systematic investigation of backward transfer. Thus, another iteration of the design-based research study was needed to explicitly target productive backward transfer as an instructional goal.

I conjectured that two particular instructional supports would direct what students noticed in the context of quadratic functions in ways that would lead to productive backward transfer onto their prior knowledge of linear functions. The first instructional support was a strong emphasis on covariational reasoning. The rationale behind this support was that, as reported earlier, both students exhibited changes in covariational reasoning. Matt went from reasoning with two quantities—altitude and time—in the pre-interview to reasoning primarily with altitude in the post-interview. In contrast, Kala went from reasoning primarily with altitude to reasoning with altitude and time. It was hoped that an emphasis on attending to both quantities in quadratic contexts during instruction may promote reasoning with both quantities in linear contexts.

There are several ways that emphasizing covariational reasoning might be accomplished. One way would be to engage students in an exploration of nested independent-variable intervals. For example, if students were exploring nested time intervals of a quadratic distance-time function, like those shown in Figure 1.6, it may promote noticing of both time intervals and distance intervals rather than just distance intervals.
Another way to emphasize covariational reasoning would be to have students explore end-to-end independent variable intervals of shrinking size. For example, in a quadratic distance-time context, students could be asked to find the changes in distance for end-to-end 1 s intervals, for end-to-end .5 s intervals, for end-to-end .1 s intervals, etc. Shrinking the size of the intervals may help students attend to both the intervals of time and the intervals of distance. These ideas for emphasizing covariational reasoning were used in the instructional intervention.

A second instructional support was to emphasize point-by-point comparisons between quadratic functions and the linear functions that are embedded within them. By embedded linear functions I mean: (a) linear functions that start at a point on a quadratic and then maintain a constant rate (see Figure 1.7a), and (b) linear functions that run between two points on a quadratic function (see Figure 1.7b). By comparing

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Questions:
- Find the changes in distance when the time changes from 0.50 to 1.50 s.
- Find the change in distance when the time changes from 0.60 to 1.00 s.
- Find the change in distance when the time changes from 1.00 to 1.40 s.
- Find the change in distance when the time changes from 0.80 to 1.20 s.

Figure 1.6. Sample activity for focusing attention on covariational reasoning by exploring nested intervals.
a quadratic function to one of these embedded linear functions, on a point-by-point bases, students might develop a better understanding of how quadratic functions behave and at the same time develop a deeper understanding of linearity.

Emphasizing these point-by-point comparisons may be helpful to a student like Matt who, in his post-interview, identified the rate of climbing of the airplane as the “acceleration.” His response suggests that he had not sorted out that acceleration was not the rate of change but how a rate of change changes. I observed similar thinking by Jared, another student from the pilot data. In the pre-interview, Jared interpreted

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**Figure 1.7.** (a) embedded linear function that starts at a point on quadratic and maintains constant rate; (b) embedded linear function between two points on a quadratic function.
the pattern in the data (see Figure 1.1 for data) as “I think it’s climbing at the same rate.” In the post-interview, he interpreted the pattern as “like the acceleration.” Perhaps making point-by-point comparisons between quadratic functions and the embedded linear functions, may help students like Matt and Jared come to see the difference between constant rates of change in linear functions and changing rates of change within quadratic functions. Therefore, students in the dissertation instructional intervention were given activities where they made point-by-point comparisons between quadratic functions and embedded linear functions.

The aim of this dissertation study was to produce productive backward transfer with instruction that was primarily focused on quadratic functions, not on linear functions. Therefore, the comparisons between linear and quadratic functions in the instructional intervention were limited to those linear functions that are embedded within quadratic functions. These embedded linear functions often come up naturally in the course of quadratics instruction at the upper grades (e.g., in calculus courses) as the linear function of the secant line between two points and the linear function of the tangent line at a point. While the concepts secant and tangent are too advanced for most 8th graders, these concepts are accessible informally as linear functions embedded in quadratic distance-time functions. For example, if one object is moving according to a quadratic function between two points, then the linear function of a secant line could be modeled by an object moving at a constant rate between the same two points (over the same time interval). Similarly, if one object is moving according to a quadratic function at a given point, then the linear function of the tangent line could be modeled by an object moving at a constant rate from that point (and time) on.
Because the focus of instruction was on quadratic rather than linear functions, the entire embedded linear function was not generated, just the points necessary for making comparisons.

In summary, two instructional supports were used to direct students to notice features of quadratic functions, which were conjectured to support productive reasoning with linear functions. These supports were based on the following conjecture for promoting backward transfer:

Activities set in the context of quadratic functions that promote covariational reasoning and that involve making point-by-point comparisons between quadratic functions and the linear functions embedded within them will help learners develop an understanding of quadratic functions and at the same time deepen their understanding of linearity by (a) promoting covariational reasoning in linear contexts and, (b) by making critical features of linear functions more salient.

This conjecture guided the design of the dissertation instructional intervention.

**Research Questions and Significance**

In this section, the research questions that drive this study are presented. Each question will be connected back to aspects of the rationale presented above. Then, the significance of addressing these questions will be discussed. The three research questions are:

1. When the targeted conceptual understandings of an instructional unit are conceived of as being built upon particular prior knowledge, when and in what ways does the newly constructed knowledge lead to productive changes in prior knowledge (productive backward transfer)? When and in what ways does it contribute to unproductive changes (unproductive backward transfer)?

2. In what ways are backward transfer (influences by a person’s newly constructed knowledge on prior knowledge) and forward transfer (influences by a person’s newly constructed knowledge on how they come to understand novel situations) related?
3. What are the transfer processes by which classroom instruction leads to productive backward transfer?

**Question 1.** This question asks about the productive and unproductive influences of newly acquired quadratic function knowledge on prior knowledge about linear functions. The motivation for this question emerges out of the summarizing statement made above regarding the gap in research on backward transfer. In particular, because linguistics research is primarily focused on backward transfer along a lateral dimension for naturalistic language learning, there is a gap in the research on backward transfer along the vertical dimension for school-based mathematics learning.

Additionally, the preliminary findings suggested that it may be useful to examine productive backward transfer. This is in contrast to linguistics research, which has focused primarily on unproductive backward transfer. While the study of negative effects could be informative to the field of mathematics education, the primary goal of the intervention and subsequent analysis was to promote and investigate positive effects. Only if productive backward transfer had not been produced during the intervention, would the negative effects have become the principle focus of analysis.

**Question 2.** This question asks about the relationship between backward transfer of newly acquired quadratic functions knowledge onto prior linear functions knowledge and the forward transfer of newly acquired quadratic functions knowledge onto thinking about novel quadratic functions problems. This question emerges out of the argument presented above that backward transfer should be situated within
transfer. As argued, analysis of pilot data suggested that the phenomenon labeled *backward transfer* aligns with features of forward transfer according to an AOT approach. These observations suggested that backward and forward transfer may be related in important ways.

**Question 3.** This question asks about the transfer *processes* by which classroom instruction would result in productive influences by newly acquired quadratic functions knowledge onto prior linear functions knowledge. Out of the set of potential backward transfer processes that were considered, noticing emerged as the best candidate for accounting for the pilot data evidence. To address this question, I began by looking at how the features of the quadratic functions instruction supported noticing, and then, how noticing may have led to a deepening of student understanding of linearity. However, during analysis, I left open the possibility that other processes may also be useful to consider.

**Significance.** I have organized this section on the significance of answering the research questions into the following three categories: (a) the significance of the relationship between prior knowledge and newly constructed knowledge, (b) the significance for the theoretical conceptualization of transfer, and (c) the significance for pedagogy. These categories do correlate to a degree with the research questions (i.e., the first category reflects the significance of answering the first research question; the second category reflects the second question, etc). However, as I will show, the significance of answering a particular research question does in certain instances overlap with other categories.
Significance of the relationship between prior and newly constructed knowledge. One component of the significance of answering the research questions is that it may add to the field’s understanding of the relationship between prior and newly constructed knowledge. In particular, this dissertation could contribute to the understanding of how prior knowledge might be influenced by newly acquired knowledge. This component of the significance of answering the research questions has been subdivided into two levels, a content-general level and a content-specific level.

At the content-general level, because this dissertation study looks carefully at what is happening to prior knowledge, findings from the study could be significant for a wide range of learning theories because most learning theories account for new knowledge being built up from prior knowledge. For example, according to the ACT-R information processing theory, new chunks of knowledge can be created by applying rules to existing chunks (Anderson, 1996). According to Piagetian constructivism, new cognitive schemes can result in modifications to older schemes (Sáenz-Ludlow, 1994). According to a socio-cultural perspective, generalizations are built on generalizations (Vygotsky, 1987). Interestingly, there exists little in these theories that specifically addresses what happens to the prior knowledge. Therefore, it is significant that the dissertation study addresses the relationship between prior knowledge and new knowledge from a new and potentially informative angle.

At the more content-specific level, this study investigates the ways that learning about quadratic functions might influence beginning algebra students’ conceptions of linear functions. School curricula are typically organized so that linear
functions instruction occurs prior to quadratic functions instruction. For example, the San Diego Unified School District recommends that Algebra 1 instruction cover linear functions from October to December of a given school year and quadratic functions from February to March (http://prod031.sandi.net/depts/math/modules/alg_pacing_trad.pdf). Therefore, the research questions address an issue that potentially impacts every student’s learning of algebra.

Also, in many states, including California, the state mandated testing occurs in late spring, on the heels of the instruction on quadratic functions. For example, in 2010, California schools were mandated to administer the standardized tests within 10 days before and 10 days after the day when 85% of the instructional year was complete (http://www.cde.ca.gov/ta/tg/sa/0910testdates.asp). In San Diego Unified School District, this meant that 2010 testing was administered between May 2 and May 22 (http://old.sandi.net/comm/schools/calendars/0910traditional.pdf). Therefore, addressing the relationship between prior and newly constructed knowledge has significance for the ways that students perform on standardized tests and thus, for school and school district accountability.

In summary, one component of the significance of answering the dissertation study research questions is that it could inform the field about an aspect of building on prior knowledge that is not often considered. Furthermore, this could have implications for how Algebra 1 students are taught about linear and quadratic functions and for how these students score on standardized testing at the end of Algebra 1.
**Significance for theoretical conceptualization of transfer.** According to Barnett and Ceci (2002), “there is little agreement in the scholarly community about the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms” (p. 612). Furthermore, the traditional approach to transfer has been shown to have several serious shortcomings (Lave, 1988; Beach, 1999). The dissertation study addresses a few of these problems with the way that transfer is traditionally conceived and thus is theoretically significant.

Beach (1999) identified three challenges to the traditional model of transfer. The first challenge is that transfer is conceptualized as a limited and narrow part of learning. Beach (1999) provides several possible scenarios that would not traditionally be counted by experimenters as transfer but that he argues should count. One such example is learning that occurs prior to both a learning task and the subsequent transfer task. The dissertation study may help to address this challenge because, despite backward transfer being included in the definition of transfer in a limited number of transdisciplinary studies, primarily in linguistics, backward transfer is not part of the traditional conceptualization of transfer in education research.

The second challenge is that transfer is often conceived of as a static process. Beach (1999) argues that transfer is traditionally seen as a process of applying knowledge to a new situation in a way that the knowledge and the situation do not change. Lave (1988) made a similar critique, arguing that transfer is misconceived when experimenters take it to be the unmodified “transportation of tools for thinking from one situation to the next” (p. 37). Several researchers have demonstrated empirically how learners can modify the transfer situation before they make
connections with prior experiences (Bransford & Schwartz, 1999; Lobato & Siebert, 2002; Wagner, 2006). The dissertation study may contribute to the conceptualization that transfer is a dynamic process by showing how the knowledge gained prior to a targeted learning experience can change in either productive or unproductive ways as a result of the transfer of learning.

The third problem is that the traditional conceptualization of transfer implies a “launch model” (Beach, 1999, p. 109) of development. Beach (1999) argues that the traditional conceptualization of transfer involves the assumption that the initial learning situation determines what a person will do on a future task and what the trajectory for the future development of knowledge will be. This problem is closely linked to the previous problem of transfer being conceived of as static. The dissertation study may provide insight into the development of prior knowledge that would not be accounted for with the launch model.

Besides helping to expand the definition of transfer and addressing problems with the traditional conceptualization of transfer, this study extends the contexts in which the noticing has been used as an explanatory frame for the occurrence of transfer. Specifically, this dissertation study builds upon and extends the focusing framework from Lobato and colleagues (2011a) by offering an explanation for both forward and backward transfer effects. In summary, the theoretical significance of this study is that it addresses several key difficulties with how transfer is traditionally conceived, and as a by-product, could also help to extend noticing as a transfer process.
**Pedagogical significance.** At the content-general level, the significance of answering the third research question is that it investigates ways that new content might be taught so that prior knowledge is productively influenced. This seems pedagogically significant because, from personal experience as a teacher and from talking with other teachers, I believe that most teachers grapple with the issue of not wanting to confuse their students’ existing knowledge when introducing a new more complex topic. For example, a teacher might be concerned that in teaching his or her students about multiplying fractions, they may think about adding fractions differently. One goal of the study was to uncover one or more guiding principles for instruction that might cut across mathematics content areas with respect to foundational prior knowledge changing as result of learning something new.

At the content-specific level, the significance of this study is that it addresses key topics within Algebra 1, which is often described as a gatekeeper for future educational and economic success (Ladson-Billings, 1998; Moss & Cobb, 2001; National Research Council, 1996, cited in Knuth, Stephens, McNeil, & Alibali, 2006). Furthermore, linear and quadratic functions assume a prominent place among the topics in Algebra 1. For example, according to the California Mathematics Content Standards for Algebra 1, seven of the 25 standards deal specifically with linearity and six standards deal with quadratics (http://www.cde.ca.gov/be/st/ss/documents/mathstandard.pdf). It is also a common assumption among educators that knowledge of quadratics can be built on mastered knowledge of linear functions (Movshovitz-Hadar, 1993; Ohlsson, 2009). What the dissertation study addresses is significant for algebra teachers, since, as will be shown later, it provides instructional principles for
teaching linearity and quadratics within an Algebra 1 course. In summary, the pedagogical significance of this study is that it provides general principles by which to address backward transfer and helps inform teaching of linearity and quadratics within Algebra 1 courses.
CHAPTER 2:
LITERATURE REVIEW

This chapter is organized into two main sections. In the first section, I review the literature pertaining to transfer and backward transfer, organized within a conceptual framework for studying backward transfer. In the second section, I review the literature pertaining to the mathematics relevant to this study. At the end of the chapter, I revisit the research questions in light of the research presented in the review.

REVIEW PERTAINING TO FORWARD AND BACKWARD transfer

A conceptual framework for conceiving of and investigating backward transfer has been developed for the dissertation study by adapting the eight dimensions used by Lobato (2003) to compare the theoretical assumptions behind the traditional versus the actor-oriented approaches to transfer. The framework has two main categories: (a) the conception of backward transfer and, (b) the investigation of backward transfer (see Figure 2.1). The conception category is further subdivided into the definition of backward transfer (and the resulting implications for where invariance is located), and the processes that underlie backward transfer. The investigation category has also been further subdivided into the researcher point of view and the research design (i.e., the research questions that are addressed and the approach to causality that is used).

This conceptual framework for backward transfer serves three purposes. First, it articulates what I believe are the most defining aspects of thinking about and investigating backward transfer. Second, the framework serves as the structure by which the review of literature pertaining to transfer and backward transfer will be
organized. Third, weaving the review into the framework allows the ideas about backward transfer that drove the dissertation study to be situated within the larger literature on the transfer of learning. The fact that this conceptual framework for investigating backward transfer was adapted from dimensions used to classify perspectives on forward transfer reflects the assumption underlying this study that forward and backward transfer are related.

The purpose of this review is to contextualize both the perspective and the approach used to study backward transfer. Therefore, this review of transfer will not be exhaustive but will provide a range of research within which the dissertation study will be situated. Specifically, various approaches to transfer will be compared to the

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6 Thousands of articles, chapters and books have been published on the topic of transfer in the past 100
actor-oriented transfer approach (Lobato, 2006, 2008a, 2008b; Lobato & Siebert, 2002), which is the perspective used in the dissertation study.

According to the conceptual framework, the transfer literature will be presented in two sections. The first section contains literature pertaining to conceptions of forward and backward transfer. The second section contains literature pertaining to investigating forward and backward transfer.

This part of the review focuses on the various ways that researchers define transfer and conceive of the processes that support its occurrence. Much more literature exists about forward transfer than about backward transfer. In fact, some of the categories of the conceptual framework are not very well addressed in the literature on backward transfer. In those cases, inferences will be made about backward transfer based on existing literature. For each category of the framework, I will discuss how the approach used for the dissertation study aligns with the literature.

**Conceptions of the Phenomenon**

Two important aspects of a conception of forward transfer and backward transfer are (a) how transfer is defined, including where it is located, and (b) the processes that underlie transfer. Each aspect is discussed in turn.

**Definitions of transfer.** From a traditional information processing perspective, transfer is typically defined as knowledge acquired in one context applied to another context (Singley & Anderson, 1989). A behaviorist variant of this...
There have been numerous critiques of the traditional definition of transfer (e.g., Beach, 1999; Evans, 1999; Greeno et al., 1993; Lave, 1988; Tuomi-Gröhn & Engeström, 2003; van Oers, 2004). First, the notion of “context” in the traditional transfer definition is treated as the task presented to the students and is analyzed independently of students’ purposes and construction of meaning in situations, which does not account for the situated nature of transfer (Greeno et al., 1993; Lave, 1988). Second, these definitions involve the assumption that transfer is measured by applying normative models of performance, which does not account for many of the ways in which novices generalize their learning experiences (Lave, 1988). Third, the traditional definition attributes transfer to identifying similarities across situations (e.g., Thorndike’s identical elements theory; Detterman, 1993) but not to discerning differences between situations in transfer (Marton, 2006). Fourth, these definitions characterize transfer as purely a cognitive phenomenon, rather than accounting for the contribution of social interactions, cultural artifacts, and practices to the organization and support of the generalization of learning (Tuomi-Gröhn & Engeström, 2003).

In response to these and other criticisms, a number of alternate definitions of transfer have been developed. Greeno et al. (1993) defined transfer as the extent to which participating in one activity influences successful participation in another activity in a different situation. This definition is consistent with situated cognition, in that knowing is relative to situations rather than being an invariant property of the individual (Brown, Collins, & Duguid, 1989).
Another alternate definition of transfer comes from Lave (1988) who suggested that transfer is “how people establish relations of similarity between the problems they encounter in different settings” (p. 44). This definition responds to the criticism that what counts as transfer is normative or expert-like performance on a transfer task. This criticism emerged out of the observation that even though expert-like transfer was difficult to produce (Lobato, 1996; Detterman, 1993) people experience continuity of activity across situations (Beach, 1999; Lave, 1988). Defining transfer so that what counts is what ordinary people do—Lave (1988) calls them “jpfs [just plain folks]” (p. 23)—rather than what experts do allows researchers to observe aspects of transfer that remain hidden when applying the traditional definition.

A third alternate definition comes from Marton (2006), who defined transfer as “relations between what people learn and can do in different situations” so that transfer is “a function of perceived differences (and similarities) in two or more situations and between those situations” (p. 512). This definition is a response to Marton’s criticism that identifying similarities between situations was the primary focus of the conceptualization of transfer, while discerning differences was largely ignored. In other words, he argued that not only similarities connect contexts but small differences as well. He gave the example of students throwing a shuttlecock at a target from different angles. In order for a child to transfer their knowledge for hitting the target at a particular angle to a novel situation with a new angle, they would need to discern the dimension of angle variation and also discern how the angle had changed.
Several definitions of transfer have also been offered from a socio-cultural perspective. Tuomi-Gröhn and Engeström (2003) defined transfer as the collective activity of developing a model for new forms of practice. Evans (1999) characterized transfer as building bridges between communities of practice. Finally, Beach (1999) defined transfer as the transformation of “knowledge, skill and identity” (p. 112) across social organizations. This set of definitions emerged out of the criticism that the traditional definition characterizes transfer as entirely a cognitive phenomenon and ignores any social aspects that might be associated with transfer. The first two of these definitions account for transfer as a social phenomenon, while the third definition coordinates the individual and the social aspects of transfer.

A fifth approach comes from Lobato (2008a), who defined transfer as “the influence of a learner’s prior activities on his or her activity in novel situations, which entails any ways in which learning generalizes (as opposed to that learning being restricted to the initial context of learning)” (p. 171). This definition is identified with an *actor-oriented transfer* (AOT) approach to transfer (Lobato, 2008a) and includes some features of the other four alternate definitions. The evolution of the AOT approach can be traced back to an initial effort to extend the theoretical ideas of Greeno et al. (1993) to empirical studies (Lobato, 2003). Also, the AOT approach definition includes features of the second and third alternate definitions because the connection-making between situations is seen as involving the process of similarity-making as well as discerning differences (Lobato, 2008b). Finally, AOT approach definition includes features of the fourth set of definitions because transfer influences are conceived of as being distributed across mental, material, social, and cultural
planes (Lobato, 2008a).

**Definitions of backward transfer.** There are three main conceptions of backward transfer in the literature. The first is illustrated by Gentner, Loewenstein and Thompson (2004), who defined backward transfer as applying a learned principle to a previously acquired example or case. This definition aligns with the way that forward transfer is traditionally conceived. The subjects in their study, a group of professional management consultants, spontaneously retrieved old examples of negotiations from previous experiences and reinterpreted them in light of a newly learned principle of negotiation. Chen (2006) and Su (2001) both defined backward transfer as bilingual speakers applying their newly acquired second language (L2) strategies in processing their first language (L1). In the study by Su (2001), advanced L2-English speakers used English strategies to process meanings in L1-Chinese. Su also described this as “L2 strategies being carried over to L1 processing” (p. 107). In all three examples, the researchers appeared to define backward transfer as new knowledge being applied to old contexts.

The second conception of backward transfer is that it is an influence backward. For example, some linguists define backward transfer as an L2 influence on L1 processing (Felser, Roberts, Marinis, & Gross, 2003; Gürel, 2007; Tsimpili, Sorace, Heycock, & Filiaci, 2004). For example, Tsimpili et al. found that backward transfer occurred when interpretable features of language were specified in L1 but were
underspecified in L2.\textsuperscript{7} Uninterpretable features of a language are features like the grammatical case of a word, whose meaning is not specified in the sentence.

Gürel (2000, cited in Gürel, 2007) observed backward transfer in a study with native Turkish speakers living in North America. Surprisingly, in a subsequent study Gürel (2007) found no evidence of backward transfer from L2 to L1. Comparing these two contradictory studies led Gürel to conclude that not only does the presence of L2 influence backward transfer, but so do psycholinguistic factors, such as the amount of exposure the native speakers continue to have to their native language. This finding aligns with the earlier argument by Greeno et al. (1993) and Lave (1988) that a conception of transfer should account for the situation.

Camarata, Nelson, Gillum and Camarata (2009) defined backward transfer as training in one modality (e.g., expressive language or receptive language) yielding gains in another modality. They found that improvement in either a person’s language production mode or their comprehension mode (e.g., through training), resulted in a developmental shift upwards in the online patterns of social-communicative interactions that then supported advances in the other mode.

While explicit reference to backward transfer is frequently made in linguistics and cognitive science, there are only isolated instances where it is implied in mathematics education literature. For example, Marton (2006) addressed backward transfer (what he called retrospective transfer) in his paper on discerning differences.

\textsuperscript{7} Interpretable features are features such as the distinction between the agent and the patient of a sentence which can be read or interpreted by the cognitive system.
He described backward transfer as new experiences affecting a previously acquired image in a way that highlights features of the image that differentiate it from the new knowledge. To illustrate, consider Gibson’s (1969, cited in Marton, 2006) example that when someone is shown a red ball, they initially form an image of the ball, with all its features coexisting. However, if then shown a white ball, the person’s previous image of the red ball will change so that the color of the ball is pushed to the foreground. In other words, the new experience influences the person’s previous image.

The third conception of backward transfer involves organizations instead of individuals. Specifically, backward transfer is defined as knowledge being passed from a secondary organization back to the primary organization (Buckley, Clegg & Tan, 2003; Millar & Choi, 2009). Buckley et al. (2003) looked at the backward transfer of technological knowledge into and out of foreign-invested enterprises. One finding from this study showed that in order for backward transfer to occur, there needed to be a congruence of goals among the participants, and the participants needed to possess the necessary aptitudes and abilities. A second finding was that a knowledge-creation strategy was shown to be necessary for the backward transfer of explicit or formal knowledge to occur, while the backward transfer of tacit knowledge occurred independently of a strategy.

Similarly, Millar and Choi (2009) defined backward transfer as the movement of either explicit or tacit knowledge from a multinational corporation subsidiary to its headquarters. Several psychological factors were found to be essential for backward transfer to occur, including a psychological contract among coworkers that backward
transfer would occur, as well as intrinsic motivation for all concerned that it would occur. In this example and in the example from Buckley et al. (2003), the conception of the role of new knowledge was that it gets transmitted back from a younger, less established organization to its older, more established parent organization. This way of conceptualizing backward transfer is somewhat parallel to the conceptions held by Evans (1999) and Tuomi-Gröhn and Engeström (2003) of collective forward transfer.

The definition of backward transfer that most closely aligns with the analysis of my pilot data is that of newly acquired knowledge influencing prior knowledge. One reason is that this definition was linked with both productive effects (e.g., Camarata et al., 2009) and unproductive effects (e.g., Tsimpli et al., 2004). A second reason is that backward transfer occurred both with explicit training to try to produce an effect (Camarata et al., 2009) and without training (e.g., Gürel, 2000, cited in Gürel, 2007). These characteristics fit the transfer phenomenon identified in the pilot data.

*The Location of Invariance for Transfer.* As shown above, the traditional conception of transfer is that knowledge remains invariant as it is transferred from one context to another. This raises several important questions such as the following: Invariance is a property of what or whom? Where is invariance located? Is invariance necessary for transfer? There are two main ways of thinking about invariance with respect to transfer. The prevailing view among cognitive scientists is that invariance is in the mental representations of individuals (Singley & Anderson, 1989). An alternate view is that the invariance that is associated with transfer is distributed across
the environment (Greeno et al., 1993). In this section, I review both views and elaborate the perspective adopted for this dissertation study.

In the mainstream cognitive perspective, transfer is normally thought of as a process that involves invariance of mental representations (e.g., Gick & Holyoak 1983; Reed, 1993; Singley & Anderson, 1989, cited in Greeno et al., 1993). According to this view, a mental representation is acquired during initial learning to support a certain action in a given situation. If that initial representation is adequate to support the same action in a new situation, transfer may occur (Gick & Holyoak, 1983, cited in Greeno et al., 1993). Mental representations that are invariant across several situations are also referred to as production rules or schemata. For example, symbolic schemata are the mental representations that capture the meaning of written mathematical formulas and that are invariant across situations (Reed, 1993, cited in Greeno et al., 1993). Transfer may also occur if representations obtained in the initial situation sufficiently overlap with the representations acquired in the new situation (Singley & Anderson, 1989). The implication of this view is that the invariance of transfer is a psychological phenomenon (Lobato, 2003).

However, Greeno et al. (1993) argue that mediation via invariant mental representations is not the only way that transfer occurs. Specifically they maintain that transfer can also be explained by invariance across activities as individuals interact with material resources and with other people in situations. When the particular ways in which an agent interacts with a situation remain the same as an initial learning situation is transformed into a transfer situation, transfer may occur.

An important part of this alternate view is that there exist supports for
particular activities, called *affordances* and *constraints*, that are created by the properties of materials in a given situation, (Greeno et al., 1993). The classic example of an affordance, from Gibson (1979/1986, cited in Lobato, 1996), is a chair that affords sitting. This example raises the additional point that affordances are not just self-attributes of particular materials and things but are the attributes of the relationships between materials and agents, since, by the previous example, “a chair affords sitting for a human but not for a hippopotamus” (Lobato, 1996). Greeno et al. (1993) further elaborate that there are also conceptual affordances such as a rectangle that affords multiplying the dimensions to find the area.

Greeno et al. (1993) argue that affordances (and constraints) play an important role in transfer. The invariant interactions across situations, described above, are supported by affordances. When transfer occurs from one situation to another, this implies that there are affordances across the two situations that support the same actions. Greeno et al. (1993) explain that affordances can be mental representations but they are more often supports that are distributed throughout the environment. Therefore, this alternate view is that invariance is distributed among objects, people and other aspects of a situation.

The following example illustrates Greeno’s affordances and constraints perspective on transfer. In a study by Hendrickson and Schroeder (1941, cited in Greeno et al., 1993)\(^8\), involving boys shooting an air gun at a target submerged in

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\(^8\) This experiment is a recreation of a classic experiment by Scholckow and Judd (1908, cited in Greeno et al., 1993). Greeno et al. (1993) refer to the re-creation rather than the original because the recreation provided a better description of methods and results.
water, boys in the experimental group receive instruction about the refractive properties of water prior to shooting at the target, while boys in the control group received no instruction. Results showed that the boys that participated in the instruction did significantly better. One way to interpret these results is that the boys in the experimental group acquired a mental representation during the instruction that significantly overlapped with the mental representations they formed for the shooting task which resulted in the knowledge formed during instruction being applied to the task of shooting at the submerged target.

Greeno et al. (1993) suggest that there are alternate interpretations involving invariant affordances that could also account for the improved performance. For example, a boy may have noticed that in order to hit the center of the target, the apparent path of the bullet in the water forms a certain angle with the actual path of the bullet before it enters the water. This property is an affordance because it supports hitting the center of the target. It is distributed across the situation (i.e., across target, water, bullet, gun, boy, etc) and if the depth of the water were to change, the angle, and hence the affordance, would remain invariant. This invariant affordance could support transfer of the knowledge of how to hit the submerged target’s center at one depth to hitting the target’s center at other depths. Greeno et al. (1993) go on to explain that a possible reason that the boys in the experimental group benefitted is because the talk about bending rays of light during instruction attuned them to the affordance of the angle formed by the apparent and actual path of the bullet. It would then have been more likely for the experimental group than the control group to use that affordance.
A similar interpretation to that made about the dart-throwing experiment could be made of the pilot data presented in Chapter 1. For example, it is plausible that the ordered data table, with its numerical values being spatially situated one on top of the other, afforded the actions associated with finding the first and second differences. This kind of affordance would be invariant across data that is linear and that is quadratic. Moreover, it may have been that the quadratics instruction attuned Matt to the affordance of finding first and second differences in the linear context, even though reasoning proportionally would have been more productive.

The AOT approach has its origins in the affordances and constraints perspective (Lobato, 2003). Both perspectives agree that invariance is distributed across materials and things in the situations. However, Greeno et al. (1993) make no explicit distinction between affordances as defined by an expert and idiosyncratic affordances as seen by each individual. In contrast, the actor-oriented perspective emphasizes that it is the “personal perception of affordances” (Lobato, 2003, p. 19) that influences what transfers. Lobato describes this as explaining transfer by accounting for both generalizing assimilation (Piaget, 1977, cited in Lobato, 2003) and the structuring of the situation by the relevant artifacts. In the context of mathematics instruction, the relevant artifacts are often related to the mathematical tasks. For example, Lobato, Ellis and Muñoz (2003) showed that graphing calculators provided structure to graphing activities: the graphing calculators required students to enter the scale for the \( x \)-axis which directed focus to the scale of the \( x \)-axis. This structuring resource was consequently related to the transfer effects reported in the study that some students associated the slope of a linear function (incorrectly) with the scale of
Lobato et al. (2011a) further elaborate that, according to the AOT approach, the personal perceptions that emerge, called *centers of focus*, are socially organized by the following features of a mathematics learning situation: (a) *focusing interactions* (discourse practices, such as gesture, talk and drawing diagrams), (b) *features of mathematical tasks*, and (c) *the nature of the mathematical activity* (collectively-held structures, such as the routines that govern the ways that students and teachers participate).

The distributed nature of transfer is also emphasized in Beach’s (1999) *consequential transitions perspective*. From this perspective, transfer is distributed across the transformations of social roles, knowledge and activities. For example, Beach counted as transfer, the changing roles of Nepalese students as they transitioned to becoming shopkeepers. However, rather than accounting for transfer with the invariance across situations, Beach accounted for transfer with continuity (and transformations) across situations, where continuity is seen as emerging out of the dialectic between the individual and the context. For example, in a study of machinists transitioning from mechanical to computerized machining, Beach described the dialectic created by changes in the machinists’ activities and changes in the technology as somewhere between a pure continuity and a pure discontinuity.

The kinds of consequential transitions that Beach (1999) wrote about, such as the transition from being a student to being a shopkeeper, are on a much larger scale than the transitions that students would experience when participating in 16 lessons on quadratic functions. In other words, students’ identities and social roles in given
social organizations would likely not be greatly affected by the relatively short instructional intervention. Therefore, the AOT approach seemed more appropriate for the dissertation study.

**Location of invariance for backward transfer.** My literature search did not reveal any research of backward transfer in which the location of invariance was made explicit. One reason for this may be that backward transfer has been relatively unproblematic for researchers to produce, particularly in the field of linguistics research (e.g., Su, 2001; Tsimpli et al., 2004). Thus, perhaps it was not deemed necessary to make explicit the assumptions regarding invariance. By examining the explanations offered by researchers for the occurrence of backward transfer, I have inferred that some researchers locate invariance in the psychological plane and others locate it in the social plane.

Locating backward transfer in the psychological plane resonates with the “overlapping mental representations” perspective on forward transfer described above. However, the composition of these mental representations is not always as clearly articulated for backward transfer. For example, Chiang (2003) found that phonological training of five- and six-year-old children in L2-English, led to greater phonological awareness in L1-Chinese. Chiang attributed this to the presence of an underlying abstract ability. In other words, Chiang argued that behind the two different abilities that bilingual speakers possess, there is an invariant mental process. Chen (2006) described the backward transfer of L2 onto L1 as generalizing underlying discourse competence, where generalizing discourse competence is defined as “the ability to produce context-reduced academic prose in both L1 and L2 as a function of a
common underlying cognitive academic language proficiency” (Carson & Kuehn, 1992, p.159, cited in Chen, 2006). Similarly, Tsimpli et al. (2004) found backward transfer effects involving the interpretation of L1, for native speakers of Greek or Italian who are bilingual in English. In particular, they found backward transfer effects from L2 onto L1 for interpretable (i.e., semantic/pragmatic) aspects of grammatical subjects but not for uninterpretable (i.e., syntactic) aspects. I interpreted this as invariance being attributed to an overlap in mental representations across languages.

Several backward transfer researchers described invariance as a psychological and physiological phenomenon. Kelso and Zanone (2002) found that subjects’ natural ability to coordinate the swinging of two body parts (two arms, two legs or one arm and one leg) could be altered by getting them to practice swinging their limbs at a particular relative phase difference. For example, if subjects were trained to swing an arm and a leg at a phase difference of 75°, then on future trials when asked to swing limbs at different phase differences, performance was attracted to 75°, when compared to performance before training. Kelso and Zanone made reference to the invariance, explaining it as “a high-level but neurally instantiated dynamic representation [italics added] of skilled behavior that proves to be largely effector independent, at least across anatomically symmetric limbs” (p. 776).

A different assumption regarding invariance is that backward transfer is distributed across the social plane. Buckley et al. (2003) found that knowledge flow from affiliates back to headquarters of a parent firm depended, in large part, on the congruence of goals of all parties involved. In other words, congruence of goals is an
aspect of invariance across both organizations that supports backward transfer. Millar and Choi (2009) explain that for knowledge to flow from a subsidiary back to headquarters located in another country, one essential factor is that the perceived psychic distance—the level of perceived attachment—between organizations in two different countries be minimal. When the organizations are psychically close, backward transfer is more likely to be successful. Psychic distance appears to reflect the degree to which cultural features are invariant from one culture to another. Porte (2003) found that speakers in a closed community with a common L1 and L2 often exhibit backward transfer effects from L2 to L1. He explained part of this effect as being the result of a tacit agreement between speakers within the community that sanction code manipulation of L1. Thus, the agreement between speakers functions like an agreed-upon cultural invariance, which supports L2 use in L1 situations.

In summary, situating the location of invariance in the psychological plane is common for traditional studies of forward transfer and backward transfer alike. Alternative transfer perspectives often assume a distributed view of transfer across mental, material, and social planes. Similarly, some studies of backward transfer have located invariance in both the physiological and social planes. The lack of research on invariance for backward transfer being distributed across a material plane is likely a function of the particular interests of the fields (e.g., linguistics), as opposed to an indication that material resources of the environment are inconsequential for backward transfer.

Processes. In this review, I take transfer processes as the explanatory accounts offered by researchers for how and why transfer takes place. The traditional transfer
paradigm traces its conception of transfer processes back to Thorndike’s (1908, cited in Detterman, 1993) theory of identical elements, in which transfer occurs when learning and transfer situations share common elements (typically interpreted as shared features of physical environments). In later mainstream cognitive approaches to transfer, common elements were reformulated as mental representations, and a variety of cognitive transfer processes were identified, including (but not limited to) analogical mapping, constraint satisfaction, and encoding specificity (Gentner, Loewenstein, & Thompson, 2003; Nokes, 2009; Sternberg & Frensch, 1993). As alternative models of transfer have emerged, so have explanations of how social environments and artifactual resources afford and constrain the generalization of learning (Engle, 2006; Lobato, 2006; Lobato et al., 2011a).

In this dissertation, I offer an explanatory account of the occurrence of backward transfer by examining conceptual links between particular classroom practices from an instructional unit on quadratic functions and related changes in the students’ reasoning with linear functions (in Chapter 6). In an effort to identify potential processes that might account for the occurrence of backward transfer, I first review processes from outside transfer research. There are a number of well-known processes, not specifically tied to transfer research, which appeared to be viable backward transfer processes. It seemed prudent to address these processes first and show why they fall short of fully accounting for the backward transfer phenomenon. Then, I elaborate why *noticing*, the process used to account for transfer from an AOT approach, holds promise for explaining backward transfer.
Potential backward transfer processes. Seven potential backward transfer processes are elaborated in this section (see Table 2.1). The following four processes occur without the explicit acquisition of new knowledge: (a) forgetting (Ebbinghaus, 1885/1913, cited in Estes, 1987), (b) representational redescription (Karmiloff-Smith, 1992), (c) knowledge-in-pieces (Izsák, 2005; Smith et al., 1993), and (d) folding back (Pirie & Kieren, 1994; Pirie, Martin & Kieren, 1996). The other three processes explicitly involve the acquisition of new knowledge: (e) the recency effect (Glenberg, Bradley, Kraus, & Renzaglia, 1983; Glenber, & Swanson, 1986), (f) resubsumption/assimilation hypotheses (Ausubel, 1963; Ohlsson, 2009; Shuell, 1986), and (g) retroactive inhibition (Isurin & McDonald, 2001; Mayr & Keele, 2000). Each is considered in turn.

The first potential backward transfer process is forgetting. Hermann Ebbinghaus said, “All sorts of ideas, if left to themselves, are gradually forgotten” (1885/1913, p. 62, cited in Estes, 1987). Using himself as the subject, he studied learning of nonsense syllables. His data showed that the relationship between time and the retention of memory could be modeled with a smooth, monotonic, decreasing function which first decreases rapidly and then gradually levels off and which has been called a retention curve and a forgetting curve (Rubin & Wentzel, 1996). For example, Ebbinghaus found that on average he forgot more than 60% of the nonsense syllables within the first eight hours after learning them, and forgot less than 20% over the next 30 days (Baddeley, 1999).
Table 2.1. Processes that were considered as potential explanations of backward transfer effects.

<table>
<thead>
<tr>
<th>Processes that occur before or separate from acquiring new knowledge</th>
<th>Processes</th>
<th>Researchers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Forgetting</td>
<td>Ebbinghaus</td>
</tr>
<tr>
<td></td>
<td>Representational Redescription</td>
<td>Karmiloff-Smith</td>
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<tr>
<td></td>
<td>Knowledge-in-Pieces</td>
<td>Smith, diSessa &amp; Roschelle</td>
</tr>
<tr>
<td></td>
<td>Folding Back</td>
<td>Pirie, Martin &amp; Kieren</td>
</tr>
<tr>
<td>Processes that occur in conjunction with or after acquiring new knowledge</td>
<td>Recency Effect</td>
<td>Glenberg et al.</td>
</tr>
<tr>
<td></td>
<td>Resubsumption/Assimilation Hypothesis</td>
<td>Ohlsson Ausubel</td>
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<tr>
<td></td>
<td>Retroactive Inhibition</td>
<td>Isurin &amp; McDonald Mayr &amp; Keele,</td>
</tr>
</tbody>
</table>

It is possible that what looks like backward transfer might be partially accounted for with forgetting. In other words, perhaps changes in prior knowledge that are attributed to backward transfer are due to forgetting. For example, the difficulties that Matt experienced may have been the result of forgetting aspects of linear functions knowledge in the intervening 15 days between interviews. However, as stated above, the rate of decay of memory is greatest immediately after the memory is created. According to Ebbinghaus’ retention function, if the significant change in understanding about linear functions could have been explained by a significant amount of forgetting, Matt’s knowledge about linear functions would need to have been created immediately prior to the pre-interview. This was not the case. We know that Matt and his fellow 8th graders had studied aspects of linear functions already in
earlier grades, as well as in the months prior to the interview. Therefore, it is likely that memory decay of linear functions in the time between interviews was minimal.

A second potential backward transfer process is *representational redescription* (RR). Karmiloff-Smith (1992) defines RR as the recoding of representational information in a different form (e.g., recoding proprioceptive knowledge into linguistic knowledge, linguistic knowledge into spatial knowledge, etc). Each recoding of knowledge is more condensed and more explicit than the last form. Karmiloff-Smith describes the personal example of her learning how to solve a Rubik’s Cube. Initially, she developed a proprioceptive method for solving the puzzle. However, she had not yet developed an explicit understanding of what she was doing. Therefore, she could not explain to her daughter how to solve the puzzle. Later, she began to recognize certain states of the cube. This more explicit form of the understanding was developing through RR. Eventually, she was able to explain to her daughter how to solve the puzzle.

There are several features of Karmiloff-Smith’s construct that align with backward transfer. For example, as understanding becomes more explicit, performance may temporarily drop because the learner begins to rely less on automated actions and more on explicit knowledge. Thus, RR could potentially account for the unproductive backward transfer observed with Matt. Additionally, RR can occur in the midst of internal stability, spontaneously, and off-line. These features might account for why Matt exhibited an effect that appeared to occur spontaneously, unconsciously, and without any apparent perturbation.

Despite these promising features, it does not appear that RR fully accounts for
Matt’s reasoning. As demonstrated in the first interview, Matt already had a fairly explicit understanding of linear functions. For example, he was able to explain why the data were linear, using proportional reasoning. This suggests that the effect was not the result of a movement from implicit to more explicit understanding.

The third potential process that might explain backward transfer comes from Smith et al.’s (1993) theory of knowledge-in-pieces. According to this theory, learners build on and refine their novice conceptions rather than replace them with more expert-like conceptions. For example, in a study by Izsák (2005), a fifth-grade student showed evidence of refinement and reorganization of prior knowledge when she went from thinking that to-scale drawings and unit squares were needed to determine areas of rectangles, to thinking that unit rectangles and not to-scale drawings of rectangles could be used as long as the unit rectangles had the same row and column structure that unit squares would have on to-scale drawings.

It is possible that the knowledge-in-pieces perspective may account for the productive backward transfer of Kala, which was described in Chapter 1. Perhaps, by learning about quadratic functions she was able to reorganize her thinking about linear functions. However, it seems more difficult to imagine that refinement and reorganization are principally responsible for Matt’s unproductive backward transfer, particularly since he showed a fairly sophisticated understanding of linearity in the first interview.

The fourth potential process is folding back. Pirie and Kieren (1994) and Pirie, Martin and Kieren (1996) define folding back, in the context of their theory of the levels of growth of understanding, as retreating to prior levels of understanding in the
face of encountering a problem that is not immediately solvable. Because a learner’s actions at an inner level during folding back occur with the intention of informing actions at an outer level, the actions will be different than those the learner performed when initially attaining that level. As a result, the understanding at the inner level thickens (hence the folding metaphor) and becomes more sophisticated and enriched. Once the inner level thickens, it supports further growth of understanding at the outer levels. I categorized this process as not explicitly involving acquiring new knowledge because, according to Pirie and Kieren (1994), new knowledge is not yet acquired when folding back occurs.

It is possible that backward transfer is an instance of folding back. There are several aspects of folding back that seem to align with backward transfer. First, both ideas assume that understanding develops non-linearly. Specifically, both backward transfer and folding back are about prior knowledge not being fixed but continuing to evolve even as students grapple with more sophisticated concepts. Second, both ideas assume that the relationship between prior knowledge and new knowledge is dynamic. For example, according Pirie et al. (1996), what happens at the outer level of understanding influences what happens at inner levels and vice versa. Similarly, backward transfer involves new knowledge influencing prior knowledge.

Despite the significant alignment, there are several reasons why folding back and backward transfer do not seem to be synonymous constructs. First, folding back is defined as a step in acquiring new knowledge, whereas backward transfer appears to be a consequence of acquiring new knowledge. Second, folding back is invoked by the teacher, the instructional material, a peer, or oneself (Martin, 2008), while
backward transfer appears to occur spontaneously without invocation. Third, folding back involves “knowing that you know what you need to know” (Pirie, Kieren & Martin, 1996, p. 147), and backward transfer does not. In other words, folding back is a deliberate and intentional process whereas backward transfer might be unintentional. Finally, folding back is primarily productive while the processes identified in the pilot data resulted in both productive and unproductive backward transfer.

As stated above, the next three potential backward transfer processes involve an explicit association with accumulating new knowledge. The recency effect is defined as best recall of the most recent information even after a considerable and filled retention interval (Bjork & Whitten, 1974; Glenberg et al., 1980; Glenberg & Swanson, 1986). The size of the recency effect is measured by comparing accuracy of memory of the stimuli presented first with accuracy of memory of stimuli presented second. The recency effect has been shown to impact not only short-term but also long-term memory. For example, Glenberg et al. (1983) showed that the recency effect occurred over time spans of anywhere from 4 s to 7 days. Furthermore, they found that the size of the recency effect was positively correlated to the size of the ratio of the interpresentation interval (time between presentation of first and second stimuli) to the retention interval (time between presentation of second stimuli and recall test). In other words, the longer the time between two stimuli, when compared to the length of time from the second stimuli to the recall test, the more likely a subject is to remember the second stimuli more accurately than the first.

One might argue that backward transfer is an instance of the recency effect. In particular, it is possible that what looks like an influence on prior knowledge by new
knowledge may simply be new knowledge being better recalled than prior knowledge. However, the recency effect involves changes in ability to recall prior knowledge, whereas the phenomenon of interest for this dissertation study involves changes in understanding.

Two related processes are resubsumption and assimilation.⁹ According to Ausubel’s (1963) assimilation hypothesis, assimilation is described as the processes of prior knowledge getting subsumed under a more inclusive new concept. As a result, the prior knowledge becomes less dissociable from the new knowledge. According to Ohlsson (2009), resubsumption occurs when newly acquired knowledge contradicts prior knowledge, and as a result, some phenomenon or experience that was previously understood with the prior knowledge, comes to be understood with the newly acquired knowledge instead. It is possible that resubsumption or the assimilation hypothesis can account for how the two students in the pilot data seemed to be able to access a specific aspect of linear functions understanding during the pre-interview but not during the post-interview. For example, perhaps Matt’s explicit understanding of what makes a set of data linear was subsumed by his conception of quadratics so that it was no longer dissociable. However, quadratic functions for 8th graders would be a fairly different topic from linear functions. It is doubtful that Matt saw quadratic functions as a concept that could subsume linear functions, or that he saw quadratic and linear functions as contradictory.

The final potential process, retroactive inhibition (also known as backward

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⁹ Ausubel’s assimilation hypothesis is different from Piagetian assimilation.
inhibition) is defined as new information interfering with retrieval of older information (Isurin & McDonald, 2001) or as a new task set suppressing the previous task set (Mayr & Keele, 2001). Mayr and Keele proposed that retroactive inhibition functions “as a counterforce to the persistent-activation property of control settings and would thus ‘clear the slate’ for currently relevant task sets” (p. 5). Studies have shown that when an experimental group learns material first from one list and then from a second list, while a control group learns material from the first list only, the experimental group does more poorly on recall of items from the first list compared to the control group (e.g., Barnes & Underwood, 1959; Blank, 2005; Hübner, Dreisbach, Haider, & Kluwe, 2003; Isurin & McDonald, 2001; Mayr & Keele, 2001; Melton & Irwin, 1940).

There are several theories about how retroactive inhibition occurs (Estes, 1987). One theory is that it is the result of competition between memory traces that have common elements (McGeoch, 1932, cited in Estes, 1987). Hübner et al. (2003) found that the greater the competition, the greater the inhibition. Since the linear and quadratic functions that Matt explored shared common elements (e.g., involving relationships between independent and dependent variables, tabular representation, etc), it is plausible that his memory traces of linear and quadratic functions were competing. Nevertheless, retroactive inhibition does not seem like a plausible mechanism to explain productive effects like those exhibited by Kala.

In summary, I reviewed seven processes from outside the transfer literature that might account for backward transfer. Each of these processes seemed to insufficiently account for either productive or unproductive backward transfer effects.
Transfer processes. The argument in Chapter 1 for why the phenomenon of interest should be situated within transfer research was based on how closely the definition of transfer from an AOT approach aligned with the observations that came from the pilot data. Part of that argument was that noticing, which has been posited by Lobato et al. (2011a) as a transfer process, shows promise as an explanatory frame to account for the occurrence of both productive and unproductive backward transfer. In this section, Lobato et al.’s conceptualization of noticing is presented, as is the focusing framework, which is an effort to coordinate what individual’s notice with the social organization of noticing. Then, noticing is compared to other transfer processes that have also been identified by taking alternate approaches to transfer.

Noticing refers to selecting, interpreting and working with particular mathematical features or regularities when multiple sources of information compete for students’ attention (Lobato, et al, 2011b). While noticing can involve orienting (turning toward a sensory signal, such as directly one’s gaze to a graph of a quadratic function), it is closer to what Posner and Petersen (1990) call executive attention, namely the selection of certain information in the presence of competing sources of information. As an individual psychological process, noticing is rooted in one’s comprehension of a situation and can be captured by the initial aspects of reflective abstraction, such as creating and re-presenting mental records of one’s activities, isolating particular characteristics of mathematical objects or events from the experiential flow, and identifying regularities among the records related to the learner’s goals (Campbell, 2001; von Glasersfeld, 1995). For example, Piaget (1952) used the notion of “focus of attention” (or “centration”) to describe the tendency of
young children to notice only one salient aspect of the situation at a time, to the exclusion of other potentially relevant aspects.

Because this noticing process is not directly accessible, Lobato et al. (2011a, 2011b) examine the observable products of noticing instead, what they call *centers of focus*. *Centers of focus* are the properties, features, regularities, or conceptual objects to which individual students attend. An example of a center of focus for a quadratic function is the change in values of the dependent variable, meaning that students’ likely orient their gaze toward the values of the dependent variable (possibly to the exclusion of the independent variable), mentally form differences between consecutive values of the dependent variable and then attend to those changes.

However, Lobato and colleagues (2011a, 2011b) conceive of noticing not only as an individual, psychological process but also one that is socially organized. According to Goodwin (1994), there are “socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group” (p. 606). In his seminal paper, *Professional Vision*, Goodwin (2004) demonstrated how people from different professions (e.g., police personnel or archaeologists) view a particular event (e.g., a videotape of the Rodney King beating or the color of dirt at a dig site, respectively) through perceptual frameworks that are developed within those professions. Correspondingly, Lobato et al. (2011a, 2011b) examine socially organized ways of seeing and understanding events that are distinctive to particular classrooms in schools. To that end, they use the construct of *focusing interactions* to examine the discursive practices that give rise to particular centers of focus.
Discourse here is conceived of broadly to span a range of communicative modes, including gesture, talk, and inscriptions. Two particular discourse practices involved in the social organization of noticing are *highlighting* and the use of *coding schemes* (Goodwin, 1994). Highlighting refers to visible operations upon external phenomena, such as labeling, marking, and annotating, which can shape the perceptions of others by making certain features of the perceptual field prominent. A coding scheme (what Lobato et al. call *naming* or *renaming*) refers to the use of a category of meaning or a classification scheme, which in turn, can provide a lens that is located in a larger organizational system than the individual. An example of a focusing interaction involving quadratic functions would be the discursive practice of highlighting tabular displays by putting brackets between rows of dependent variable values (see Matt’s inscriptions on Figure 1.2). By adding these brackets, the students and teacher may direct their own attention and the attention of others toward changes in the dependent variable. Lobato et al. conceive of these particular discursive practices as social processes of noticing.

It is important to note that discursive practices do not occur in a vacuum but rather are situated in the *mathematical tasks* that are utilized and in the *nature of the mathematical activity* that are present in a classroom. As such, both can play a role in the social organization of noticing. Specifically, *mathematical tasks* form the backdrop for discourse practices because they are the situations that students and teachers discuss. Features of those tasks afford and constrain what centers of focus emerge. For example, a task that involves students drawing diagrams of real-world quadratic function contexts affords students the opportunity to notice relationships
between measureable quantities (Dougherty, 2008).

The nature of the mathematical activity refers to the participatory organization that establishes the roles and expectations governing students’ and teachers’ actions (Cobb & Yackel, 1996) that contributes to the emergence of particular centers of focus (Lobato et al., 2011a). For example, in a classroom where it is common practice that students justify their thinking, a center of focus for quadratic functions will likely be different than in a classroom where justifying is not expected or encouraged. In the first classroom, the center of focus may incorporate student ideas. In the second classroom, the center of focus may be on the ideas from a mathematical authority, such as ideas from the teacher or a text book.

In sum, noticing is posited as an umbrella process, comprised of both individual and social sub-processes. Together, centers of focus, focusing interactions, task features, and the nature of mathematical activity form Lobato et al.’s (2011a) focusing framework, which provides a way to coordinate individual and social aspects of noticing. The focusing framework is reminiscent of Cobb and Yackel’s (1996) emergent perspective, without the positing of collective constructs and with noticing as the unifying theme (Lobato & Hohensee, 2008). Individual noticing processes are taken as beginning aspects of reflective abstraction, which are observable as centers of focus—the products of students re-presenting perceptual images to themselves (i.e., as indicators of what was noticed). Social noticing processes are taken as the particular discursive practices that give rise to centers of focus, which in turn are inextricably situated within the use of particular mathematical tasks and within the nature of the mathematical activity present in the classroom.
The focusing framework is illustrated in Figure 2.2. The centers of focus are represented by the circular targets. Each black dot represents an individual student. Several dots are clustered on a particular target to represent students who share a similar center of focus. The ovals that surround each target represent the focusing interactions that direct student attention toward particular centers of focus. The octagons that surround the focusing interactions represent particular tasks. When the task changes, the focusing interactions and the centers of focus may change, as indicated by the arrows. The large rounded rectangle represents the nature of the mathematical activity that cuts across tasks. Of course, the nature of mathematical activity could also change. If so, it could result in changes to the focusing interactions, the centers of focus and the ways that individual students group themselves around particular foci.
As stated earlier, the AOT approach to transfer emerged as a response to critiques of the traditional approach. Differences between the AOT approach and the traditional approach are further reflected in the kinds of processes that each approach uses to account for transfer. Generally speaking, the traditional cognitive approach to transfer posits a process of matching or overlapping schemata (Anderson, 1996; Gentner & Markman, 1997; Greeno et al., 1993; Haskell, 2001; Holyoak, 1984, cited in Greeno et al., 1993; Reed, 1993; Singley & Anderson, 1989). More fine-grained processes also include: (a) encoding specificity and organization, (b) discrimination, and (c) analogical reasoning (Anderson, 1989, cited in Singley & Anderson, 1989; Babb & Crystal, 2006; Chi & Glaser, 1985; Haskell, 2001; Nokes & Ohlsson, 2005; Sternberg & Frensch, 1993, cited in Lobato, 1996; Tulving, 1985; Tulving &
Thomson, 1973). The way that information is encoded and organized in memory can contribute to whether or not schemata overlap will lead to transfer. The way that learners discriminate between prior knowledge that is and is not relevant for transfer also influences whether transfer will occur. From this perspective, transfer is the process of applying an analogy to a source schema which results in a mapping from the source to a target schema. Additionally, analogical reasoning results in a more decontextualized source schema.

There are at least three important differences between processes that are used to account for transfer from a traditional perspective and the process of noticing. First, the traditional perspective processes do not account for distributed or social aspects of the generalization of learning, while noticing does. Second, the traditional perspective processes do not account for dynamic changes to schemata, while the interplay between components of the focusing framework does. Third, the traditional processes account for expert-like performance while noticing accounts for influences on thinking whether the thinking is expert-like or not.

Besides noticing, there are at least three additional processes that have been identified by alternate approaches to transfer. Greeno et al. (1993) explained transfer as an *attunement* to invariant affordances and constraints. When an individual is attuned to the affordances and/or constraints of a new situation, which are invariant from a previous situation, then one’s ways of participating in the activities associated with the previous situation may transfer to the new situation. This process emerged as a response to the critique that traditional transfer researchers were separating problem-solving from everyday life, and that they were not accounting for the situatedness of
transfer (Lave, 1988; Greeno et al., 1993). Invariant affordances and constraints are most closely aligned with the mathematical tasks component of the focusing framework because the focus is primarily on the material world.

Engle (2006) offered the mechanism of *framing* to account for transfer. *Framing* is described as a process of determining the social acceptability of using certain knowledge in a given situation. Framing is also involved in the way that knowledge is initially acquired. If knowledge acquisition is framed in a way that links the current context with another context, transfer to the other context will be more likely. For example, in a study of students learning about whale endangerment, Engle observed students transferring their graded and multicausal explanations about whales to other animal endangerment contexts. Engle accounted for this transfer as an instance of framing: the teacher framed the exercise of creating an explanation of whale endangerment by positioning the students as the authority about whale endangerment and by conveying to students that their opinion served a wider community beyond the classroom. Framing emerged as a transfer process out of efforts to adapt a situated cognition perspective to the transfer of learning, which responds to the criticism that the traditional approach did not account for contributions from the social context in which transfer occurs.

Noticing and framing share the characteristic that both attend to social aspects of transfer in classroom situations. In fact, noticing can be thought of as an extension of the framing mechanism because it accounts specifically for what students notice mathematically in mathematics classrooms and how discursive practices contribute to what gets noticed (Lobato et al., 2011a).
Beach (1999), who renamed transfer as *consequential transitions* and who redefined it as the transformation of knowledge and identity, offered the process of *participation* as a way to account for the social aspects of transfer. Moreover, he identified four participation patterns that each result in a particular form of consequential transitions: (a) moving between preexisting activities is a lateral transition, (b) participating in two activities simultaneously is a collateral transition, (c) changing participation within an activity is an encompassing transition, and (d) participating in a simulated activity is a meditational transition.

Noticing and Beach’s process of participation share the feature that both account for social aspects of transfer. However, the two processes account for transfer at different grain sizes: noticing can account for small transfer effects at the classroom-learning level, while the participation that leads to consequential transitions can account for transformations that involve larger changes in personal identities and social activities.

Within the limited literature on backward transfer, the processes that are used to account for backward transfer are primarily cognitive. For example, Chen (2006) found an effect when L2 speakers had limited exposure to their L1. He explained backward transfer as a rule-governed process and said that learners compare language structures in L1 and L2 so as to decide which structure is simpler.

This section concludes with a brief discussion about how noticing as a transfer process and the focusing framework was used in the dissertation study. Lobato et al., (2011a) showed that the focusing framework could be employed to explain what was noticed in the classroom and to link that to transfer (as defined by the AOT approach).
In answering the third dissertation question, the framework was employed in a parallel way to try to account for the backward transfer effects produced by the instructional intervention (see Chapter 6 for details). In other words, first backward transfer effects were identified as changes in reasoning from pre- to post-interview (the results of which are presented in Chapter 4). Second, what students noticed mathematically during the instructional intervention was identified by analyzing the classroom data for emerging centers of focus. Third, the classroom data were analyzed for the focusing interactions, the features of mathematical tasks, and the particular mathematical activities that appeared to give rise to the centers of focus. Finally, conceptual connections were identified between the backward transfer effects found in the post-interview and what students noticed in the instructional intervention. General methodological approaches to the study of backward transfer presented in this dissertation, as well as contrasting approaches to the investigation of transfer in a variety of research perspectives, are considered in the next section.

**Investigation of Transfer and Backward Transfer**

As discussed earlier, several critiques have been made of the way that transfer has been defined by the traditional approach, and from those critiques alternate definitions were developed. However, it may not be entirely clear from the traditional and alternate definitions of transfer what the implications of the various definitions are for transfer research (Lobato, 2008a). Therefore, in this section the various methodological approaches to the investigation of transfer will be reviewed.

This part of the review is organized by the sub-categories belonging to the *investigation* category of the conceptual framework (see Figure 2.1). Specifically, the
point-of-view and the methodology that the researchers have used to investigate transfer will be presented. The methodology sub-category is further sub-divided into two sub-categories: (a) the kinds of research questions that researchers of transfer have asked through their research, and (b) the approach that researchers of transfer have used to establish causality. The approach used by the dissertation study will be situated within this review. It will also be shown that, in most cases, the methods used to investigate transfer align with the ways in which researchers conceptualize transfer. However, it will also be shown that there are some studies in which this is not the case.

**Point-of-view used to study transfer and backward transfer.** The way that a researcher defines transfer is often related to the point-of-view (POV) that the researcher adopts. In particular, the POV relates to the perceived agency of transfer and the unit of analysis that is utilized. Therefore, different POVs often lead to very different interpretations of transfer data. The three POVs represented in transfer research are (a) the individual expert, (b) the individual actor/learner, and (c) the collective. Researchers take on one POV exclusively or use different POVs for different purposes within the same study.

Most traditional transfer studies adopt the individual expert’s or observer’s POV. This means that a researcher studying transfer focuses on assessing whether or not subjects recognize what has been predetermined by an expert to be the surface differences and structural similarities between learning and transfer tasks (Lobato, 2008a). This was not a POV with which researchers have characterized themselves but rather a label that emerged in critiques of the traditional approach to transfer. One
critique was that research from a traditional approach amounts to subjects trying to match the researcher’s expectations (Lave, 1988). A second critique was that the scant amount of transfer that researchers were able to produce with a traditional approach was irreconcilable with prevailing beliefs that the generalization of learning experiences is ubiquitous and that students do transfer their schooled learning outside of school (Detterman, 1993; Barnett & Ceci, 2002; Lobato, 1996).

In response to these critiques, Lobato (2003) used Mackay’s (1969, cited in Lobato, 2003) notion of observers and actors to distinguish between the expert’s and the learner’s POV in transfer research. As a result, she helped explain why traditional studies have produced so little transfer, namely that by using an expert’s POV, researchers ignored ways in which learning generalized, which could only be discerned when taking the actor’s POV (Lobato, 1996).

To illustrate an expert’s POV and the resulting underestimation of the amount of transfer produced, consider Matt’s responses to the swimming fish task from the post-interview as presented in Chapter 1 (see Figure 1.3). In his calculation of the acceleration, Matt failed to account for the 2 s intervals. Instead, he found the 2nd differences between intervals 2 s apart and concluded that the acceleration would be “40 feet per second per second.” From an expert’s viewpoint, Matt did not exhibit transfer in this episode because his performance was incorrect, and thus, not expert-like (i.e., the acceleration should have been 10 ft/sec/sec). However, in the instructional session Matt had many experiences with finding 2nd differences, experiences that appeared to be influencing his thinking in this episode. Therefore, using the expert’s POV to interpret this data would result in an underestimation of the
amount of transfer produced (i.e., not fully account for the ways in which Matt
generalized his learning experiences).

When transfer researchers predetermine the norm or standard by which transfer
will be counted, it is an indication that they are assuming an expert’s POV. This POV
has dominated transfer research (e.g., Brown & Kane, 1988; Camarata et al., 2009;
Gentner, Loewenstein, & Thompson, 2003; Gick & Holyoak, 1980; Hendrickson &
Schroeder, 1941; Kotovsky, Hayes, & Simon, 1985; Singley & Anderson, 1989;
Wertheimer, 1945/1959). For example, in a study by Reed, Ernst and Banerji (1974),
subjects learned the solution to the missionary-cannibals problem, which involved
determining how five missionaries could safely cross a river on a boat that holds three
people if five cannibals were on the other side. After subjects learned the solution,
they were given a new problem involving jealous husbands, which could be solved
using a similar solution to the missionary-cannibals problem. Correctly solving the
jealous husbands problem was taken as evidence of transfer. Despite the solution to
both problems being virtually identical, this study found very little evidence of
transfer.

In contrast, the AOT approach was named for taking an alternative POV,
namely that of the individual actor or learner. This means that there is no
predetermination about what relations of similarity will be counted as such between a
learning and transfer task. Instead, any similarity that an individual generates between
contexts counts as transfer (Lobato, 2003). Evans (1999) assumed an actor’s POV
when he observed that during some questions about a newspaper graph, a student in
the study made a connection with college mathematics and during other questions, the
student made a connection with finance mathematics. Because this POV is less restrictive in terms of what counts as transfer, researchers typically find more evidence of transfer using this POV than using the expert’s POV (Lobato, 2008a, 2008b). For example, Lobato (1996) analyzed interview data of students finding slope for house roofs and playground slides, using first the expert’s POV and then using an actor’s POV. With the first POV, evidence of transfer was found in only 40% of the students. With the actor’s POV, every student provided evidence of transfer.

There are three reasons why adopting an actor’s POV for the study of backward transfer in the dissertation study is important. First, analysis of the pilot data revealed incidences of non-expert-like backward transfer, which were important to capture despite being unproductive. Second, even if productive backward transfer is obtained in the dissertation study, it is unlikely that 8th graders’ understanding of linear functions will be expert-like. Finally, in a design-based research study, it is important to consider how students’ non-normative generalizations are afforded and constrained by the design innovations because such information can inform the next iteration of the design-based intervention (Lobato, 2003).

Despite the actor’s POV being more inclusive than the expert’s POV in terms of what counts as transfer, the actor’s POV does not focus on transfer at the collective level. In fact, at the collective level it may be more difficult to articulate the distinction between normative and non-normative performance because cultures and communities establish their own norms. However, there are researchers that do account for transfer from a collective POV. For example, Beach (1999) operated from a collective POV to account for the changing relationship between machinists and
their work when computer-controlled machines began to replace mechanically-controlled machines. He interpreted these changing relations between a community of practice and their social activity as transfer and called them *consequential transitions*.

An actor’s POV was adopted for the dissertation study because the primary interest was to investigate all changes in reasoning, not just changes that result in expert-like performance, at the individual, as opposed to the collective level. The kinds of puzzling findings that were evident in the pilot data suggested that students were making associations between learning and transfer tasks that would not be revealed using an expert’s POV. It was hoped that taking the actor’s POV for the dissertation study would lead to findings that add to the field’s understanding of how newly acquired knowledge and prior knowledge interact. Of course, this also depends on whether the rest of the research design aligns with such a goal. In the next section, the various research designs that have been used in the study of transfer are reviewed.

**Research designs for studies of transfer and backward transfer.** The POV that the researcher assumes is often related to two aspects of a research design, namely the research questions posed in a transfer study and the underlying approach to causality. These two aspects of the research design will be presented in turn, beginning with research questions. The dissertation study’s research questions and approach to causality will be situated within this presentation.

**Research questions.** The research questions that drive much of transfer research from an expert’s POV are “Did transfer occur?” (Lobato, 2003, p. 19) and “What conditions produce transfer?” (Lobato, 2008a, p. 173). For example, Su (2001) asked, “Will there be backward transfer [i.e., transfer from L2 to L1] in any of these
learner groups?” In another example, Singley and Anderson (1987) investigated the transfer of text-editing skill from one computer program to another. They asked, “Would transfer be positive, negative, or non-existent? How would transfer among the line editors compare with transfer from the line editors to the screen editor? . . . Could transfer effects be localized to particular sub-skills and pieces of knowledge? What would be the nature of these components?” (p. 228). Even when additional questions are asked, transfer, from a traditional perspective, is typically treated as “an unproblematic consensual construct; the only challenge being the facilitation of its occurrence” (Lobato, 2006, p.435).

The question “Did transfer occur?” does not align with an actor’s POV because, from this POV, transfer is ubiquitous (Lobato, 2003). In contrast, researchers operating from an actor’s POV may ask, “What are the images by which learners construct two situations as similar?” (Lobato, 2008a, p. 173). For example, Lobato and Siebert (2002) “look[ed] for evidence of personal connections that the student made between the wheelchair ramp situation and his experiences in the teaching experiment” (p. 89). In an example from backward transfer research, Kim and Montrul (2004) asked, “How is knowledge of Korean binding as an L1 affected by knowledge of English as an L2 in Korean heritage speakers (early and late bilinguals) who are immersed in an English context?” (p. 306; emphasis added).

A second research question that aligns with an actor’s POV is “How does the environment structure the production of similarity?” (Lobato, 2008a, p. 173). This question is somewhat different than the question about personally constructing relations of similarity because it addresses the instructional environment. However, it
too aligns with an actor’s POV because the goal of this question is to identify conceptual connections between classroom events and the changes in students’ reasoning, regardless if the reasoning is expert-like or not.

The research questions for this dissertation study also align with an actor’s POV. In particular, the first two research questions ask (a) when and in what ways do students construct personal relations of similarity between the quadratic functions context and the linear functions context, and (b) how are the personal relations of similarity that are constructed in backward and forward transfer related. The third research question also aligns with the actor’s POV, because it asks about how the quadratics instructional environment supports the personal relations of similarity that students create between quadratics contexts and linear contexts.

Research questions that focus on individual relations of similarity (i.e., like those used by the actor-oriented approach), would be inadequate to account for transfer that occurred at the level of Beach’s (1999) consequential transitions. The kinds of questions that would be appropriate for examining this type of transfer would focus on “the continuity and transformation of knowledge, skill, and identity across various forms of social organization” (Beach, 1999, p. 112). These kinds of questions would align with taking a collective POV. For example, in a study of Nepalese students becoming shopkeepers and vice versa, Beach (1999) investigated “how adolescents’ and adults’ arithmetic reasoning changed during transitions between school and work in a Nepali village” (p. 120).

By asking questions that account for both changing communities and changing forms of social activities, aspects of transfer that are not accounted for by the actor-
oriented approach, can be addressed. For example, questions that align with the collective POV address how “leading activities, the power structure, and social status play a role in transfer” (Lobato, 2008a, p. 175). On the other hand, this approach, unlike the actor-oriented approach, does not account for psychological processes involved in the generalization of learning.

**Research methods and approach to causality.** Traditional transfer studies typically rely on experimental design. To answer questions of whether or not transfer occurred or to determine conditions that facilitate transfer, researchers typically provide instruction regarding a solution method, procedure, or principle to an experimental group but not a control group. Then, both experimental and control groups are given a transfer task. Statistical comparisons between the groups’ performances on transfer tasks determine the kinds of probabilistic conclusions that can be made regarding the likelihood that transfer has occurred. Such methods rely largely on the use of quantitative analysis of performance measures. When qualitative methods are used to determine what transferred (e.g., what mappings people construct between initial learning tasks and transfer tasks), researchers typically rely on models of expert performance and pre-determine the acceptable mappings (see for example, Bassok & Holyoak, 1989).

To examine transfer processes empirically, traditional transfer studies typically rely on what Maxwell (2004) refers to as a *regularity* view of causality. From this perspective, causality cannot be observed directly but rather is established via a systematic relationship between inputs and outputs. By utilizing experimental methods, transfer researchers capitalize on the logic of stochastic causality to make
claims regarding the effectiveness of particular training activities on students’ ability to perform transfer tasks and inferences regarding the occurrence of particular transfer processes. For example, Nokes (2009) tested multiple mechanisms of transfer (i.e., analogical transfer, knowledge compilation, and constraint violation) by assigning participants to one of three training groups or to a control group. Each training approach corresponded to a proposed transfer mechanism. To test the mechanism of constraint violation, for instance, the training provided participants with instruction on how to find and continue a pattern in a sequence of symbols, using a set of specific constraints along with the general method of generating an initial solution, evaluating how the solution fits the constraints, and then revising the solution if any constraints were violated. Using the constraint violation process on a novel task was taken as evidence of transfer. One way that Nokes determined whether or not this process was used was with solution time data: longer solutions times for the constraint group indicated that several iterative cycles were needed to produce correct solutions, and hence that constraint violation had been utilized.

In contrast, in the type of classroom-based research conducted for this dissertation study, identifying a single element to vary (such as a training approach or curriculum) while trying to fix all other elements neglects the social reality of classrooms, namely that “important aspects of thinking emerge in interactions among people and between people and other material and informational systems in their environments” (Hatano & Greeno, 1999, p. 647). Following Lobato et al. (2011a), I identified a cluster of related phenomena to investigate—discursive practices, affordances and constraints of mathematical tasks, and the nature of the mathematical
activity—relying on an analysis of events and the processes that connect them conceptually (what Maxwell calls *process causality*). The process-oriented approach is defined as an analysis of the conceptual links by which some events influence other events (Maxwell, 2004).

The AOT approach takes a process-orientation using ethnographic methods to investigate both *what* transfers and *how* (Lobato, 2006). In particular, design-based research is used in conjunction with transfer tasks in pre- and post-interviews (Lobato, 2008a). The transfer tasks are initially created to share structural features with learning activities, but differ across surface features, from an expert’s POV. However, when the students’ responses to the tasks are analyzed, the researcher sets aside the expert’s POV and assumes the actor’s POV so as to be sensitive to the students’ often idiosyncratic ways of interpreting and connecting learning and transfer situations.

To examine what has transferred, four kinds of evidence are gathered (Lobato, 2008a): (a) students show a significant change in reasoning on transfer tasks from pre-to post-interviews, (b) students demonstrate limited knowledge of transfer tasks during the pre-interview, (c) plausible relationships of similarity are identified between the student’s reasoning on the transfer tasks in the post-interview and some activity in the instructional intervention, and (d) the student’s changed conception during the post-interview is associated with some aspect of the instructional intervention rather than due to the spontaneous construction of knowledge during the post-interview.

Ethnographic methods are used to analyze the data, since establishing connections between learning and transfer situations from an actor’s POV necessitates a focus on the nature of students’ reasoning (as opposed to a focus on performance measures).
To examine transfer processes, classroom data are analyzed using the focusing framework (Lobato et al., 2011a). There are two goals (a) to determine whether the changes in reasoning on the transfer tasks in the interviews are conceptually connected to what the student noticed during the instructional intervention (centers of focus), and (b) to explain how the centers of focus emerged by establishing conceptual connections between the centers of focus and particular discursive practices (focusing interactions), features of mathematical tasks, the nature of the mathematical activity in the classroom.

**Review Relevant to the Mathematics of Study**

The review of relevant mathematics has been organized into three sections. The first section contains a review of the research literature pertaining to the relationship between linear and quadratic functions. It is important that this relationship be established because the idea that knowledge of linearity could be influenced by quadratics knowledge is based on the assumption that there is a relationship. The second section contains a review of the research literature pertaining to the big ideas about linear and quadratic functions and to what is known about adolescent understanding of these topics. This review informed the design of the linear and quadratic interview tasks and the design of the quadratics instruction. It also informed the analysis of student thinking during the interviews. The third section addresses how teaching complex topics like quadratic functions could be approached, as well as what motivates introducing these kinds of topics so early on. The information contained in this section also informed the instructional design.
Relationship Between Linear and Quadratic Functions

Three-fourths of the Principles and Standards in School Mathematics (PSSM; NCTM, 2000) and California State Mathematics Standards (California Department of Education, 2006) for algebra involve linear and quadratic functions. Clearly, the two topics are central to algebra. However, thus far in this document, it has been assumed that there is a relationship between learners’ knowledge of linearity and their knowledge of quadratics. A more explicit case for why this relationship exists will now be made. In particular, it will be argued that there are several reasons why linear and quadratic functions are related. First, they both involve functions. Second, they both are part of the mathematics of change. Third, they both involve covariation. Each aspect of the relationship will be discussed in turn.

**Functions.** One reason why linear and quadratic functions are related is that both are functions. The concept of function is traditionally defined as the relationship between two sets A and B, where each element in A, the domain, is assigned to exactly one element in B, the range (Confrey & Smith, 1991, 1995; Giordano, 2008; Leinhardt, Zaslavsky & Stein, 1990; Vinner, 1983). This definition is often associated with a *correspondence approach* to function, which involves a heavy emphasis on determining the algebraic rule (Confrey & Smith, 1995). An alternate approach is the *covariational approach* where the emphasis is on coordinating the changing values of two quantities (Confrey & Smith, 1995). More will be said about this approach in the section on covariation.

The National Council of Teachers of Mathematics (NCTM) and mathematics
education researchers have argued that the function concept is a core idea that should run from the early grades through grade 12 as a common thread in mathematics education (NCTM, 2000; Leinhardt et al., 1990; Metcalf, 2007). In support of this goal, linear and quadratic functions normally figure prominently in pre-algebra and beginning algebra (Brenner et al., 1997; Zaslavsky, 1997). Furthermore, Movshovitz-Hadar (1993) argues that knowledge about quadratic functions can be built on prior knowledge about linearity. For example, students could be shown that a quadratic function is composed of the product of two linear functions (Buck, 1995; Curan, 1995; Giordano, 2008; Movshovitz-Hadar, 1993).

Linear and quadratic functions are also related in terms of how the topics within a typical algebra course are sequenced. Ordinarily, the sequencing of topics is such that quadratic functions are taught soon after linear functions (Movshovitz-Hadar, 1993; Zaslavsky, 1997). Sequencing the curriculum in this way increases the potential for building knowledge of quadratics on knowledge about linearity. Nevertheless, what often happens is that linear functions and quadratic functions are treated as distinct and unrelated topics (Movshovitz-Hadar, 1993).

Other research has shown that there can be a complex interaction between knowledge of linear functions and knowledge of quadratic functions. For example, studies have found that it is common for students to be overly reliant on linearity and to overgeneralize to quadratic contexts (Dreyfus & Eisenberg, 1983; Ellis & Grinstein, 2008; Karplus, 1979; Lovell, 1971; Markovits et al., 1983, cited in Zaslavsky, 1997; Zaslavsky, 1997).

Linear and quadratic functions are also related in the ways that they are
conceived. As stated above, there are two ways that functions can be conceived: (a) as a correspondence when conceived as “any correspondence between two sets that assigns to every element in the first set exactly one element in the second set (the Dirichlet-Bourbaki definition)” (Vinner & Dreyfus, 1989, p. 360), or (b) as a dependency relation when conceived as “a dependence relation between two variables; \(y\) depends on \(x\)” (Vinner & Dreyfus, 1989, p. 360). The difference between the two conceptions is that conceiving of a function as a dependency relation means thinking that there is directionality to the relationship (i.e., a change in the independent variable produces a corresponding change in the dependent variable) while conceiving of a function as a correspondence means thinking about the assignment of one element to another without a particular directionality from one set to the other (Kieran, 1993). In other words, directionality is an important feature of the dependence conception but not of the correspondence conception.

Both linear and quadratic functions can be conceived of as correspondences between sets or as dependency relations. In the pre-interview of the pilot data presented earlier, Matt appeared to rely on the dependency of the plane’s altitude on time, when he said, “in 2 seconds it climbs 7 feet” (line 10) and “1 second it should be 3½” (line 20). However, in the post-interview he did not emphasize directionality, but referred primarily to the dependent variable in both quadratic and linear contexts. Instructional experiences may lead students to focus on one particular conception over the other (Freudenthal, 1982, cited in Kieran, 1993). Davis (1982, cited in Kieran, 1993) argued that children as young as 10 years old should be introduced to the dependency conception. The dependency of distance on time was emphasized in the
dissertation study.

Linear and quadratic functions can also be related by what they represent. For example, linear distance-time functions represent motion with constant speed, while quadratic distance-time functions represent motion with constant acceleration (and linearly increasing speed). This relationship was evident in Matt’s responses because he confused constant speed and constant acceleration.

Students’ self-generated drawings also reflect aspects of the relationship between linear and quadratic functions. For example, in other data collected in conjunction with the pilot data, one student drew a representation for a quadratic function and a linear function (see Figure 2.3). The student, Evelyn, used little squares to represent distance. For the quadratic function, she drew increasingly tall columns of boxes, where each square represented 1 m of distance and each column represented 1 s of traveling time. For the linear function, she drew a series of columns, all the same height.

In another example, Ling used number-line diagrams to represent quadratic and linear functions (see Figure 2.4). For the quadratic function, the spacing of the number-line tick marks labeled with accumulated distances and times became progressively farther apart. For the linear function, the tick marks were approximately equally spaced. Notice that in both examples, the student’s linear representation shares similarities with their quadratic representations. Therefore, the way that these students represented one kind of function was related to the way they represent the other kind of function.

In the previous paragraphs, the case was made that linearity and quadratics are
related because they both involve functions. Next, I explain why these two concepts also intersect as two aspects of the mathematics of change.

Mathematics of change. A second reason that linear and quadratic functions are related is because both qualify as topics that fall under the category of mathematics of change. The mathematics of change is prominently featured in the NCTM Standards: “Instructional programs from prekindergarten through grade 12 should enable all students to . . . analyze change in various contexts . . . Algebra encompasses . . . the mathematical study of change” (2000, p. 36).

One way to define the mathematics of change is as the systematic study of aspects of our experiences that are embodied in quantifiable variation (Kaput & Roschelle, 1999). In other words, the mathematics of change is about concepts like rate of change, accumulation, limits and continuity (Kaput & Roschelle, 1999; Noble
et al., 2001). In the context of the dissertation study, linear functions involve constant rates of change while quadratic functions involve linearly increasing rates of change and constant rate of change of the rate of change. While an understanding of some concepts associated with the mathematics of change will not fully develop in students prior to taking calculus (Stewart, 1990), other aspects are accessible in earlier grades (Vahey, Tatar, & Roschelle, 2004).

Rate of change for most students is initially defined in the context of linear functions (Roschelle et al., 2000). Quadratic functions are one of the first contexts that students experience to be associated with a non-constant rate of change. Stroup (1996, cited in Roschelle et al., 2000) argued that linear functions portray too simple an example of rate of change to make the important features relevant and that students need to be exposed to more complex functions. This suggests that quadratic functions could act as a contrasting case (Bransford & Schwartz, 2001) to linear functions,

Figure 2.4. Ling’s representations of quadratic and linear function data.
revealing features of rate of change that may be hidden when only linear functions are explored. Revealing these hidden features may influence one’s prior knowledge of rate of change and thus, result in backward transfer.

At the same time, a complete understanding of continuously changing rates of change, as is the case for quadratic functions, is likely inaccessible for most 8th graders because to understand continuous variation one must understand limits (Lobato, Hohensee, Rhodehamel, & Diamond, in press; Roschelle, 1991). Roschelle et al. (2000) response to this dilemma is to engage students in explorations of piecewise linear approximations of curved functions. This approach, which was used in the dissertation study, establishes another path by which linearity and quadratics could be associated.

In summary, quadratic functions provide a useful context for introducing students to changing rates of change and, at the same time, involve concepts whose full understanding is likely inaccessible for 8th graders. Linear piecewise approximations of quadratic functions, which were used in the dissertation study, are a more accessible context in which the rate of change changes.

**Covariation.** A third reason that linearity and quadratics are related is that both involve covariation. *Covariation* is defined as coordinating the movement between successive values of one variable and the corresponding successive values of another variable (Carlson et al., 2002; Confrey & Smith, 1994, 1995; Saldanha & Thompson, 1998). Initially, covariational reasoning involves forming two sets of elements and then setting the individual elements in direct correspondence to each other (Piaget, 1952). At more advanced levels, covariational reasoning becomes much
more complex but also mathematically powerful (e.g., as in reasoning about the Fundamental Theorem of Calculus; Thompson, 1994a).

According to Confrey and Smith (1995), a covariational approach to functions emphasizes the relationship between two related sets of data rather than the algebraic rule for finding outputs from inputs. They argue that a covariational approach to quadratic and linear functions provides greater access to thinking about rate of change and increments of change. Furthermore, they argue that using the covariational approach to identify the constant second differences, is central to understanding quadratic functions.

Clement (1989), Nemirovsky (1996) and Saldanha and Thompson (1998) connect covariation with the mathematics of change. Clement makes the distinction between static versus dynamic covariation. He defines the static case as the limited ability to see “correspondences between individual static values of the variable” (p. 5). In contrast, he defines the dynamic case as the ability “to see the changes in two quantities at once” (p. 5). Clement found that a static view of covariation increased the likelihood of misconceptions, such as conceiving of a graph as a picture. Nemirovsky (1996) makes a similar distinction between pointwise contexts of covariation, where the variables only assume discrete values, and variational contexts, where the variables vary on a continuum. He argues that at least some of the problem contexts that students are exposed to should be of a variational form.

Saldanha and Thompson (1998) describe functional covariation as “holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously” (p. 1). They further characterize it as forming a multiplicative object with the two quantities
and realizing that for any value of one quantity, the other quantity also has a value. According to Saldanha and Thompson (1998), covariation is developmental. In the early stages of development, the quantities exist one at a time, first one and then the other. Eventually, both exist at the same time. At the final stages of development, all intermediates between values of two continuous quantities are understood to exist simultaneously. They call this *tight coupling*. The development of covariational reasoning has also been described as a shift from an action or mapping conception of function to a process conception (Carlson, Jacobs, & Larsen, 2001; Silverman, 2006; Thompson, 1994b).

Saldanha and Thompson (1998) found that developing an understanding of functions as two covarying quantities is non-trivial. For example, in their study in which students reasoned about a car travelling between two cities, students were able to avoid covariation by reasoning separately with the distance between the car and each city, without tightly coupling the two quantities. Therefore, the development of covariational reasoning will likely be a challenge for 8th grade beginning algebra students.

Carlson et al. (2001) captured the developmental nature of covariation with a conceptual framework that accounts for the mental actions, and the related behaviors, involved at different stages of development (see Figure 2.5). They conceived of the development of covariational reasoning as going hand-in-hand with the development of the concept of rate. At the second highest level (MA4), covariational reasoning supports coordination of average rates of change in the dependent variable with uniform changes in the independent variable. At the highest level (MA5),
covariational reasoning supports the coordination of instantaneous rates of change of the dependent variable with continuous changes in the independent variable of a function. Level MA5 would be the level required to fully understand rates of change in the context of quadratic functions, while level MA4 would be sufficient to understand rate of change in the context of linear functions (or linear approximations of quadratic functions). Based on an informal estimate, most of the eighth grade students that participated in the dissertation study were likely at the MA3 to MA4 level (i.e., they had a limited ability to engage in mental actions associated with coordinating average rates of change with uniform increments of change in the input variable).

In summary, three reasons were presented for why linear and quadratic functions are related. First, both are functions. The way that one conceives of linear functions is likely related to the way that one conceives of quadratic functions and vice versa. Second, both types of functions are conceived as part of the mathematics of change, with its emphasis on an analysis of rates of change. Furthermore, the mathematics of change of the piecewise linear approximation of quadratic functions may be more accessible to 8th graders than the mathematics of changes of quadratic functions. Third, linear and quadratic functions both involve covariation. The ability

**Figure 2.5. Carlson et al.’s (2002) covariational framework.**

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>MA1</td>
<td>Coordinating the value of one variable with changes in the other</td>
</tr>
<tr>
<td>MA2</td>
<td>Coordinating the direction of change of one variable with changes in the other variable</td>
</tr>
<tr>
<td>MA3</td>
<td>Coordinating the amount of change of one variable with changes in the other variable</td>
</tr>
<tr>
<td>MA4</td>
<td>Coordinating the average rate-of-change with uniform increments of change in the input variable</td>
</tr>
<tr>
<td>MA5</td>
<td>Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.</td>
</tr>
</tbody>
</table>
that one has to engage in covariation with linear functions is likely related to the way that one engages in covariation with quadratic functions and vice versa.

**Big Ideas and Student Conceptions**

In this section, a review is presented about what the mathematics education literature says about the big mathematical ideas from linear and quadratic functions. This review will focus primarily on those aspects of linear and quadratic functions that pertain to the dissertation study (e.g., a focus on tabular over graphical representations). Whenever possible, what is known about the kinds of beginning conceptions algebra students form with respect to the big ideas from linearity and quadratics will be presented.

**Linear functions.** The three big ideas about linearity addressed in the mathematics education literature that most pertain to the dissertation are: (a) ratios, (b) rates of change, and (c) proportional reasoning.

**Ratios.** There are several ways to conceive of ratios. One way is as a multiplicative comparison between two quantities (Thompson, 1994c; Lobato & Ellis, 2010). The mental operation of making a multiplicative comparison involves asking, “How many times greater is one thing than another?” (Lobato & Ellis, 2010, p. 18). Thompson (1994c) distinguishes between comparing quantities multiplicatively, which results in the formation of a ratio, and other mental operations such as comparing quantities additively, which do not result in the formation of a ratio.

Alternatively, a ratio may be formed by composing two quantities into a new unit or entity that can be operated on (Lamon, 1995; Lobato & Ellis, 2010). This is often seen as a rudimentary but foundational form of ratio, which some researchers
categorize as *pre-ratio reasoning* (Lesh, Post, and Behr, 1988). Researchers often take partitioning and iterating as evidence of forming a composed unit. Partitioning is the process of subdividing the new quantity into equal parts, while iterating is the process of repeating the new quantity. For example, Lobato and Siebert (2002) provide an account of an 8th grader’s reasoning with composed units. To explain that walking 10 cm in 4 s was the same speed as walking 2.5 cm in 1 s, the student partitioned 10 cm and 4 s unit into four equal 2.5 cm and 1 s units. Then, he iterated the 2.5 cm and 1 s unit four times to reproduce the 10 cm in 4 s unit. This provides some evidence that the student had joined the quantities of distance and time to form a 10:4 unit, which could then be operated on by iterating and partitioning.

In traditional math textbooks, ratio is often defined in ways that fall short of capturing the essential understandings necessary to form a ratio (Lobato & Ellis, 2010). For example, in the Discovering Algebra textbook, Murdock, Kamischke and Kamischke (2002) define a ratio as “a comparison of two quantities, often written in fraction form” (p. 699). This definition does not specify the type of comparison that is needed (e.g., multiplicative rather than additive). Furthermore, simply writing two numbers in fraction form does not guarantee that the student has mentally formed a ratio.

Language further complicates the issue of defining a ratio. As Kaput and Maxwell-West (1994) point out, in everyday experience, language encodes some ratios as entities, (e.g., speed, unit price, or gas milage), while other ratios are named by referring to the two quantities or parts that comprise the ratios, (e.g., the ratio of girls to boys). Attempts to name the attribute measured by these latter ratios may
seem awkward or contrived (e.g., “girlness”).

As stated above, forming ratios between the inputs and corresponding outputs of linear functions is an important part of characterizing linear functions. However, forming ratios is not a trivial matter, and student conceptions of ratios often fall short of the criteria of making multiplicative comparisons or of forming composed units. Simon & Blume (1994) argued that the construction of a ratio as the appropriate measure of an attribute, which they call *ratio-as-measure*, is a complex process. Clark and Kamii (1996) found that the multiplicative thinking behind ratio reasoning appears as early as Grade 2 but develops very slowly. Thus, if students are first introduced to linear functions in the middle grades, it is likely that aspects of their reasoning with ratios and proportions are still emerging.

Thompson’s (1994c) case study of a fifth grader being interviewed about speed illustrates the complexity that students face when constructing ratios. Thompson found that his subject was not making multiplicative comparisons between distance and time but was conceiving of speed as a distance, or what Thompson called a *speed-length*. For example, when the child was asked about a turtle which moved at 20 ft/s and rabbit which moved at 30 ft/s, the student stated that the turtle would always be 10 ft behind the rabbit.

If a student reasons additively or reasons with only one quantity in situations in which ratio reasoning is appropriate, it will affect the way the student experiences instruction on linear functions. Conversely, the exploration of linear functions could play a major role in helping students develop ratio reasoning. Therefore, ratios are a big idea from linear functions. The dissertation study investigated whether there was
an influence from the instructional intervention on students’ ability to form ratios in the context of linear functions.

**Rates of change.** As stated earlier, the mathematics of change links linear and quadratic functions, and an important aspect of the mathematics of change are rates of change. Thompson (1994c) defines rates as reflectively abstracted constant ratios. By this, he means that a rate is formed when the mental operation of making a multiplicative comparison becomes less directly tied to the original experience and is assimilated to higher levels of thought (Piaget, 1980, cited in Thompson 1994c). This also means that the multiplicative comparison used to form a ratio is preserved without it being tied to any particular ratio. Similarly, Lobato and Ellis (2010), conceive of a rate as “a set of equivalent ratios” (p. 42). Thus, a rate defines a linear function of the form \( y = mx \) (Thompson, 1994c).

Thompson (1994c) argued that the conventional definitions of rate incorrectly place the distinction between ratio and rate in the situation rather than in the way the learner conceives of the situation. For example, Ohlsson (1988, cited in Thompson, 1994c) defined rate as a ratio of a quantity to a time and Vergnارد (1988) defined rate as a comparison of two unlike quantities. For example, 3ft to 8 ft would be a ratio but 3ft for 1 min would be a rate. In contrast, characterizing ratio in terms of how one thinks about it (e.g., as a multiplicative comparison or as a composed unit) makes the “same unit versus different unit” distinction less important. Lobato and Ellis (2010) point out that Thompson’s use of rate is consistent with how rate is used in calculus. For example, the derivative of a function at a particular point is conceived of as an instantaneous rate of change at that point, regardless of what the particular quantities
By investigating student conceptions of rate, researchers have found that forming a rate can present challenges for students (Thompson & Thompson, 1994), especially when one of the two quantities is not time (Thompson, 1994a). For example, it is common for students to form a few equivalent ratios through simple operations such as halving and doubling, long before being able to conceive of a set of infinitely many equivalent ratios (Kaput & Maxwell-West, 1994; Lamon, 2007). Thompson also suggests that when students speak of rate in the expression “distance equals rate times time,” they are likely not conceiving of rate as a reflectively abstracted ratio but are engaging in what Hayes (1973, cited in Thompson, 1994a) called *symbol speak*. Because a rate invokes a linear function, these challenges to understanding rate could also be challenges to understanding linear functions. The dissertation study looked for backward transfer influences on students’ conceptions of rate.

**Proportional reasoning.** Proportional reasoning has been referred to as the capstone of the elementary curriculum and as the cornerstone of algebra and beyond (Lesh, Post, & Behr, 1988). According to Inhelder and Piaget (1958, cited in Lamon, 1993), the ability to reason proportionally distinguishes formal operational thought from concrete levels of thought. Despite these strong endorsements for the importance of proportional reasoning, Lamon (1993) reports that many teachers treat proportional reasoning as the ability to set up and solve a proportion equation using a technique such as cross-multiplication. However, as Lobato and Ellis (2010) point out, it is
possible for students to set up and solve proportion equations correctly without understanding what proportions represent.

Other researchers define proportional reasoning by the kind of understanding it entails. Karplus, Pulos, and Stage (1983) define proportional reasoning as “a term that denotes reasoning in a system of two variables between which there exists a linear functional relationship” (p. 219). Lesh, Post and Behr (1988) characterize it as attending to the covariation of two quantities and recognizing that when they change simultaneously, they maintain a multiplicative relationship. Lamon (1994) adds that “students’ ability to think about a ratio as an invariant composite unit and work simultaneously with both its composite units in a double-matching process (covariance) illustrated the kind of understanding we would like them to have about the meaning of ratio and proportion” (p. 112).

Proportional reasoning is complex and develops slowly over a number of years (Lamon, 1993). At the beginning levels of development, students may iterate a composed unit. For example, forming a composed unit of 2 pieces of candy and 8¢ and then reasoning that 4 pieces cost 16¢, 6 pieces cost 24¢, 8 pieces cost 32¢ is an example of iterating. This is also referred to as a build up strategy (Lesh et al., 1988; Lamon, 1993, 2007).

At higher levels of proportional reasoning, students form what Steffe (1988) calls multiple levels of inclusion relations. For example, suppose a student is asked how much candy she can buy with $30 when 4 lbs costs $6. She may form a composed “four-six” unit and use a build up strategy until she has reached 20 lbs for $30. Steffe described this kind of reasoning as operating at a single level of inclusion.
Suppose instead the student compares the $6 to the $30, conceives of $30 as five
groups of $6 and then treats the “4 lbs for $6” as a unit of which she needs five.
According to Steffe, if the student then conceives of the “20 lbs for $30” as a new unit
which is comprised of five groups or units, this new unit is at a different level of
inclusion because the student has formed a unit of units. Steffe describes this as a
hallmark of multiplicative (and proportional) thinking.

In summary, three big ideas about linear functions are ratios, rates of change
and proportional reasoning. As I showed, these three concepts are interrelated: rates
are reflectively abstracted ratios and proportional reasoning involves working with
equivalence classes of ratios. There are other big ideas from linear functions (e.g.,
algebraic and graphical representations of linear functions, slope, etc). However, these
other ideas will not be discussed here because the dissertation study focused primarily
on identifying backward transfer effects that involve students’ conceptions of ratios,
rates of change and proportions. Next, several big ideas about quadratic functions are
presented.

**Quadratic functions.** As with the previous section, the scope of this review
will be limited to the big ideas from quadratic functions that pertain to the design of
the instructional intervention that was used in the dissertation study. The scope is
further limited by an underrepresentation of mathematics education literature about
quadratic functions, especially compared with the existing literature on linear
functions.

The majority of research on learning about quadratic functions focuses on
students making connections between algebraic equations and graphical
representations (e.g., Coburn, 2000; Dreyfus & Halevi, 1990, 1991, cited in Curran, 1995; Eraslan, 2008; Goldenberg, 1988, cited in Curran 1995; Metcalf, 2007; Movshovitz-Hadar, 1993; Owens, 1992; Winicki-Landman, 2002). This emphasis on connecting equations and graphs in quadratic contexts reflects the research focus that emerged in the 1980s and 1990s in which being able to move flexibly across algebraic representations was taken as evidence of having an understanding of functions (DeMarois & Tall, 1999; Leinhardt et al., 1990; Moschkovich, Schoenfeld & Arcavi, 1993; Yerushalmy & Schwartz, 1993).

Lobato and Bowers (2000) have questioned what being able to move between representations reflects about a person’s understanding of the relationship to be represented. Yet, there is a dearth of research on beginning-level conceptions that addresses aspects of understanding quadratic functions other than graphical representations. The two big ideas about quadratic functions that are most relevant for my study are (a) changing rates of change and (b) the rate of change of the rate of change. These ideas are the focus of only a handful of studies.

**Changing rates of change.** Quadratic functions are likely one of the first contexts in which students explore non-constant rates of change. Often these beginning explorations will occur in contexts of motion. For example, in a study by Roschelle, Kaput and Stroup (2000), in which students explored constantly increasing speed, a middle school student went from being able to find the distance covered with constant speed to discovering how to find distance covered with linearly increasing speed. This suggests that beginning conceptions of changing rates could be accessible to beginning algebra students. Bowers and Doerr (2001) showed that exploring
changing rates using simulated motion can lead to important understandings that even learners who have more formal understandings may be missing. Their study looked at pre- and in-service mathematics teachers’ conceptions of constant speed and increasing speed, using the SimCalc Mathworlds™ software. One of the mathematical insights that emerged for the teachers was that there was a difference between average and instantaneous speed. Hauger (2000) suggested that to help students learn to distinguish between average rates of change and instantaneous rates of change, they should explore successively smaller intervals of the independent variable of a quadratic function with the data recorded in two columns of an EXCEL spreadsheet and the average rate of change recorded in the third column.

Lobato et al. (in press) identified five conceptual learning goals (CLGs) that are instructional targets designed to move students toward a sophisticated understanding of quadratic functions. This work emerged from the analysis of data collected in the first two design-based iterations, of which the dissertation study is the third iteration. The first three conceptual learning goals (CLGs) identified by Lobato et al. (in press) are related to the big idea of changing rates of change. CLG 1 states that students should be able to reason with changes in the dependent variable values of a quadratic function. This learning goal was based on the observation that, in the context of quadratic position-time functions, a significant number of students attended to accumulated distances rather than elapsed distances when determining whether or not the speed of a falling ball was changing. CLG 2 states that students should be able to give equal prominence to the changes in the dependent variable and the changes in the independent variable. The motivation for this CLG was the observation that, in the
quadratic position-time context, it was more difficult for students to attend to elapsed times than elapsed distances. Finally, CLG 3 states that composing a sequence of changes in the dependent variable and corresponding changes in the independent variable is a good intermediate goal for beginning algebra students toward developing a conception of constantly changing rates of change. The students with this understanding reasoned in more productive ways about changing speeds than students who had not composed these quantities.

The big idea that rates of change in quadratic functions are changing is a difficult idea for beginning algebra students. However, the studies above suggest that students can make progress that can prepare them for developing a more sophisticated conception in the future.

**Rate of change of the rate of change.** A big idea in the teaching and learning of quadratic function is that the rate of change of the rate of change of a quadratic function is constant. In the context of a quadratic distance-time function, this means that the quantity acceleration is constant. Very few studies have looked at beginning-level conceptions of rates of change of rates of change.

In the initial stages of planning for the first iteration of the design-based study on quadratics, of which this dissertation study is a part, the researchers consulted the field of physics education research to determine what was known about teaching the concept of acceleration. Basson (2002) describes acceleration as a key concept in physics, likening it to the concept of function in mathematics. He argued for the importance of sequencing concepts so that instruction builds up to an exploration of acceleration and proposed the following sequence of topics: (a) position and time, (b)
length and time intervals, (c) displacement and distance, (d) speed and velocity, and (e) acceleration. The dissertation study generally follows Basson’s trajectory.

The other two conceptual learning goals from Lobato et al. (in press), shed light on the specific understandings that are pivotal for students to understand with respect to acceleration. In particular, CLG 4 states that students should be able to recognize that the difference between two rates of change is a rate of change, and not a dependent variable value. In the context of a quadratic distance-time function, this means being able to recognize that the difference between two speeds is a speed and not a distance. This learning goal was motivated by the observation that a majority of the students in the study treated the difference between two speeds as a distance (70% of the students displayed this problem). CLG 5 states that students should be able to compose the change in the average rate of change in the dependent variable with the corresponding change in the independent variable to form a constant rate of change of rate of change of a piecewise linear approximation of a quadratic function. In the context of a quadratic distance-time function, this means being able to compose a change in the average speed with the corresponding change in time to form constant acceleration. This CLG emerged from the observation that students in the study often misinterpreted acceleration units (e.g., meters per second per second) as a distance composed with a time and were unclear about why an extra time component was part of the units. Furthermore, students struggled when presented with data using non-unit time intervals, suggesting that they had not composed changes in speed with changes in time.

The CLGs provide a greater elaboration than what is contained in science and
mathematics education standards of what students need to understand about acceleration and quadratic functions. Furthermore, the CLGs outline productive ways of conceiving of quadratic functions that might be accessible to beginning algebra students. In contrast, science standards emphasize the proportional relationship between acceleration and force instead of what it means to understand the concept of acceleration itself (American Association for the Advancement of Science, 2008; California Department of Education, 2003; National Research Council, 1996), while math standards often provide general principles rather than specifying the particular understandings that students need to know about quadratic functions (California State Board of Education, 2006; NCTM, 2000).

Content-general Issues Pertaining to Backward Transfer

The review of issues pertaining to mathematics has thus far focused on the specific content of linear and quadratic functions. In this section, the literature pertaining to aspects of learning mathematics that are not content-specific but that are relevant for this study of backward transfer are presented. In particular, the literature that addresses quantitative reasoning, the use of student-designed drawings and algebra in early grades is presented.

Quantitative reasoning. As stated above, the students participating in the dissertation study were too young to develop a sophisticated understanding of quadratic functions, especially as it pertains to calculus concepts like instantaneous rates of change (Kaput, Carraher, & Blanton, 2008). This issue is not unique to teaching quadratic functions, as will be shown later in the discussion of the Algebra
for All movement. Some researchers have advanced the idea that being able to conceive of quantities and then reason with and manipulate them is foundational for reasoning algebraically (Carraher, Schliemann, & Schwartz, 2008; Dougherty, 2008; Smith & Thompson, 2007). This implies that an emphasis on quantities in the context of quadratic functions may help to prepare students for more sophisticated understandings later on. This is particularly relevant because an emphasis on quantitative reasoning guided the design of the instructional intervention from which the pilot data came. Consequently, it was in the context of quantitative reasoning that backward transfer was first observed.

There are several definitions related to quantitative reasoning that distinguish it from numerical reasoning. A quantity is defined as one’s conception of a measurable attribute of an object, event, or situation (e.g., conceiving of distance, weight, how fast something travels, and so on; Smith & Thompson, 2007; Thompson, 1988, 1994c.). Quantities are not numbers; yet a particular value of a quantity can be numeric (e.g., 3 feet, 20 pounds, 6 miles/hour, etc). Quantification is the process of measuring and assigning numerical values to quantities (Thompson, 1994c). Quantitative operations are defined as the mental actions by which a new quantity is formed from two or more existing quantities (Thompson, 1994c). In contrast, arithmetic operations are operations on numerical values. In other words, quantitative operations are non-numerical. Quantitative reasoning is defined as reasoning with quantities, with the relationships between quantities and with quantitative operations (Smith & Thompson, 2007). Using arithmetic operations to compute numerical values is not seen as quantitative reasoning (Smith & Thompson, 2007; Thompson, 1994c). This definition

Researchers advocate for the use of quantitative reasoning as a foundation for algebra (Carraher et al., 2008; Dougherty, 2008; Smith & Thompson, 2007). According to Smith and Thompson (2007), quantitative reasoning is the root on which numerous approaches to algebra can grow. As stated above, quantitative reasoning makes the quantities the center of focus. It also draws heavily from everyday experiences. Smith and Thompson (2007) argue that this kind of emphasis is not limiting because the problems that can be solved using algebra can also be solved using quantitative reasoning. Furthermore, they argue that by engaging students in quantitative reasoning, students will develop reasoning that is flexible and general but that does not yet necessitate symbolization. Quantitative reasoning prepares students for algebra because it provides them with the conceptual content necessary for them to engage in meaningful algebraic representation and manipulation.

Carraher et al. (2008) recommend that teachers immerse students in contexts where students need to make sense of the relations between quantities and need to make generalizations. They showed that when students worked with indeterminate amounts (e.g., number of candies in a sealed box) they moved from thinking about specific numerical values to thinking about functional relations between quantities. Emphasizing the relations between quantities helped young students shift from arithmetic to algebraic thinking. Carraher et al. (2008) described the relations between quantities as the content which, like a thread, algebra strings together. Echoing a similar point made by Smith and Thompson (2007), Carraher et al. contend that
without this content first being constructed, the thread (algebra) has nothing to string together.

Other researchers have begun to experiment with the construction of curricula that use quantitative reasoning as a central principle of design. For example, in Dougherty’s (2008) approach to algebraic thinking (the Measure Up project), which derives from the work of Russian educators and psychologists (e.g., Davydov, 1975a, 1975b, cited in Dougherty, 2008), children as young as first graders were taught to compare objects and measure things against each other. Furthermore, students were asked to name the quantities that they had identified. In a deviation from Smith and Thompson (2007) and Carraher et al. (2008), students in the Measure Up project were asked to symbolize comparisons of quantities using letter symbols. Manipulation of letter symbols was shown to begin quite naturally (Dougherty, 2008). Students explored how to make unequal amounts equal, change two equal quantities while maintaining equality, and change two equal quantities while maintaining inequality. This was often accomplished non-numerically, with numbers being introduced only when required.

In traditional classrooms, specific cases are normally discussed first and then general principles are abstracted later. The Measure Up project reversed this order, with ad hoc units and relations between quantities and units being considered first and numerical values being introduced later. The project found that this order and the resulting emphasis on the relationships between units and quantities encouraged students to develop flexible quantitative reasoning. Flexibility in thinking included being able to (a) split quantities into separate parts so that they could be represented in
multiple symbolic ways, (b) draw part-whole diagrams of quantities, (c) use letters to represent nonspecified generalized quantities, and (d) see quantitative prenumerical relationships.

The dissertation’s instructional intervention emphasized quantitative reasoning by incorporating many of the recommendations reviewed above. Specifically, students were encouraged to think about the relations between quantities, such as the relationships between accumulated and elapsed distances and times, and between speed and acceleration (see CLGs presented above). Tabular displays were used to present data in order to help students see relations between quantities.

One additional aspect of quantitative reasoning was addressed by Dougherty (2008). Specifically, she explained that student-constructed diagrams help students make sense of the relationships within situations. In the next section, several papers on student-constructed diagrams are reviewed.

Diagrams. The traditional perspective on representations is the symbol systems orientation (Nemirovsky, 1994, cited in Sherin, 2000). From this perspective, external representations stand for something in the represented world. Knowing what the symbols contained in the representations correspond to in the represented world is the central issue in understanding representations (Palmer, 1997, cited in Sherin, 2000). Goodman (1976, cited in Sherin, 2000) and Kaput (1987, cited in Sherin, 2000) align with this perspective because they defined symbol schemes and symbol systems as collections of symbols and the rules for combining them. Lehrer (2002) also aligns with this orientation in defining a representation of space as a “correspondence between the world in which they [his students] walk and the
inscription of that world on paper” (p. 69). A symbol systems orientation is often associated with a research focus on specific schooled representations (e.g., how students learn and use line graphs) and on the difficulties students have with representations (e.g., confusion between intervals and points or between slope and height; Leinhardt et al., 1990).

As stated above, a large body of research has focused on flexible movement across conventional representations (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993; Schoenfeld, Smith, & Arcavi, 1993; Yerushalmy & Schwarz, 1993). However, Dougherty (2008) raised the concern that the exclusive use of traditional representations such as graphs or equations may put the focus on the structure of the representation and not on the mathematical relationships being represented. Consequently, Dougherty advocates for students to create their own diagrams to support a developing understand of the relationships between quantities.

An alternate perspective on representations is the meta-representational competence (MRC) perspective. Also known as representational literacy, MRC is about representations in general, not about specific schooled or scientific representations (Azevedo, 2000; diSessa, 2002; diSessa & Sherin, 2000). The following are four meta-representational competencies: (a) being able to invent representations, (b) being able to critique representations, (c) understanding the general and specific functions of representations, and (d) having the ability to learn new representations (diSessa, 2002). This perspective involves seeing the correspondences between symbols that refer to entities and the entities that they refer to as not remaining constant “over any extended episode of symbol use” (Sherin,
Several research studies have looked specifically at the development of representational competency (e.g., Deloache & Burns, 1994; Piaget & Inhelder, 1956, cited in Sherin, 2000). For example, Piaget and Inhelder explored children’s ability to copy geometric figures. A number of other researchers have looked more broadly at how the nature of children’s drawings changes with age, tracking the development from scribbles to more adult productions (Freeman, 1980; Goodnow, 1977; Kellogg, 1969; Lange-Kuttner & Thomas, 1995, cited in Sherin, 2000). These researchers focus on capabilities instead of difficulties and don’t assume that what the representation refers to will be necessarily identified (Sherin, 2000). This kind of research often involves students constructing novel or invented representations.

A second alternate perspective on representations is the *notations-in-use* perspective (Meira, 2002). From this perspective, the production of representations is the central issue. Cultural conventions, such as systems of notation, shape the practices from which they were developed (Meira, 2002). Symbols are meditational tools that can trigger and sustain mathematical activities while impeding other activities (Hall, 1996; Meira, 1995). The meanings of particular symbols shift over time. The notations-in-use orientation is associated with a research focus on the shifts in the meaning and the nature of inscriptions, the interaction between representational competence and the social and material world, the affordances of particular representations for certain activities, and the function of representations within given activities (Meira, 2002).

However, most of the research on student drawings does not connect drawings
with student conceptions of quantities. An exception is the study by Lobato, Hohensee and Diamond (2011) on student representations and conceptions. Lobato et al. found that their inferences about the students’ conceptions of key aspects of quadratic distance-time functions aligned with the processes students used to construct diagrams representing quadratic distance-time data for a falling ball. For example, the students who appeared to form the quantity speed by composing elapsed distances and corresponding elapsed times were the same students who drew each elapsed distance and the corresponding elapsed time label on their diagram, one right after the other and right beside each other.

Drawn diagrams were emphasized in the instructional intervention of the dissertation study to help students make sense of quantities. This is particularly important for a complex topic like quadratic functions, where the quantities and relationships between quantities are numerous. Drawing diagrams was also utilized in the individual interviews because, as shown above, they provide a rich source of data for investigating student conceptions. In the results section, I will show that changes from pre- to post-interview in the student-drawn diagrams suggested a backward transfer effect.

**Algebra in the early grades.** Thus far, literature has been presented that argues for the use of quantitative reasoning and self-constructed diagrams as a way to prepare students in early grades for algebra. However, this raises the question of whether content like quadratic functions needs to be introduced to students in early grades (i.e., Grade 8 and earlier). In this section, an argument is presented for introducing algebra topics early on.
There have been two main motivations behind the movement to introduce algebra in middle school (or earlier). First, there is a concern that U.S. students are behind their peers in other countries. According to the Third International Mathematics and Science Study, only 20% of U.S. 8th graders take Algebra 1, and a full one-third attend schools that don’t even offer algebra (Schmidt, 2002). Second, algebra acts as a filter, preventing at-risk students from accessing many career opportunities (Kaput, 1995; National Council of Teachers of Mathematics, 2000). These two problems have led researchers, educators, and policymakers to recommend that algebra concepts be introduced earlier (Davis, 1985, 1989, cited in Carraher, Schlieman, Brizual, & Earnst, 2006; Kaput, 1998, cited in Carraher et al., 2006; LaCampagne, 1995, cited in Carraher et al., 2006; Lins & Gimenez, 1997, cited in Carraher et al., 2006; Mason, 1996; NCTM, 2000; RAND Corporation, 2003; Schoenfeld, 1995; Usiskin, 1987, cited in Chazan, 1996; Vergnaud, 1988).

Many states, including California, have responded by adopting the policy that all students must have taken algebra by the end of eighth grade (e.g., California State Board of Education, 2006). However, middle school algebra courses often provide little more than traditional experiences focused on symbol manipulation. An added complication is that middle school teachers are often unprepared to teach algebra (Steen, 1999). Therefore, new policies are needed that take into account expanded notions of what middle school algebra should be, including that algebraic reasoning is about reasoning through problem situations in ways that involve representing, generalizing, and justifying (National Research Council, 2001). Recent expanded notions of algebra have set the stage for research that involves a reexamination of the
content and pedagogy of algebraic experiences that are accessible to students in the early grades (e.g., see Kaput, Carraher, & Blanton, 2008). The dissertation study responds to the need to reconsider algebra for middle school students by exploring ways to introduce complex algebraic concepts for conceptual understanding so that a foundation is built on which a more sophisticated understanding can be constructed.

Revisiting the Research Questions

In Chapter 1, the following research questions were first presented:

1. When the targeted conceptual understandings of an instructional unit are conceived of as being built upon particular prior knowledge, when and in what ways does the newly constructed knowledge lead to productive changes in prior knowledge (productive backward transfer)? When and in what ways does it contribute to unproductive changes (unproductive backward transfer)?

In the context of students in introductory algebra courses learning about linear and quadratic functions, this first question asks, when and in what ways does newly constructed knowledge about quadratic functions lead to productive changes to prior knowledge about linear functions?

2. In what ways are backward transfer (influences by a person’s newly constructed knowledge on prior knowledge) and forward transfer (influences by a person’s newly constructed knowledge on how they come to understand novel situations) related?

In the context of the algebraic content targeted in this dissertation research, this second question asks, in what ways is the backward transfer of learning experiences with quadratics functions onto students’ understanding of linear functions related conceptually to the transfer of those same learning experiences to novel situations involving quadratic functions?
3. What are the transfer processes by which classroom instruction leads to productive backward transfer?

In the algebraic context of this study this third question asks, what are the processes that explain how quadratics instruction promotes productive influences on one’s prior knowledge of linear functions? Each question is elaborated using ideas from the literature review on forward and backward transfer and on the mathematics relevant to the dissertation study.

**Question 1**

Question 1 addresses the changes to students’ prior knowledge of linear functions, as a result of participating in instruction on quadratic functions. Addressing this question involves examining changes in the ways that students conceive of linear functions, constant rates of change, and coordinating co-varying quantities. For example, it may be that backward transfer manifests itself as a change from thinking about linearity as a correspondence to thinking about it as a dependency relation. Alternately, it may be that students shift from being able to coordinate the changes in two variables when one of the quantities changes by 1 unit to being able to coordinate changes in two variables when both quantities change by greater than or less than 1 unit.

An AOT approach was taken to respond to Question 1. In other words, any influences from the quadratic functions unit on students’ reasoning about linear functions were counted as instances of backward transfer, regardless of whether or not they fit the way an expert understands linear functions. Particular attention was directed toward examining students’ thinking about the big ideas from linear functions.
(e.g., forming ratios, reasoning proportionally), their quantitative reasoning and their diagrams. In looking for changes in thinking, verbal reports, written inscriptions, and gestures were all considered.

**Question 2**

Question 2 addresses the associations that may or may not exist between backward transfer and forward transfer. Recall from the preliminary analysis of pilot data, that Matt’s thinking about linear and quadratic functions seemed to be related. For example, Matt appeared to exhibit a focus primarily on the dependent variable in both linear function and quadratic function contexts. In other words, the focus on the dependent variable cut across function types. Other common threads in ways of thinking about linear and quadratic functions were examined in the dissertation study. Identifying these common threads helped reveal what students learned about the big ideas of quadratic functions during instruction (i.e., changing rates of change, constant rate of change of the rate of change) that influenced their thinking about linear functions (backward transfer) as well as their reasoning about novel quadratic function contexts (forward transfer).

**Question 3**

Question 3 addresses the instructional supports for productive backward transfer of newly constructed knowledge about quadratic functions onto prior understandings of linear functions. Lobato et al.’s (2011a) focusing framework was used to identify the particular perceptual or conceptual mathematical objects that individual students noticed during instruction (their centers of focus), as well as the socially-situated features of the learning environment that helped shape what they
noticed (the focusing interactions, the features of the mathematical tasks and the
nature of the mathematical activity). I also looked for ways that the centers of focus
were related conceptually to the findings of backward transfer from the analysis of the
interviews (i.e., Question 1). Thus, in my response to Question 3, I accounted for
noticing as both a psychologically- and socially-organized process and found evidence
supporting the claim that noticing is a backward transfer process.
CHAPTER 3:
RESEARCH METHODS

In the previous two chapters, it has been shown that the phenomenon of interest, backward transfer, has not been well-studied in the context of mathematics education. Moreover, a case has been made for why this phenomenon is related to the more prominent concept of forward transfer and why an actor-oriented transfer (AOT) approach would best serve this study. In this chapter, the methods used to examine backward transfer in the dissertation study will be presented.

As stated earlier, the dissertation study was the third iteration of a design-based research study. The previous two iterations were part of ongoing NSF funded research by Dr. Joanne Lobato (REC-0529502). The design of the dissertation study was informed by those earlier research efforts. However, the third iteration also contained several unique features because backward transfer was the primary phenomenon of interest in this iteration.

This chapter is organized into three sections. In the first section, I discuss the relationship between the research questions and the research design. In the second and third sections, I describe the data collection and data analysis methods.

Addressing the Research Questions

The purpose of the dissertation study was to investigate the following three questions:

1. When the targeted conceptual understandings of an instructional unit are conceived of as being built upon particular prior
knowledge, when and in what ways does the newly constructed knowledge lead to productive changes in prior knowledge (productive backward transfer)? When and in what ways does it contribute to unproductive changes (unproductive backward transfer)?

2. In what ways are backward transfer (influences by a person’s newly constructed knowledge on prior knowledge) and forward transfer (influences by a person’s newly constructed knowledge on how they come to understand novel situations) related?

3. What are the transfer processes by which classroom instruction leads to productive backward transfer?

To investigate the nature of backward transfer (Question 1), the research design included (a) a pre-interview, (b) a design-based intervention consisting of instruction on quadratic functions that, it was conjectured, would lead to productive backward transfer in the form of enhanced understanding of linear functions, and (c) a post-interview. For this study, it was important to obtain detailed baseline information about students’ prior knowledge of linear functions before the start of the intervention, because the main emphasis was on how students’ knowledge of linearity changes. Therefore, the pre-interview focused primarily on an in-depth exploration of students’ understanding of linearity and less on assessing students’ incoming knowledge of quadratics.

The post-interview was conducted the day after the last instructional session. Like the pre-interview, this interview contained an in-depth exploration of students’ understanding of linearity because a primary goal was to investigate how their understanding of linear functions changed as a result of the instruction on quadratic functions.
The post-interview also included two quadratic functions tasks. One task was closely aligned to the intervention and helped establish that students learned something about quadratics during the intervention. The other task constituted a classic lateral transfer task. Specifically, it addressed ideas regarding quadratic functions set in a novel context (i.e., a context not explored in the instructional intervention; Lobato, 2008a). Student responses to this task were used to determine if there was any evidence of forward (lateral) transfer.

While control groups (i.e., groups who receive no instruction) are important for many forward transfer studies, one was not included in this study for two reasons. First, it was not a goal of the study to obtain stochastic evidence that backward transfer occurred. Instead, by taking an AOT approach to investigate backward transfer, the goal was to qualitatively compare the nature of students’ reasoning during learning experiences to their thinking on transfer tasks and to infer the connections that they appeared to be making between the two. Second, the effectiveness of the instructional materials per se were not being evaluated. Instead, the focus was on the nature of the influence of the instructional materials on students’ reasoning about quadratic and linear functions.

The instructional intervention constituted the learning experiences in this transfer study. It was crucial that a design-based intervention be used rather than an intervention based on typical instruction on quadratic functions because the mathematical experiences that were conjectured to promote productive backward transfer are not commonly found in traditional instruction (Steffe & Thompson, 2000). Furthermore, there were no known curricular units that emphasized quantitative
reasoning with quadratic functions prior to the development of the first iteration of the intervention by the larger ongoing design-based research study. By conducting a third iteration of that unit, the dissertation study was able to further develop this instructional approach to quadratic functions.

In Chapter 4, findings will be presented that address the first research question. These findings will show that students’ learning experiences with quadratic functions during the instructional intervention influenced their prior knowledge of linear functions (i.e., backward transfer). The specific logic of argumentation on which these findings were based is an adaptation of Lobato’s (2008a) method for demonstrating the transfer of learning from an AOT perspective. In particular, there was evidence in the data supporting the following four-step argument for each finding of backward transfer:

Step 1: There was significant change, from pre- to post-interview, in the student’s conceptualization of and reasoning about linear functions in the backward transfer tasks.

Step 2: The student’s reasoning on the pre-interview quadratic tasks demonstrated their limited knowledge of quadratic functions coming into the study, meaning that there was evidence that the students learned about quadratic functions during the instruction rather than entering the study with that knowledge.

Step 3: A plausible conceptual connection was established between the student’s reasoning on the linear backward transfer tasks in the post-interview and some activity during the design-based instruction.

Step 4: There was evidence to make a case that the changed reasoning on the backward transfer tasks was not due entirely to the spontaneous construction of knowledge during the interview, but was linked to what the students learned during the instruction.

To categorize students’ reasoning about linear functions, I used a mixed approach (Miles and Huberman, 1994): some categories were derived from the literature (e.g.,
iterating composed units), and other categories were induced using open coding from grounded theory ((e.g., coordinating non-proportional and proportional relationships; Strauss, 1987).

In Chapter 5, findings will be presented that address the second research question. These findings will show that students’ learning experiences with quadratic functions during the instructional intervention also influenced their reasoning on novel quadratic function tasks (i.e., forward transfer). Furthermore, it will be shown that the findings of backward transfer presented in Chapter 4 and the findings of forward transfer presented in Chapter 5 are related. Forward transfer was established using a similar claim structure as identified above for backward transfer. Additionally, a mixed approach was utilized to categorize students’ understanding of quadratic functions (Miles and Huberman, 1994), with some categories derived from the analysis of students’ reasoning during previous iterates of the design-based instruction, while other codes were induced using open coding (Strauss, 1987). Then axial coding from grounded theory (Strauss & Corbin, 1990) was used to identify relationships among categories and conceptual connections between forward and backward transfer (see the data analysis section below for more details).

In Chapter 6, findings will be presented that address the third research question. These findings will show that noticing provides explanatory power to account for occurrences of backward transfer. This question was addressed by looking for the conceptual connections between the ways in which students generalized their learning experiences with quadratic functions to tasks involving linear functions (i.e., backward transfer) and the ways in which the instructional environment directed
students toward noticing some mathematical properties of the perceptual and/or conceptual field and not others. Specifically, there are two parts to the logic of the argumentation, and both rely on use of the *focusing framework* from Lobato et al. (2011a). First, conceptual connections were established between the changes in students’ reasoning on the linear tasks in the interviews (evidence of backward transfer) and what individual students noticed mathematically during instruction (their *centers of focus*). Second, conceptual connections were identified among the centers of focus that emerged during instruction and the discourse practices (*focusing interactions*), features of *mathematical tasks*, and aspects of the *nature of mathematical activity* that appeared to give rise to the centers of focus.

Each component of the focusing framework is associated with a separate analysis of the data. In the first analytic pass of the classroom data, the *centers of focus* were identified using open coding (Strauss & Corbin, 1990). In the second analytic pass, focusing interactions that occurred during the time in which each center of focus emerged were examined using a priori codes for discourse practices, such as *naming, highlighting, and quantitative dialogue* (Goodwin, 1994; Lobato et al., 2011a). In the third pass, the affordances and constraints of the *features of the mathematical tasks* that appeared to influence what students noticed mathematically were examined (à la Watson, 2004). Finally, in the fourth pass, the nature of the mathematical activity of each classroom was identified through a description of the particular expectations governing students’ and teachers’ roles that seemed to contribute to the emergence of centers of focus (following Cobb, Gresalfi, & Hodges, 2009; and Cobb, Wood, Yackel, & McNeal, 1992).
In general, the dissertation study was geared toward theory generation instead of theory testing (Sloane & Gorard, 2003). This reflects the newness of the actor-oriented transfer perspective and the lack of existing theories about backward transfer in the mathematics education literature.

Data Collection

In this section, I begin with a description of the participants and the setting for the dissertation study. I follow this with a description of the design-based research model on which the instructional intervention for dissertation study was based. I conclude with a description of the interview instrument that was used.

Participants and Setting

There were two types of participants, the students and the teacher. I will describe each in turn. I also describe the setting for the 16-lesson instructional intervention and the pre- and post-interviews.

Students. The students in this study were 7th and 8th graders who were recruited from a local middle school. The original goal was to recruit eight to ten students. The rationale behind having this many students was that it would be sufficiently large to allow the teacher the flexibility to create both whole-class discussions and small-group explorations, and at the same time, sufficiently small to permit the teacher to also assume the role of researcher, making observations in situ. During recruiting, only seven students volunteered. However, this number of subjects was similar to the number that participated in one of the two design cycles of the larger study. Therefore, I felt that seven students would suffice to achieve the goals of the study.
In previous iterations of the design-based research project, participating students already had multiple experiences with linear functions in their regular mathematics classes prior to the interventions, but few (if any) experiences with quadratic functions. This was possible because the studies took place in the spring, after the students’ regular math classes had already covered linear functions but before or during when quadratic functions were being covered. The dissertation study was conducted at the end of the school year and so all students had already participated in an introductory unit in their regular math classes on quadratic equations. However, it was unlikely that the students had explored quadratic functions with a quantitative-reasoning approach because an examination of the textbook revealed an emphasis on procedures for solving quadratic equations, including the quadratic formula, as well as a brief exploration of graphical representations. Furthermore, the pre-interview quadratic function items revealed that the students had very little prior knowledge of the big idea that guided the instructional unit, namely that the rate of change of the rate of change of a quadratic function is constant.

In the preliminary analysis of the pilot data, it was observed that the students who displayed an average understanding of the mathematics in the pre-interview tasks, were the same students who showed the greatest change in thinking from pre- to post-interviews on the linear function task. This observation aligns with findings by Chen (2006), that backward transfer from L2 to L1 occurred for Chinese ESL speakers who had attained an average level of English proficiency (Level 2), but did not occur for speakers who had attained a low- or high-level proficiency (Level 1 or 3). These findings suggested that to maximize the amount of observable backward transfer, the
majority of participants should be those who come to the study with some understanding of linearity but whose understanding needs further development. Thus, the goal for the dissertation study was to recruit more students with an average level of understanding about linearity than students with a high or low level of understanding. As will be seen in the results section, three of the seven students that participated in the study were categorized as high-level proportional reasoners, three were categorized as mid-level, and one was categorized as low-level.

The initial plan was to recruit a group of participants that were balanced on gender and that reflected the distribution of ethnicities at the school (the school population was 1334 students with 50% Hispanic, 20% Filipino, 15% White, 7% Asian, 7% African-American, and 1% Pacific Islander). However, this was not possible given the small sample size (n=7). Of the seven participants, two were female and five were male; two were African-American and five were Hispanic.

**Teacher.** The author of this dissertation served as the teacher. One reason for this was because the author had been closely connected to the previous two iterations of the larger design-based research study. By serving as the teacher, the researcher was able to ensure that the dissertation study closely aligned with and further developed the approach used in the previous iterations of teaching quadratics to middle school students.

A potential drawback was that the researcher became part of the data. However, as will be explained below, this approach is consistent with design-based research, in which the researcher/teacher distinction often gets blurred in order to facilitate the refinement of the instructional intervention while the study is being
conducted. This is seen as particularly important for developing a better understanding of phenomena that are not well understood (Cobb et al., 2003; Design-Based Research Collective, 2003).

**Setting.** The same school site was utilized as for previous iterations of the larger design-based research study. Having a past history with the school meant that relationships had already been formed between the research team for the larger study and the personnel at the school. These relationships were important because the successful recruitment of students and the smooth implementation of the instructional intervention depended in large part on the cooperation of school personnel. However, using the same school site as in previous iterations was also in keeping with the cyclic nature of the design-based research model.

**Design-Based Research Study**

To situate the methodology that will be used for the instructional intervention, a short review of how design-based research is described in the education literature is presented. Then, the following features of the intervention will be presented: (a) the conjecture regarding how to promote productive backward transfer instructionally, (b) the conceptual learning goals for a beginning unit on quadratic functions, (c) the general pedagogical approach, and (d) other logistical details of the intervention.

**Review of design-based research.** This methodology is a relatively new approach to studying innovation (Dede, 2005), which draws from the field of engineering (Hjalmarson & Lesh, 2008). One only needs to go back twenty years to find the original proponents of design-based research (DBR) in Collins (1992) and Brown (1992) (Kelly, 2003; Cobb et al., 2003; Sandoval & Bell, 2004). However, this
review focuses on more recent thinking on DBR. It is organized into three sections, exploring, in turn, the features, benefits, and limitations of DBR.

**Features of DBR.** There are three main features of DBR. First, DBR involves iterative design or design cycles (Cobb et al., 2003; Sandoval, 2004; Hoadley, 2004). Rather than testing one implementation of an intervention, as would be done in an experiment, DBR interventions are implemented and revised multiple times (Bannan-Ritland, 2003). Bannan-Ritland further elaborates that each cycle involves an informed exploration, an enactment of an intervention and then an evaluation of the enactment at both local and global levels. The use of iterations has a different purpose than replication in an experiment. In DBR, conducting multiple iterations of an intervention is seen as fundamentally important feedback for developing a nuanced understanding of the particular phenomenon of investigation (Hoadley, 2004; Lobato, 2003).

Second, DBR is simultaneously about generating theories of learning and improving instructional practices (Cobb et al., 2003). According to the Design-Based Research Collective (DBRC, 2003), while an initial theory about a particular learning phenomenon is developed prior to the first iteration of an intervention, the theory undergoes a refinement (cf. Sandoval, 2004; Kelly, 2003), and new theories are generated *in situ* (Cobb et al., 2003). The result is that the theories are more immediately sharable with educational practitioners (Hoadley, 2004; DBRC, 2003). It also means that while the intervention is taking place, the researcher is developing a deepening understanding of the learning phenomenon (Cobb et al., 2003).

The theorizing that was undertaken in the context of the dissertation study was
about backward transfer and transfer processes. Specifically the dissertation study theorizing extends the ideas about forward transfer and noticing that emerged from the larger study into the domain of backward transfer.

At the same time as theories are developing and new theories are being generated, the instructional intervention is evolving into something that is educationally more valuable (Bannan-Ritland, 2003). Because DBR allows for the intervention to change and evolve, idiosyncratic barriers to effective instruction can be identified during the study and remedied. This means that researchers can sometimes bypass expensive studies (Hoadley, 2004). It also means that there is a high likelihood that DBR interventions will be used in practice (Bannan-Ritland, 2003). Thus, DBR balances the tension between explanatory goals (i.e., developing theory) and performance goals (i.e., developing effective interventions; DBRC, 2003).

Third, in DBR the distinction between researcher and education practitioner becomes blurred (Hoadley, 2004; Kelly, 2003; McCandliss, Kalchman, & Bryant, 2003; Bannan-Ritland, 2003). Before and during the intervention, the researcher and educator collaborate closely (McCandliss et al., 2003). For example, they have regular debriefing sessions in order to make revisions to the intervention. Therefore, instead of shielding his or her perspective from the study, the researcher becomes a key participant in the intervention (Hoadley, 2003). Similarly, because the educator participates in the co-construction of the intervention, he or she becomes a co-investigator (Collins, 1992). In some cases, the researcher and educator is the same person (Cobb et al., 2003) as was the case in this dissertation study.
**Benefits of DBR.** There are several benefits to using DBR. One benefit is that theory and practice are intimately connected (Hoadley, 2004; Sandoval & Bell, 2004; DBRC, 2003). This becomes a benefit when one considers that education research has often been criticized for not producing “usable knowledge” (Lagemann, 2002, cited in Sandoval & Bell, 2004, p. 199). As stated above, theory that emerges or that is refined in a DBR study has relevance for practitioners because it is grounded in and directly applicable to practice (DBRC, 2003).

A second benefit is that a DBR investigation allows researchers to go deeper than just designing and testing interventions. This methodology facilitates an exploration of the relationships between theories, artifacts, and practices (DBRC, 2003). Furthermore, DBR methods can be used to generate plausible causal accounts by developing theories about processes (DBRC, 2003; Cobb et al., 2003; Sandoval, 2004). Brown (1992) explains this difference between DBR and traditional experimentation in education as a shift from focusing on the products of learning to focusing on the processes of learning. Therefore, this feature of DBR aligns well with taking a process-orientation to causality (Maxwell, 2004).

A third benefit is that DBR is ideally suited for learning conditions that are not well understood (DBRC, 2003). Cobb et al. (2003) describe DBR as “test beds for innovation” (p. 10) where new forms of learning are studied. These innovations include new educational technologies, complex approaches to classroom teaching (Sandoval & Bell, 2004) and innovative educational environments (Brown, 1992).

**Limitations of DBR.** There are also several limitations to DBR methods. One limitation is that the theories generated by DBR are not universally generalizable
across contexts (Hoadley, 2004). Instead, DBR generates localized, domain-specific theories (Sandoval, 2004; Cobb et al., 2003; diSessa, 1991, cited in Hoadley, 2004). If a researcher chooses to conduct DBR, then they must realize that there is a trade-off between being able to generalize results across contexts and being able to immediately apply the knowledge that is generated to the context in which it was generated (Hoadley, 2004). One reason a researcher might make the choice to use DBR would be because they are operating under the assumption that learning is fundamentally situated (Sandoval, 2004; Cobb et al., 2003). A second reason would be that not enough is yet known about the learning phenomenon in question, in order for the researcher to have conjectures which he or she feels confident about testing with an experiment or randomized clinical trial (Cobb et al., 2003). This second reason applies to the dissertation study because little is known about backward transfer in the context of learning mathematics.

A second limitation is that the DBR researcher, being both critic (researcher) and advocate (education practitioner) of the performance outcomes of the intervention, threatens the construct validity of the study (DBRC, 2003). In other words, by deliberately intervening in the study, a researcher’s own agenda might influence the results (Hoadley, 2004). For example, in this study, the instructional intervention design changed based on the daily debriefing sessions between the teacher/researcher and the observers.

There are several ways to safeguard against this threat to validity. One safeguard is for the researcher to explicitly document and monitor his or her agenda and intervention strategies (Hoadley, 2004). This will allow others to assess the merits
of any results. A second safeguard is for the researcher to triangulate using multiple sources of data, as well as to look for alignment of theory, design, practice and measurement over time (DBRC, 2003). For the dissertation study, this meant aligning the design of quadratics instruction, the theory about backward transfer of quadratics function knowledge, the instruction of the quadratics unit, and the measurement of backward transfer. A key component of this alignment was to provide an instructional intervention that constituted authentic quadratic functions instruction. This meant that even though the production of productive backward transfer onto students’ prior knowledge of linear functions was a primary goal, it was critically important that the instruction that was provided encouraged students to develop productive ideas about quadratic functions.

However, Hoadley (2004) also contends that while construct or measurement validity is under threat in DBR, the overarching systemic validity is under greater threat in traditional education research. Hoadley argues that there is systemic validity when theories are fairly tested and the results are communicated in a way that informs practice. In other words, a study that is systemically valid is one that informs the original questions that motivated it. To illustrate systemic validity, consider Robinson’s (1998) report that despite seventy years of research that shows the ineffectiveness of tracking (e.g., Oakes & Guiton, 1995; Wells & Serna, 1996, cited in Robinson, 1998), this research has to a large extent failed to inform practice and has failed to resonate with practitioners and stakeholders (e.g., administrators, policy-makers, parents). Robinson (1998) contends that the tracking research was largely ignored is because “the empiricist press to produce formal theory is counterproductive
when it preempts inquiry into the detail of practitioner reasoning” (p. 22). By
Hoadley’s (2004) standard for systemic validity, traditional studies do not measure up. According to Hoadley, a major strength of DBR is that it can maintain a high level of systemic validity because the researcher is intimately involved in carrying ideas from theory creation to enactment to revision.

For the dissertation study, this means that while the results from the intervention may not be generalizable to other contexts without further investigation (perhaps with traditional experimentation), the results might be immediately useful to education practitioners within similar contexts as those present in the instructional intervention.

**Conjecture about Promoting Backward Transfer**

As stated above, the instructional intervention was the third iteration of the larger design-based research study. Consistent with the first two iterations, the same actor-oriented approach to transfer was adopted and similar quadratic functions instruction was use. However, in keeping with the DBR methodology, the intervention for this study was a refinement the first two interventions based on findings such as the CLGs presented above (Lobato et al., in press). The previous two iterations were focused on gathering information about forward transfer for quadratic functions, whereas this study investigated both forward and backward transfer, with more focus on the latter. Therefore, the conjecture that guided the design of the instructional intervention for the dissertation study differed from previous conjectures, namely in that it addressed producing productive backward transfer effects on prior knowledge of linearity while simultaneously promoting the development of productive
Activities set in the context of quadratic functions that promote covariational reasoning and that involve making point-by-point comparisons between quadratic functions and the linear functions embedded within them will help learners develop their understanding of quadratic functions and at the same time productively influence their prior understandings of linearity.

Contained within this conjecture are two instructional supports for promoting productive backward transfer (i.e., promoting reasoning covariationally and promoting point-by-point comparisons between quadratic functions and embedded linear functions). These instructional supports will be discussed next.

**Covariational reasoning.** One instructional support was to engage students in activities that promote covariational reasoning in the context of quadratic functions. There were at least three ways that this principle was used to guide the design of the intervention. First, the use of *language of change* was promoted. This means that an attempt was made to establish the sociomathematical norm (Cobb & Yackel, 1996) that the result of the comparison of two values of one quantity is referred to as a change in the quantity from one of the values to the other. This is an important part of covariational reasoning because covariation involves attending to the changes in one quantity in relation to the changes in another quantity (Carlson et al., 2002).

Second, students were encouraged to attend to changes in the dependent variable for various sized changes in the independent variable (i.e., greater and smaller than 1 unit) and for nested changes in the independent variable (e.g., students found the speed for a quadratic distance-time function within the following intervals of time: from 2 to 3 s, from 2 to 2.5 s, from 2.5 to 3 s and from 2.25 to 2.75 s; see Figure 1.6.
for a similar example). It was conjectured that in attending to changes for various sizes of intervals and changes for overlapping intervals, students would be more inclined to engage in covariational reasoning than if the changes in the independent variable were always the same (see Appendix A for a sample lesson involving various interval sizes).

Third, it was also conjectured that engaging students in an exploration of data in which the dependent variable of a quadratic function, instead of the independent variable, was spaced at equal intervals, could help students to develop an attentional focus that was more evenly balanced between the changes in the independent variable and changes in the dependent variables. The intention was to use all three strategies presented above (see Appendix B for an articulation of the instructional goals for each lesson). However, because of time constraints only the first two strategies were used. Therefore, the third strategy could be tested in a future iteration of the design-experiment.

Making point-by-point comparisons. The second instructional support was to encourage students to make point-by-point comparisons between quadratic functions and the linear functions that are embedded within them. By embedded linear functions, I mean, (a) linear functions that share one point with a quadratic function and that have a constant rate of change that matches the quadratic function’s rate of change at that point (see Figure 1.7a), and (b) linear functions that share two points with a quadratic function and that have a constant rate of change that matches the quadratic function’s average rate of change between the two points (see Figure 1.7b). An example of students making these comparisons was when, during the
intervention, they considered the constant rate of change of distance with respect to time that would be necessary for an object to start and end at the same position and time as an object moving according to a quadratic distance-time relationship. It was conjectured that this kind of activity, in which students compared an embedded linear function, on a point-by-point bases, to the quadratic function in which it was embedded, would help students gain a deeper understanding about quadratic functions and, at the same time, deepen their understanding of linearity and promote iterating and partitioning a constant rate (see Appendix B for lesson goals). Note that students only explored the second kind of embedded linear function in the instructional intervention (i.e., Figure 1.7b).

**Instruction focused on quadratic functions.** To protect the validity of the findings regarding backward transfer, the intervention needed to constitute authentic quadratic functions instruction rather than linear functions instruction disguised as quadratics. At the same time, it was assumed that the development of quadratic functions would rely on linear function concepts as a foundation. Consequently, there were activities in the instructional intervention that explored constant rates of change, but this was always in the service of developing an understanding of quadratic function concepts. Additionally, the instruction was developed as a replacement unit for quadratic functions in a middle school, as opposed to an introduction to both linear and quadratic functions. There were two features of the instructional design that helped ensure a focus on quadratic functions.

The first feature was that the design of the dissertation intervention was based on the previous iterations for the larger study. In the previous iterations, the goal had
been to design an instructional unit about quadratic functions at beginning levels. The following aspects of the interventions from the first two iterations were incorporated into the design of the intervention for the dissertation study: (a) a distance-time quadratic function context was used, (b) students were encouraged to draw self-designed representations of problems involving quadratics as a way to make sense of the relationships between quantities, (c) the conception of quadratics that was promoted was that the change in speed for equal-sized successive time intervals was constant (i.e., acceleration was constant), (d) the sequencing of concepts in the instruction started with distance and time, and was followed by increasing speed, changes in increasing speed, and acceleration in that order, and (e) computer simulations of distance-time quadratic functions were used (i.e., SimCalc Mathworlds™ software).

Second, the instruction was designed to target the five conceptual learning goals (CLGs) that emerged from the previous iterations (Lobato, et al., in press). In the context of quadratic distance-time functions, the five CLGs are: (a) students differentiate elapsed distances from accumulated distances and reason with the elapsed distances as a quantity, (b) students give equal attention to the elapsed times and elapsed distances, (c) students compose a set of elapsed distances and times to form a set of increasing speeds as an approximation of continuously increasing speed, (d) students conceive of changes in average increasing speeds as speeds instead of as distances, (e) students compose the rates of change of speed and corresponding changes in time to form acceleration and understand that, for a quadratic distance-time function, acceleration is constant.
There were also differences between the intervention for the dissertation study and the interventions from the larger study. One difference was that student attention was directed to changes in quantities and covariation using the two instructional supports described above. A related difference was that students did not explore quadratic function equations and graphs in the dissertation study, focusing instead on tabular representations and computer simulations. This was because the two conjectured instructional supports for backward transfer were well-suited to tabular data and computer simulations of quadratic distance-time functions, and instructional time to pursue these backward transfer activities was created by not addressing graphs and equations. A sample lesson has been included in Appendix A to provide the reader with a sense for how these features of the instructional intervention manifested themselves.

**Interviews Instruments**

The students in this study participated in two clinical one-on-one interviews (Clement, 2000; Ginsburg, 1997). The first interview occurred the day prior to the first lesson, and the second interview occurred the day following the 16th (and final) lesson. Dr. Joanne Lobato and the dissertation author served as the interviewers. The same tasks were presented to each student, but the follow-up probes were tailored to individuals (Ginsburg, 1997).

The use of clinical interviews is consistent with taking an AOT approach. Clinical interviews are sensitive to subtle observations and can provide researchers some access to students' personal interpretations and meanings (Ginsburg, 1997), which is critical to the process of inferring the ways in which learning experiences and
transfer tasks are related from an actor’s point of view. An important feature of clinical interviews, which results in increased sensitive to subtle observations, is hypothesis testing (Clement, 2000; Ginsburg, 1997). Hypothesis testing can occur when the student in the interview provides a response that the interviewer finds puzzling. During the interview, the interviewer tries to formulate an hypothesis about what the subject is thinking. Then, the interviewer attempts to generate new data that supports or refutes his or her hypothesis, by probing the student’s thinking further.

Theorizing about the responses a student makes during a clinical interview also takes place after the interview has ended (Clement, 2000). In order to obtain a rich record of student responses that could be viewed over and over, the interviews for the dissertation study were recorded with a video camera and a table microphone. The camera operator was instructed to capture on camera the students’ verbal reports and gestures and to zoom-in on the production of inscriptions that the students made on paper.

**Pre-intervention interviews.** The main goal of the pre-interview was to establish a baseline for students’ understanding of linear functions. A secondary goal was to determine the level of quadratics knowledge the students came into the study with. Therefore, the interview consisted of five main problems: four were set in a linear context, one was set in a quadratic context that was similar to the contexts used in the intervention, and one was set in a quadratic context that differed from the contexts used in the intervention. Each interview was 75 minutes in duration. (see Appendix C for two sample interview tasks).
**Post-intervention interviews.** The post-interview format was similar to the pre-interview (i.e., clinical, one-on-one, 75-minute interviews). The problems were also similar. However, some of the numerical values and some of the contexts of the problems were different. Also, the probing of student thinking was somewhat different in the post-interview because the primary goal for the interviewer was to explore how students’ conceptions of linearity had been influenced by the intervention.

**Logistics of Data Collection**

In this section, the following logistical details for conducting the instructional intervention and collecting both classroom and video-taped data are discussed: (a) the supporting research team, (c) the time frame, and (d) the kinds of data being collected.

**Research team.** During the intervention, the seven participating students were often divided into two groups to work together on activities and explore ideas. This meant that two camcorders and two operators were needed during each lesson. Dr. Lobato and five graduate students shared the camcorder operator duties.

The camera operators served a dual role in acting as observers for each lesson as well. Having observers present was consistent with the DBR methodology (Steffe & Thompson, 2000). The observers served two purposes. First, the observers assessed whether the instruction was focused on quadratic functions. Second, the observers acted as extra sets of eyes and ears for the teacher/researcher with respect to students’ reasoning. The observers shared their observations with the teacher/researcher at the end of each class, during a debriefing session. Cobb et al. (2003) identify several goals for debriefing: (a) to deepen the researcher’s understanding of
the phenomenon, (b) to facilitate ongoing design of the intervention, (c) to further the evolution of the conjecture, and (d) to communicate what the researcher is learning so as to open it to public scrutiny. Cobb et al. also described the debriefing session as the time to interpret past classroom events and plan for future events.

**Time frame.** Data collection occurred at the end of the 2010 academic year when students at the middle school were on their summer break. During this time, no classes were in session at the school. One benefit of the students being on break was that data collection was conducted during the morning hours, which was more appealing to the 8th graders than staying after school for each session. The intervention was comprised of 16 one-hour lessons, which was approximately the same length as the instructional sessions conducted in the previous two design cycles for the larger study and approximately the same length as an instructional unit in a typical algebra class.

**Kinds of data.** The kinds of data collected from each lesson in the instructional intervention consisted of video/audio recordings of the lessons and debriefing meetings, observer field notes, student notes and curricular materials. Audio of all interviews and of select parts of the instructional intervention (75% of all classroom video data) were transcribed. The sessions were recorded by two video camcorders. One camcorder was used to capture whole-class discussion, and each camcorder focused on one of the two student groups when the students were involved in small-group work. Each table had a microphone to capture small group discussions and individual student-teacher interactions. Three hanging microphones were also
strategically placed around the room to capture whole-class discussion. An audio mixer was used to combine the inputs from the three hanging microphones.

**Data Analysis**

Cobb et al. (2003) call the analysis that happens after a design experiment is complete, a retrospective analysis. They describe it as analysis that “attempts to generate a coherent framework that accounts for these effects [effects observed during the intervention], thus making it possible to anticipate outcomes in future designs” (p. 13). This section begins with an overview of the mixed methods qualitative approach used to conduct the retrospective analysis for each research question (Miles & Huberman, 1994), followed by an elaboration of the specific ways in which the mixed methods approach was used to address the first two research questions. Then, I elaborate how the Lobato et al. (2011a) focusing framework was used in conjunction with a mixed methods qualitative approach to respond to the third research question. Finally, I address issues of reliability, validity, and disconfirming.

**Overview of the Mixed Qualitative Approach**

Each part of the data analysis involved a mixed methods qualitative approach, what Miles & Huberman (1994) describe as “partway between a priori and inductive approaches” (p. 61). Categorizing students’ understanding of linear and quadratic functions involved using some constructs from the mathematics education literature (e.g., iterating composed units; Lamon, 2007; Lobato & Ellis, 2010), while new categories were induced using **open coding** from grounded theory (Strauss, 1987) when the pre-established constructs had been exhausted (e.g., coordinating non-proportional and proportional relationships). Strauss and Corbin (1990) acknowledge
the use of literature as legitimate within a grounded theory approach, as long as one is not overly constrained by existing categories.

Sometimes analyses in this study relied more heavily on either a priori codes or inductive codes. For example, the analysis of the classroom data for focusing interactions used discursive practices identified in prior research on the social organization of noticing (Goodwin, 1994; Lobato et al., 2011a, 2011b). Also, the focusing framework, which organized and helped reduce the classroom data for analysis, derived from prior efforts by Lobato et al. (2011a, 2011b). However, most of the centers of focus (the mathematical features that individual students’ noticed) were induced via grounded theory, due largely to the newness of this type of research on student noticing in classroom settings. Because the use of techniques from grounded theory (Glaser & Strauss, 1967) played a large role in these analyses, further elaboration of the features and methods from grounded theory relevant for this study follows next.

Coding is an important feature of grounded theory analysis in which data are fractured and reorganized into categories, which are then named and described (Strauss, 1987). Developing categories of concepts is the basis for the creation of theory, which captures the researcher’s interpretation of how the categories relate to each other (Glaser & Strauss, 1967). Specifically, “Theory consists of the plausible relations proposed among concepts and sets of concepts” (Strauss & Corbin, 1990, p. 278). The emphasis on categorization and interpretative theory-building distinguishes grounded theory from a descriptive grounded approach (Miles & Huberman, 1994; Strauss & Corbin, 1994).
Two types of categorizing from grounded theory were used in this dissertation study—*open coding* and *axial coding* (Strauss & Corbin, 1990). Categorizing begins with open coding, which is where the data are broken apart into discrete incidents. Incidents that share similarities are grouped together. Each group is given a category name so that it can be thought about and talked about as a concept. Part of the reason for naming/categorizing is that it reduces the data from many individual incidents to a much smaller number of categories.

In contrast to open-coding, which fractures the data, *axial coding* puts the data together again by establishing connections between categories (Strauss & Corbin, 1990). Axial coding is about specifying the following features of the phenomenon of interest: (a) the causal conditions, (b) the context, (c) the intervening conditions (broader context in which the phenomenon occurred), (d) the actions/interactions, and (e) the consequences. These specifying features define categories and subcategories. These features have normally been already identified during open coding and now need to be related to the phenomenon. Axial coding also involves making statements about relationships between categories and then verifying those statements within the data.

Another key feature of the grounded theory approach is the *constant comparison* method (Glaser & Strauss, 1967; Strauss & Corbin, 1990, 1994). Making comparisons begins with comparing parts of the data to each other. Later, as the theory begins to emerge, constant comparisons are made between the theory and the data (Dick, 2005). As conjectures are generated from data analysis, they are compared to additional data and revised accordingly (Cobb & Whitenack, 1996; McClain &
Cobb, 2001). Furthermore, data occurring early in data collection are compared to
data occurring later (e.g., the first interview is compared to each subsequent interview
and the categories emerging for a particular student are compared to those for each
subsequent student and to the categories generated for previous students). In sum, the
constant comparative method informs the creation and ongoing iterative refinement of
categories developed from the data (Cobb, Stephan, McClain, & Gravemeijer, 2001).

**Analysis Addressing Questions 1 and 2**

To address the first two questions, the pre-interviews were compared to the
post-interviews. Analysis occurred using the following stages: (a) preliminary
analysis, (b) data reduction and theme creation, (c) a priori coding, (d) open coding,
(d) axial coding, and (e) establishing transfer.

During preliminary analysis, all interviews were watched and a descriptive
account was created for what each student did in response to each interview problem
(Miles & Huberman, 1994). This established an account of what happened prior to
any interpretive coding. The amount of interview data was substantial (17.5 hours in
total). Miles and Huberman (1994) recommend that to prevent data overload during
analysis of qualitative data, the data set be reduced. The preliminary analysis led to
the identification of portions of the data that were exceptionally rich, namely those
instances in which student responses about linear functions appeared to have changed
from pre- to post-interview. Specifically, the data used to answer Question 1 were
reduced from four to three linear functions tasks. The data used to answer Question 2
were not reduced since the interview protocol only contained one quadratic function
transfer task.
While watching the reduced data set a second time, common mathematical themes began to emerge (à la Bowen, 2006). It was on the basis of these themes that coding of the reduced data began. For example, proportional reasoning emerged as a common theme. Of course, the choice of themes can vary significantly across studies (e.g., one could categorize gestures, types of constructed diagrams, etc). The theme of proportional reasoning was chosen as the place to begin the coding of the dissertation data because it was relevant to answering the first research question.

The first stage of coding for the first research question used a priori codes focused on three themes that appeared to run through the students’ responses to the linear function problems (i.e., reasoning proportionally, the mathematization of linear function diagrams and the meaning of division in linear function contexts). Where possible, codes taken from the literature were applied to the data to categorize the inferred regularities in student talk, actions, gestures and inscriptions with respect to each of these themes. For the second research question, the first stage of coding focused on two themes (i.e., covariational reasoning and interpreting changes in quantities).

Once the known a priori codes were exhausted (i.e., when no known codes found in the literature accounted for observed regularities in the data), inductive codes were developed to categorize additional regularities that had not been captured by the a priori codes. For example, after exhausting the known proportional reasoning codes, a new code was developed called coordinating non-proportional and proportional relationships in a \( y=mx+b, \ b\neq0 \) context, to account for certain regularities that were observed in four of the seven students on a particular post-interview linear function
Axial coding was then used to (a) organize a hierarchy of categories (i.e., categories and subcategories), (b) verify categorization and the related associations using a constant comparison method within the data, (c) develop and refine categories, and (d) make associations between codes. Specifically, categories that pertained to non-proportional reasoning were grouped together, categories pertaining to reasoning with changes in quantities were grouped together, etc. Similarly, several subcategories about how students drew diagrams were grouped together under a common super-category of quantitative precision. The constant comparison method continued to be applied to refine categories so that they better fit the data. For example, the code targeting changes in quantities was further refined to iterating and/or partitioning changes in quantities composed units to more accurately fit the data. Finally, categories for reasoning with linear functions were associated with codes for reasoning with quadratic functions, as relationships between backward transfer and forward transfer emerged (in response to the second research question). These relationships provided evidence that forward and backward transfer were part of the same phenomenon (see Chapter 5 for the details).

Because backward transfer is defined as the influence on prior knowledge by the acquisition and subsequent generalization of new knowledge, evidence was collected with a targeted analysis of the classroom data, to support the claim that changes in students’ reasoning from pre- to post-interviews were associated conceptually with the reasoning exhibited during instruction. Following Lobato and Siebert (2002), several passes were made over the classroom transcripts/video to look
specifically for examples of student reasoning that were similar to student reasoning in the post-interview. Findings of similar reasoning in the classroom strengthened the case that a backward or forward transfer effect occurred.

**Analysis Addressing Questions 3**

Recall that the third research question is “What are the transfer processes by which classroom instruction leads to productive backward transfer?” As argued in Chapter 2, noticing appeared to be the most plausible explanatory frame by which to account for the occurrence of backward transfer. Consequently, the focusing framework, developed by Lobato et al. (2011a) was used to analyze the classroom data.

**Preliminary analysis and data reduction.** Using a similar preliminary analysis as was used for the interview data, all classroom sessions were viewed. This meant viewing more than 15 hours of video data from the main camera, which recorded all whole-class discussions and all group-work for one of two small groups, and viewing approximately 9 hours of video data from the secondary camera, which recorded some whole-class discussions and all group-work for the other small group. A descriptive account was created of the activities the students engaged in and the things that were said by students and the teacher. Again, minimal interpretations were made during this pass through the data (Miles & Huberman, 1994). However, episodes which seemed potentially rich with respect to student noticing were identified.

The amount of classroom data was substantial (a combined total of 25 hours from two cameras). To prevent data overload, the data was reduced (Miles &
The guiding principle of the data reduction process was to identify the lessons and parts of lessons that were critical for understanding the relationship between noticing and backward transfer. One way that this was done was by using the backward transfer findings from the analysis of the interview data to guide what was analyzed. Specifically, video data from the classroom in which noticing clearly appeared to be unrelated to the backward transfer findings were not included in the reduced data.

A second way that the data were reduced was by targeting instances in which the things that students noticed appeared to change (i.e., when students appeared to shift from one center of focus to another). The motivation behind targeting the analysis this way was the finding by Lobato et al. (2011a) that shifts in centers of focus in an instructional setting were related to forward transfer effects found in post-interviews. Based on this finding it was hypothesized that shifts in centers of focus may also be related to backward transfer effects.

A third way that the classroom data was reduced was by targeting those days when the two instructional supports—covariational reasoning and point-by-point comparisons—were emphasized. The motivation behind targeting the analysis this way was the hypothesis that these instructional supports would promote productive backward transfer.

**Analysis using the focusing framework.** As described above the focusing framework has four components (Lobato et al., 2011a): (a) centers of focus, (b) focusing interactions, (c) features of mathematical tasks, and (d) the nature of the
mathematical activity. Each component coincides with a separate analytic pass through the reduced classroom data.

In the first analytic pass through the reduced data, I identified emergent centers of focus. This analysis began with open coding (as described previously) of the reduced classroom data set to categorize what students appeared to be attending to mathematically (as per Lobato et al., 2011a). As analysis proceeded across students, multiple centers of focus began to emerge, with students clustering around different foci. Thus, each center of focus was linked to the students who appear to be noticing a particular mathematical property. This analysis also relied upon making constant comparisons to refine the centers of focus (Glaser & Strauss, 1967; Strauss & Corbin, 1990). Any disconfirming evidence that could not be reconciled by revising the centers of focus was reported directly (e.g., a lone student that did not appear to notice what the other students noticed).

The second analytic pass through the data involved using a priori codes to categorize the discursive practices (focusing interactions) that occurred in and around the time in which each center of focus emerged. Three focusing interaction categories derive from Goodwin (1994)—coding, highlighting and producing graphical representations. In his seminal paper, Goodwin (1994) provided two contexts in which these discursive practices contribute to the socially situated nature of seeing meaningful events (i.e., noticing). One of these contexts was an archeology field school. Goodwin described how student archeologists at the school learned to categorize dirt using a Munsell color chart (i.e., a small chart with color samples and viewing holes). The second context was the trial of the police officers who were
charged in the beating of Rodney King. Goodwin described how a police expert for the defense used highlighting (circling small movements by King on the video) and coding to interpret these movements using a category from the police profession, namely *aggression*, which in turn, influenced how the jury came to interpret the actions of the police. Furthermore, the defense lawyers created cropped and enlarged still frames of the video (i.e., a graphical representation) to focus the jurors’ attention toward instances in which Rodney King might be interpreted as the aggressor. In these examples, Goodwin illustrated how noticing can be socially organized, through discursive practices like highlighting, coding and producing graphical representations, along with artifacts like a Munsell color chart and or a video.

Lobato et al. (2011a) prefer to use the categories of *naming* and *renaming* in place of Goodwin’s *coding schemes*, because the latter has technical meaning in the educational research community. As an example of *renaming*, Lobato et al. showed that when students in a particular class were examining a multiplicative number pattern, renaming the multiplicative pattern with the name “growth,” directed student attention away from seeing the pattern as multiplicative toward seeing the pattern as additive. Lobato et al. offered an additional focusing interaction category, namely *quantitative dialogue* (i.e., dialogue that focuses on the measurable attributes of objects). They demonstrated how this discursive practice contributed to students’ focusing on a graphical line as a mathematical object (i.e., as representing a collection of coordinate pairs) instead of as a physical object (e.g., as a mountain slope). In summary, the a priori categories of highlighting, naming/renaming, producing graphical representations and quantitative dialogue were used to analyze the focusing
interactions of the classroom data for this dissertation study.

The third pass through the reduced classroom data involved identifying a connection between the features of the mathematical tasks that students engaged in and the centers of focus. First, the questions posed to students and the activities engaged in during the lessons that comprised the reduced data set were compiled. Then, the affordances and constraints of the mathematical tasks and activities, including the representational features of problems that appeared to influence what students noticed mathematically were recorded (à la Watson, 2004). Finally, a relationship was sought between the mathematical tasks (i.e., the questions, the activities and the affordances and constraints of the tasks) and the centers of focus that emerged in the instructional intervention.

The final pass through the reduced classroom data involved analyzing the nature of the mathematical activity that appeared to be related to the centers of focus that emerged in the instructional intervention. The nature of the mathematical activity is defined as the broad participatory organization of the classroom that establishes, defines, and binds the roles and expectations of the teacher and the students. The other three components of the focusing framework are situated within the nature of the mathematical activity. Some of the relevant questions that were asked when considering the nature of the mathematical activity were: What classroom participation structures were established? How did those participation structures influence what students noticed? What expectations did students have about the role that they played in the class? How did those expectations influence their activity and what they noticed? The nature of the mathematical activity of the class was identified
through a description of the particular classroom participation structures that were established by the teacher and the students that seemed to contribute to the emergence of centers of focus (Cobb, Gresalfi, & Hodge, 2009; Cobb, Wood, Yackel, & McNeal, 1992).

**Reliability, Validity, and Disconfirming Evidence**

Several additional issues were addressed to ensure the quality of the findings from this study. One issue was the reliability of the categories that emerges as a result of inducing codes from the data as part of the partially-grounded approach that was employed. A second issue was the validity of the inferences that were made as a result of the analysis. A third issue was the way that disconfirming evidence was accounted for.

One way to ensure the reliability of categories that emerge as a result of the coding and analysis was peer review (Confrey & Lachance, 2000). Dr. Lobato served as the peer in reliability checks for the categories that emerged from this study. Specifically, for each research question, I prepared and presented PowerPoint slides of categories and evidence of those categories from the data. Dr. Lobato would then provide feedback on whether or not she was seeing the same regularities and on whether or not the category labels sufficiently captured the regularities.

The issue of validity coincides with the notion of *fit* in grounded theory. Fit deals with how closely categories and relationships among categories correspond to the incidents they represent. Assessing validity or fit requires expertise in the area of study (Confrey & Lachance, 2000). Therefore, one approach to assessing the validity of the inferences would be to periodically make a presentation to an expert as the
analysis progresses.

This approach was adopted for the dissertation study. During the interview analysis (i.e., the first two research questions), presentations were made to Dr. Lobato after data reduction and open coding had been completed and again after axial coding had been completed and inferences had been made about forward or backward transfer. During the classroom data analysis (i.e., the third research question), presentations were made after centers of focus had been categorized and again after the focusing interactions had been categorized, the mathematical tasks and the nature of the mathematical tasks had been described and inferences had been made about noticing as a backward transfer process. The presentations focused on the inferential claims that were being made and the evidence supporting those claims. Dr. Lobato gave critical feedback regarding the strength of each claim.

As previously explained, grounded theory involves the constant comparison of data to data and emerging theory to data (Strauss & Corbin, 1994). One result of the constant comparison method is that it will unearth disconfirming evidence. However, rather than using that evidence as the rationale to reject a theory, as is done in empirical research, grounded theory uses disconfirming evidence to further develop the emerging theory (Strauss & Corbin, 1990). In other words, categories and emerging theory are modified to account for disconfirming evidence. Accounting for variation and disconfirming evidence within the theory is seen as a desirable process in which complexity is added to the theory (Strauss & Corbin, 1990). The result is that the theory perfectly fits the particular data set on which it is based (Borgatti, 1996).
CHAPTER 4:
RESULTS ON BACKWARD TRANSFER

In this chapter, I present findings from analysis of the pre- and post-interview data that addresses the first research question. Recall that the first question was the following:

Question 1: When the targeted conceptual understandings of an instructional unit are conceived of as being build upon particular prior knowledge, when and in what ways does newly constructed knowledge lead to productive changes in prior knowledge (productive backward transfer)? When and in what ways does it contribute to unproductive changes (unproductive backward transfer)?

Qualitative analysis revealed several significant findings of productive backward transfer. A few isolated instances of unproductive backward transfer were also observed.

I will present evidence supporting the following three claims about backward transfer: (a) five of seven students reasoned proportionally more productively with the changes in quantities on the post-interview linear tasks than they did on the corresponding pre-interview tasks (backward transfer finding 1 [BTF 1]), (b) six of seven students’ diagrams became more mathematized (BTF 2), and (c) all students’ explanations of division became more meaning-based and less procedure-based (BTF 3). For each claim, I will also present evidence from analysis of the classroom data to show conceptual connections between the reasoning students’ exhibited on the post-interview and similar reasoning in class. After the evidence supporting these claims has been presented, I discuss the isolated instances of negative backward transfer.
BTF 1: Reasoning Proportionally with Changes in Quantities

In this section, I present evidence showing that all but one student reasoned more often with the changes in quantities in the post-interview than they did in the pre-interview (BTF 1). This increased reasoning with changes in quantities had the greatest payoff for five students, four of which had not reasoned with changes in quantities on the $y=mx+b, b\neq0$ pre-interview task but who began doing so on the corresponding post-interview task. The payoff was that they were able to reason correctly that the rate of a $y=mx+b, b\neq0$ function was constant. Three of these four were also able to coordinate the non-proportional and proportional relationships that existed between quantities that were involved in $y=mx+b, b\neq0$ functions. It will also be shown that most students exhibited more productive and/or efficient reasoning when they reasoned with changes in quantities on linear tasks of the form $y=mx$ (i.e., where the $y$-intercept was zero).

*Reasoning proportionally with changes in quantities* will be taken to mean reasoning with composed units made up of the changes in one quantity and the corresponding changes in the other quantity, where both quantities are in a functional relationship. Reasoning proportionally with changes in quantities is important in a $y=mx+b, b\neq0$ context, because no proportional relationship exists between the quantities. Therefore, reasoning proportionally with the quantities in this context is unproductive. However, a proportional relationship does exist between the *changes* in quantities for all linear functions. By reasoning proportionally with the changes in quantities in a $y=mx+b, b\neq0$ context during the post-interview (but not during the pre-interview), students provided evidence that they had experienced a deepening in their
understanding of linearity. In other words, the shift from reasoning proportionally with quantities to reasoning proportionally with the changes in quantities appears to be a case of productive backward transfer.

My presentation of evidence for this finding of productive backward transfer will begin by focusing on one student who exhibited very distinct differences in thinking across the two interviews. Then, a summary of the evidence from the other students will be presented. However, before presenting evidence supporting these findings, the coding scheme that was used to categorize the students’ ways of reasoning proportionally will be described.

**Coding Scheme**

To analyze students’ proportional reasoning, a coding scheme was created using a mixed methods qualitative approach comprised of both a priori and inductive codes (diSessa, 2002; Miles & Huberman, 1994). The coding scheme used for this finding was developed by first identifying a start list of a priori codes from the mathematics education literature on proportional reasoning (e.g., iterating a composed unit; Lesh, Post, & Behr, 1988; Lobato & Ellis, 2010). Then, other inductive codes that emerged through open coding were added to the coding scheme as a way to capture aspects of student reasoning that did not fit the a priori codes (Strauss, 1987: Strauss & Corbin, 1990).

The following five a priori codes captured most of the unproductive ways that students were reasoning (see Table 4.1): (a) reasoning univariately (Harel, Behr, Lesh, & Post, 1994; Lobato & Ellis, 2010), (b) reasoning additively across measures (Kaput & Maxwell-West, 1994), (c) reasoning additively within measures (Karplus, Karplus,
(d) reasoning procedurally (Smith & Thompson, 2007; Cramer, Post, & Currier, 1993), and (e) reasoning proportionally about non-proportional relationships ($y=mx+b, b\neq0$; DeBock, Verschaffel, & Janssens, 1998; Cramer, Post, & Currier, 1993).

Table 4.1. Coding scheme for categorizing students’ proportional reasoning

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Code for proportional reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-productive</td>
<td>Reasoning univariately</td>
</tr>
<tr>
<td></td>
<td>Reasoning additively across measures</td>
</tr>
<tr>
<td></td>
<td>Reasoning additively within measures</td>
</tr>
<tr>
<td></td>
<td>Reasoning procedurally</td>
</tr>
<tr>
<td></td>
<td>Reasoning proportionally about non-proportional relationship ($y=mx+b, b\neq0$)</td>
</tr>
<tr>
<td>Productive</td>
<td>Iterating or multiplying a composed unit</td>
</tr>
<tr>
<td></td>
<td>Partitioning a composed unit</td>
</tr>
<tr>
<td></td>
<td>Consolidating iterating and partitioning</td>
</tr>
<tr>
<td>Non-productive</td>
<td>Confounding non-proportional and proportional relationships ($y=mx+b, b\neq0$)</td>
</tr>
<tr>
<td>Productive</td>
<td>Iterating and/or partitioning a composed unit of changes in quantities</td>
</tr>
<tr>
<td>Productive</td>
<td>Coordinating non-proportional and proportional relationships ($y=mx+b, b\neq0$)</td>
</tr>
</tbody>
</table>

Reasoning univariately occurred when students reasoned with only one of the two quantities involved in a linear function in order to make conclusions about the linearity of the data. Reasoning additively across measures occurred when students found the differences between corresponding values of two quantities in a given data set and then reasoned with those differences in order to make conclusions about the linearity of the data. Reasoning additively within measures occurred when students added or subtracted a common value from both the independent and dependent variable values in an ordered pair. Reasoning procedurally occurred when students
used an algorithm or procedure that was derived from proportional reasoning (e.g., creating a linear equation, setting up and solving proportion statements, dividing quantities) but that they admitted they weren’t quite sure about. Reasoning proportionally about non-proportional relationships occurred when students reasoned proportionally with the independent and dependent variables from a $y=mx+b$, $b \neq 0$ function, despite the two variables not being in a proportional relationship.

In order to categorize the productive aspects of students’ proportional reasoning, five additional codes were added. The first three codes were a priori codes, while the other two codes were inductive codes that emerged from data analysis of the student interviews. These first three codes are about reasoning with quantities and the last two are about reasoning with changes in quantities. The five codes are: (a) iterating or multiplying a composed unit (Lamon, 1993; Lesh, Post, & Behr, 1988), (b) partitioning a composed unit (Thompson, 1994c), (c) consolidating iterating and partitioning (Lobato & Ellis, 2010; Lobato & Siebert, 2002), (d) reasoning proportionally with changes in quantities, and (e) coordinating non-proportional and proportional relationships ($y=mx+b$, $b \neq 0$). An additional unproductive code—the reverse of the last of the productive codes—called confounding non-proportional and proportional relationships ($y=mx+b$, $b \neq 0$), was also used.

The first three codes involved students reasoning with the original quantities from a linear relationship. Iterating or multiplying a composed unit or partitioning a composed unit occurred when students formed a composed unit (i.e., mentally joined values of two quantities to form a new quantity, Lobato, Clark, & Ellis, 2005; Lamon, 1995), and then iterated or multiplied the composed unit to “build up” to higher values
of the two quantities, or partitioned the composed unit to “zoom in” on lower values of the two quantities, while maintaining the multiplicative relationship that was present in the original composed unit. Consolidating iterating and partitioning occurred when students combined iterating and partitioning to transform a composed unit into a new composed unit made up of particular values of the two original quantities, while maintaining the multiplicative relationship that was present in the original composed unit.

The next three codes emerged from the interview data and involved students reasoning with the changes in the quantities instead of or in conjunction with reasoning with the original quantities. Iterating and/or partitioning a composed unit of changes in quantities occurred when students joined two changes in quantities to form a composed unit that could be reasoned with. Coordinating non-proportional and proportional relationships occurred in $y=mx+b$, $b\neq0$ contexts, when students coordinated iterating and/or partitioning a composed unit of changes in quantities so as to maintain the multiplicative relationship between the changes in quantities, and the students also adjusted their reasoning for the non-proportional relationship that existed between the quantities themselves. In contrast, confounding non-proportional and proportional relationships occurred when students either reasoned with changes in quantities as composed units but then neglected to adjust for the non-proportional relationships between the quantities or when they reasoned exclusively with the quantities as if there was a proportional relationship between them.

The proportional reasoning codes described above are not intended to be an exhaustive list. For example, none of the codes capture proportional reasoning that
involves making multiplicative comparisons between corresponding values of quantities (Thompson, 1994c). The coding scheme only captures the proportional reasoning that study participants were observed using.

**Changes in Brady’s Linear Reasoning**

The student who most clearly demonstrated BTF 1 is Brady. Evidence will be presented that shows that his post-interview proportional reasoning involved changes in quantities to a much greater extent than his pre-interview reasoning, and that his proportional reasoning was more productive in the post-interview when compared to the pre-interview. First, evidence from Brady’s responses on parallel pre- and post-interview tasks will be presented that involved similarly complex functions of the form $y=mx$, set in a context about water pumping into a pool. Then, additional evidence from his post-interview responses will be presented that illustrates Brady’s productive proportional reasoning in the context of a linear function of the form $y=mx+b$, where $b\neq 0$.

**Water pump tasks ($y=mx$).** Brady was presented with data involving water being pumped into a pool in both the pre- and post-interview. The relationship between time (independent variable) and the volume of water (dependent variable) being pumped into the pool was linear. However, the task was complicated by the fact that the independent variable was not presented at equal intervals. The water pump data were presented in tables (see Figures 4.1a and 4.1b).

---

10 All student names are pseudonyms that have been chosen to preserve gender and ethnicity.
Pre-interview water pump #1 task. During this task, Brady provided evidence of proportional reasoning that was productive for an 8th grader. This task involved data representing the function \( y = 0.75x \) (see Figure 4.1a). Brady was asked if the tabular data represented a water pump that was pumping at a constant rate or not. He noticed that several of the water volume values ended in decimals (“I keep on seeing decimals”), and guessed that the water pump might “go up by .75 or something.” Brady confirmed that this guess was the correct rate by repeatedly iterating 1 minute and .75 gallons, starting at 4 minutes and 3 gallons and ending at 10 minutes and 7.5 gallons (see Figure 4.2).

Brady made only one reference to changes in time and changes in water volume. It occurred when he explained why he chose .75 gallons: “Like 4 to 10 is 6 [points at 4 and 10 minutes in time column of table] so I added 6 to 3 [points at 3 gallons in water volume column of table] and it’s not 9. So I just thought it was .75.”

In this statement, Brady referenced the 6-minute change in time between 4 and 10 minutes and alluded to the change in water volume from 3 to 7.5 gallons not being 6 gallons. Notice also that Brady’s talk and the inscriptions in his augmented data table

<table>
<thead>
<tr>
<th></th>
<th>Time (minutes)</th>
<th>Water in pool (gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>9.75</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>13.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Time (minutes)</th>
<th>Water in pool (gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>16.25</td>
</tr>
<tr>
<td></td>
<td>36</td>
<td>22.5</td>
</tr>
</tbody>
</table>

Figure 4.1. (a) Pre-interview water pump #1 data table; (b) Post-interview water pump #1 data table.
were devoid of gallons and minutes units.

![Table]

<table>
<thead>
<tr>
<th>minute</th>
<th>gallons</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3.00</td>
</tr>
<tr>
<td>5</td>
<td>3.75</td>
</tr>
<tr>
<td>6</td>
<td>4.50</td>
</tr>
<tr>
<td>7</td>
<td>5.25</td>
</tr>
<tr>
<td>8</td>
<td>6.00</td>
</tr>
<tr>
<td>9</td>
<td>6.75</td>
</tr>
<tr>
<td>10</td>
<td>7.50</td>
</tr>
</tbody>
</table>

**Figure 4.2.** Brady’s pre-interview augmented table for water pump #1 task.

Brady’s proportional reasoning on this task was coded as *iterating (or multiplying) a composed unit* because he repeatedly iterated the composed unit, 1 minute/.75 gallons, building up from 4 minutes/3 gallons to 10 minutes/7.5 gallons. Considering that Brady was an 8th grader, the reasoning he displayed on this task was fairly productive because, as Lobato and Ellis (2010) explain, “attending to and coordinating two quantities” (p. 15) and “joining two quantities into a composed unit” (p. 18) are two essential understandings about ratios that form the foundation for proportional reasoning. Evidence presented next will show that in the post-interview Brady’s understanding of linear functions had deepened.

**Post-interview water pump #1 task.** Brady reasoned very differently in the post-interview. In particular, he reasoned proportionally with changes in quantities in ways that indicated a more sophisticated understanding of linear functions. The post-interview water pump #1 task was identical to the pre-interview task, except that a
different linear function was used ($y = 0.625x$; see Figure 4.1b). Again, Brady was asked to decide whether or not the water was being pumped into the pool at a constant rate or not. Instead of guessing at how much the water in the pool increased each minute, he found all four changes in time and the four corresponding changes in water volume (see Figure 4.3). Then, he divided each change in water volume by the corresponding change in time. Notice that on the third quotient, Brady made an arithmetic error, dividing by 5 when he should have divided by 6. Because 3 of the 4 quotients were identical, he concluded: “I think I did this one wrong [points at 3.75 gallons/5 min], and I think everything’s going the same, like constant.” Despite the arithmetic error, Brady’s attention was clearly directed toward the changes in quantities on this task.

Evidence that Brady had formed a composed unit by joining a particular change in water volume and the corresponding change in time came when he explained what dividing a change in water volume by a change in time meant. He said:

These are five gallons and these are 8 minutes . . . if you make groups . . . of 8. If you put . . . .625, if you put each one in a group. It will equal 5 . . . because you make 8 [draws 8 circles; see Figure 4.4] . . . If you put .625 in each one [writes .625 in each circle] . . . So then if you multiply .625 times 8 [uses calculator] you’ll get 5 gallons.

In this excerpt, Brady partitioned the 8 minutes into 8 parts, as represented by the 8 circles, and partitioned the 5 gallons into 8 parts, as represented by the .625 written in each circle. Thus, he treated the 5 gallon change in water volume and 8 minute change in time as a composed unit that he partitioned to find .625 gallons per minute. Notice also that his inscriptions and his talk referenced the units “minutes” and
“gallons.” Brady’s proportional reasoning on this task was coded as iterating and/or partitioning changes in quantities.

There are at least three ways in which Brady’s reasoning about linear functions appeared to have deepened compared to his reasoning on the corresponding pre-interview task. First, because Brady was able to partition composed units, it freed him from having to guess at the unit rate (i.e., Brady found the unit rate by partitioning the 5 minutes/8 gallons composed unit into 8 parts). Second, because Brady was able to form composed units with changes in quantities, it allowed him to find the unit rate at multiple points in the data set without having to using iterating the unit rate to build up from one data point to another. Third, by forming composed units of changes in
quantities and by referring to the units for the changes in quantities, Brady showed that he conceived of the changes in quantities as quantities themselves. This conclusion is based on Thompson’s (1994c) assertion that a person conceives of a new quantity when they engage in mentally operating on “already-conceived quantities” (p. 9).

This third change in Brady’s reasoning is significant because, as Lobato et al. (in press) have found, conceiving of changes in quantities as quantities is non-trivial for middle-school students.

This example of Brady reasoning proportionally with composed units made of changes in quantities was not unique to this post-interview task. In fact, a similar excerpt from each of the post-interview y=mx tasks could have been presented. In each case, Brady formed composed units from pairs of changes in quantities and reasoned with them.

**Business-cost post-interview task (y=mx+b, b≠0).** Brady’s reasoning on this task further highlights the ways in which reasoning proportionally with changes in quantities was productive for him. This task involved a linear relationship between the number of employees in a business (independent variable) and the monthly cost of running the business (dependent variable; see Figure 4.5).

There are two reasons why this task was unique compared to the other tasks in the interview protocol. First, the task was presented in graphical form rather than in tabular or word-problem form. This is significant because the instructional intervention did not address graphical representations. Second, the task was complicated by an initial cost of $1000 per month when the number of employees was 0. In other words, this linear relationship had a non-zero y-intercept (i.e., the function
was $y=625x+1000)$. This is significant because the relationship between the number of employees and the monthly cost was *not* proportional, while the relationship between changes in the number of employees and the corresponding changes in the monthly cost was proportional. Consequently, to assess the linearity of the data, students needed to reason with changes in quantities. The fact that this task was unique meant that any influence from the instruction on how students thought about this task would represent a farther jump for students in terms of transferring their newly-acquired understanding of quadratics to linear contexts.

For this task, Brady was asked to determine whether or not the cost per month increased at a constant rate as more employees were hired. Consistent with his responses to the other post-interview tasks, Brady found the first three changes in the number of employees and the corresponding changes in the monthly cost between data

![Figure 4.5. Post-interview business-cost task.](image)
points and recorded them in fraction form (see Figure 4.6). Then, he reasoned with the changes in cost and the number of employees as composed units:

I think that it’s going constant . . . If you multiply this by 2 [points at \(2e/1.25K\)] . . . you’ll get 2.5K and 4e. You multiply this by 2 [points at \(4e/2.5K\)] you’ll get 5K and 8 employees. And the K stands for thousand . . . every 2 employees, she has to give away 1250 dollars, for every 2 employees . . . So I’m gonna divide this by 2 [points at \(2e/1.25K\)] . . . she will only give away 625 dollars for an employee, for a month.

In this excerpt, Brady appeared to form a composed unit by joining 2 employees with $1250, which he then doubled by multiplying each quantity by 2. He also partitioned the composed unit in half by dividing each quantity by 2 to find the cost per employee.

Brady’s proportional reasoning on this task was coded as iterating and/or partitioning composed units of changes in quantities. This reasoning reflected a deeper understanding of linearity than the reasoning he used in the pre-interview. By reasoning proportionally with the changes in quantities, Brady indicated that he conceived that there was a proportional relationship between the changes in quantities. As stated above, this is particularly important for linear functions like the one represented in this task (i.e., a linear function with a non-zero y-intercept) because there is no proportional relationship between the original quantities. Therefore, I interpreted these changes in reasoning as a case of productive backward transfer.

There was one instance of his reasoning on this task that was not productive. This reasoning occurred after Brady had determined that the cost was “625 dollars for an employee, for a month” and was asked to find the number of employees that cost $16000. To answer this, Brady divided the $16000 cost by the $625 cost for 1 employee:
I’m dividing 16000 by 625 . . . the max you could give away is 625 dollars for 1 employee. So you divide 16000 and 625 and you’ll get 25.6, but like there’s not like 2/3 of a person, so you could only give away 25.

The reasoning in this excerpt was coded as **confounding the non-proportional and proportional relationships**, because he appeared to confuse the proportional relationship between the $625 change in cost and the corresponding change in 1 employee with the non-proportional relationship between the total cost and the total number of employees. However, I did not interpret this negative result as a case of unproductive backward transfer but as a byproduct of his deepening understanding of linearity because it wasn’t until he reasoned proportionally with composed units of changes in quantities, that he could begin to consider coordinating non-proportional and proportional relationships (in $y=mx+b$, $b\neq0$ contexts). Instances of unproductive transfer will be presented later in this chapter.

The evidence from the pre- and post-interviews shows that Brady went from guessing at unit rates to partitioning composed units to find unit rates, that he went from not providing much attention to changes in quantities to treating the changes in quantities as quantities in their own right, and that he went from reasoning proportionally with composed units of quantities to reasoning proportionally with
composed units of changes in quantities. Therefore, Brady showed a productive backward transfer effect in the form of a deepening understanding of linearity from pre- to post-interview, which was particularly important in the context of a $y=mx+b$, $b\neq0$ function (Confrey & Smith, 1995). However, Brady also showed that he had not yet sorted out how to coordinate the non-proportional and proportional relationships in a linear function when the y-intercept was non-zero.

**Summary of BTF 1 for All Students**

Five of the seven students, including Brady, exhibited significant changes in their reasoning from pre- to post-interview that aligned with this finding. In other words, a productive backward transfer effect with respect to reasoning proportionally with changes in quantities appeared to have been produced for five of the students (i.e., a deepening in their understanding of linear functions).

The coding of the student pre-interviews, using the coding scheme described above (see Table 4.1), revealed that students sorted themselves into three groups. These student groupings will be used to organize the presentation of the evidence because there was a fair degree of alignment between the ways that the students reasoned coming into the study and the changes that they exhibited in their reasoning in the post-interviews.

During the pre-interviews, there were three students—George, Armando and Peter—who provided evidence of reasoning proportionally with quantities (for coding summary see Tables 4.2 and 4.3). Furthermore, George and Armando also provided evidence of reasoning proportionally with changes in quantities, which further separated them from the other students. Therefore, these three students were grouped
Three other students—Nicholas, Brady and Jenn—provided evidence of reasoning non-proportionally and proportionally with quantities in a linear relationship, during the pre-interview. Furthermore, Nicholas and Jenn provided no evidence of reasoning with changes in quantities while Brady provided one brief
instance. Therefore, these students were grouped together as mid-level proportional reasoners. The remaining student, Kendra, provided multiple examples of reasoning in non-proportional ways and only provided a brief instance of reasoning proportionally. She did provide instances of reasoning with changes in quantities but this reasoning was non-proportional. Therefore, she was categorized in a separate group as a low-level proportional reasoner.

Table 4.3. Coding of students’ post-interview proportional reasoning.

<table>
<thead>
<tr>
<th>POST-INTERVIEW proportional reasoning</th>
<th>Students grouped by proportional reasoning level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportional reasoning codes</td>
<td>High</td>
</tr>
<tr>
<td>George Armando Peter Nicholas Brady Jenn Kendra</td>
<td></td>
</tr>
<tr>
<td>Quantities</td>
<td></td>
</tr>
<tr>
<td>Reasoning univariately</td>
<td></td>
</tr>
<tr>
<td>Reasoning additively across measures</td>
<td></td>
</tr>
<tr>
<td>Reasoning additively within measures</td>
<td></td>
</tr>
<tr>
<td>Reasoning procedurally</td>
<td></td>
</tr>
<tr>
<td>Reasoning prop. about non-prop. relationship (y=mx+b, b\neq0)</td>
<td></td>
</tr>
<tr>
<td>Iterating/multiplying a composed unit</td>
<td>✔</td>
</tr>
<tr>
<td>Partitioning a composed units</td>
<td></td>
</tr>
<tr>
<td>Consolidating iterating/partitioning</td>
<td>✔</td>
</tr>
<tr>
<td>Changes in Quantities</td>
<td></td>
</tr>
<tr>
<td>Confounding non-prop. and prop. relationships (y=mx+b, b\neq0)</td>
<td>✔</td>
</tr>
<tr>
<td>Iterating and/or partitioning changes in quantities composed units</td>
<td>✔</td>
</tr>
<tr>
<td>Coordinating non-prop. and prop. relationships (y=mx+b, b\neq0)</td>
<td>✔</td>
</tr>
</tbody>
</table>
Each of the three groups of students exhibited a shift in proportional reasoning from the pre- to post-interview. The *mid-level* proportional reasoners’ exhibited the most dramatic change. In the post-interview, they all reasoned proportionally at a high-level: the non-proportional reasoning that they had exhibited in their pre-interviews was absent from their post-interviews. Instead, students in this group reasoned proportionally with changes in quantities. The *mid-level* group of students showing more dramatic changes in reasoning is consistent with the finding reported by linguistics researchers that second-language learners at a mid-level in second language proficiency, experience greater backward transfer effects (i.e., L2 influences on L1 comprehension or production) than second-language learners at a low-level in second-language proficiency and also greater than those at a high-level (Chen, 2006, p. 179).

However, contrary to the linguistics research findings, the *low-level* group also showed a significant deepening in reasoning. This deepening of understanding involved reasoning more with changes in quantities and engaging more in productive proportional reasoning with quantities. The high-level group, as a whole, showed smaller shifts in reasoning. They did reason with changes in quantities more often in the post-interviews. For Peter, this meant a substantial transformation in his reasoning about the $y=mx+b$, $b\neq0$ task. However, for George and Armando, reasoning more with changes in quantities did not translate into substantial shifts toward productive reasoning. Evidence of changes in reasoning in the *mid-level* group will be presented first, since their changes were the most dramatic of the three groups.

**Mid-level proportional reasoners.** I provide evidence to show that all students in this group, Nicholas, Jenn and Brady, exhibited reasoning that reflected a
deepening in their understanding of linearity from pre- to post-interview. I interpreted these changes in reasoning as a case of productive backward transfer.

**Pre-interview.** The students in this group engaged in non-proportional, procedural and proportional reasoning with quantities. All three students exhibited instances of reasoning that were coded as non-proportional. Jenn and Brady exhibited *univariate reasoning*, reasoning with one of the quantities or the other rather than reasoning with both quantities. Jenn also reasoned *additively across a measure space*. She found differences between successive pairs of quantities in a give data set (i.e., she subtracted time values from water volume values) and concluded that because the differences were not constant, that the rate was not constant. Nicholas’ displayed *non-proportional composed unit reasoning*. He formed a composed unit with two quantities, subtracted a common value from each part of the composed unit (e.g., he subtracted 1 from 11 minutes and 16.5 gallons) and then reasoned as if the multiplicative relationship had been preserved.

Evidence of procedurally-based reasoning was also found. For example, on a task about the cost of using a cell phone, Nicholas was asked about how much the cost of the cell phone increased each hour the cell phone was used (see Figure 4.7). In response, he wrote a proportion statement with no unknowns (i.e., 6 dollars/5 hours = 9 dollars/10 hours; see Figure 4.8). He then tried to cross-multiply and cancel common factors. He ended up with 3/5 = 40/1 and said, “I don’t think I’m doing it right.” Nicholas appeared to be trying to apply some previously-learned procedure for solving proportions for which he did not have a clear understanding.

This group also showed some evidence in the pre-interview of reasoning
proportionally with quantities. For example, Nicholas said during the water pump #1 task (see Figure 4.1a):

\[
\frac{3}{6} = \frac{9}{18}
\]

\[
\frac{3}{6} = \frac{4.5}{9}
\]

**Figure 4.8.** Nicholas’ inscriptions when setting up a proportion for the cell-phone task.

I was trying to make the 4 [minutes] and the 3 [gallons] go into the 10 [minutes] and the 7.5 [gallons]. So I multiplied them both by 2, and then I cut the three in half to get the 1.5 and then I added it.

This instance of proportional reasoning was coded as *consolidating* *iterating* and *partitioning* because Nicholas appeared to form a composed unit of 4 minutes and 3
gallons, built up the composed unit to 8 minutes and 6 gallons by multiplying each quantity by 2, partitioned the 4 minute and 3 gallon composed unit by dividing the quantities in half, and then added the result onto the 8 minute and 6 gallons to get 10 minutes and 7.5 gallons.

This group did not iterate or partition composed units of changes in quantities, although Brady provided a brief instance described earlier in which he identified several changes in quantities and used them to make a guess about the rate, and Jenn reasoned with several changes in the cost of the cell phone without simultaneously reasoning with the changes in the number of hours the cell phone was used. Nicholas was the lone student in this group to confound non-proportional and proportional relationships. During the cell-phone task, he composed a particular cost with the corresponding time and reasoned as if the relationship between the quantities was proportional: “This one’s like, the point is 12 on the monthly cell-phone cost and 10 on the hours. So it’s like 5 dollars, or 6 dollars every 5 hours for the use of the cell phone.”

Post-interview. In the post-interview, this group no longer engaged in the non-proportional or procedural reasoning described above. Instead, they reasoned proportionally with changes in quantities. For example, Brady reasoned proportionally with the changes in quantities on all four linear function tasks, Nicholas did so on both of the water pump tasks, as well as the business-cost task, and Jenn did so on the water pump #2 task and the business-cost task.

What was particularly surprising was that during the business-cost task, in which the function had a non-zero y-intercept (see Figure 4.5), the mid-level
proportional reasoners’ generally reasoned more productively with changes in quantities than the high-level proportional reasoners’. For example, Jenn reasoned proportionally with changes in quantities on the business-cost task when she found the change in cost of running the business for one additional employee by dividing a particular change in cost by the corresponding change in numbers of employees (i.e., $5000 ÷ 8 \text{ employees} = $625/\text{employee}). She repeated this reasoning with other pairs of changes in cost and corresponding changes in numbers of employees and concluded that the rate was constant.

Furthermore, when asked how many employees the business owner could afford with $16000, Jenn coordinated non-proportional and proportional relationships. First, she worked with proportional relationships to find the change in number of employees for a change in cost of $3125 by dividing that change in cost by the cost per employee (i.e., $3125 ÷ $625/\text{employee} = 5 \text{ employees}). Then, she coordinated that with the non-proportional relationships by adding the change in employees to the number of employees that cost $12875 (19 + 5 = 24 \text{ employees}). Jenn explained, “So if you were to add 5 more people because 625 goes into there [points to 3125], then you get twenty-four people. So she can only hire 24 people.” This reasoning was particularly noteworthy because it indicated that her understanding of linearity had surpassed that of the high-level group.

Jenn did, however, also temporarily confound non-proportional and proportional relationships. This occurred when she momentarily considered whether she should add the cost of running the business without any employees to the cost per employee (i.e., adding $1000 to $625/\text{employee}). Then, she checked to see if
multiplying $1625 by 4 and adding the result to $6000 for 8 employees would produce the correct cost for 12 employees. She determined that “it made it go higher than it should be.” From then on she used $625/employee as the rate. My interpretation here is that while Jenn’s reasoning had changed from pre- to post-interview, there were aspects of this new reasoning that she was still working out.

Nicholas also exhibited a significant deepening in understanding of linearity from the pre-interview cell-phone task to the post-interview business cost task. In the post-interview, he reasoned proportionally with changes in quantities:

Because it’s the 8 employees in 5000, and then from the 8 to the 12 cuts the employees in half to 4 and it cuts the cost in half as well. And from the 12 to 14, it cuts the employees from 4 to 2 and then it cuts it down to 1250 instead of 2500, and then I figured out from the 1250 in two that I divided for every, instead of every 2 employees if you divide it and make it like every one employee it’d be 625 dollars.

In this excerpt, Nicholas appeared to compose the change in cost from $1000 to $6000 with the corresponding change in number of employees. Then, he partitioned that $5000 and 8 employee composed unit several times to confirm that the cost was increasing steadily and to find the rate (i.e., “one employee it’d be 625 dollars”).

Later in the interview, Nicholas also coordinated non-proportional and proportional relationships, when he said, “So I’m multiplying 625 by 5 and it gives me 3125. And if you take 3125, from the monthly cost and 5 from the employees [points at $12875/19 employees], it gives you the 14 employees and 9750.” In this excerpt, he reasoned proportionally by multiplying a composed unit made of changes in quantities (i.e., $625/employee \times 5 = $3125 per 5 employees) and then coordinated that with the non-proportional relationship between the quantities by subtracting the
product from a given data point (i.e., $12875 for 19 employees – $3125 for 5 employees = $9750 for 14 employees).

As shown earlier, Brady’s post-interview reasoning was, in many ways, similar to Jenn’s and Nicholas’ reasoning. Like Jenn and Nicholas, Brady was able to reason proportionally, using iterating and partitioning, to determine the cost of running the business per employee (i.e., “625 dollars for an employee, for a month”). This reasoning (and other reasoning like it) was coded as reasons proportionally with changes in quantities. However, earlier it was also shown that Brady was still sorting out how to coordinate the non-proportional and proportional relationships in those contexts (e.g., he divided $16000 by $625/employee to incorrectly determine the number of employees that could be hired). This reasoning was coded as confounding non-proportional and proportional relationships.

To summarize, the students in this group shifted from reasoning non-proportionally and proportionally with linear function quantities to reasoning with changes in linear function quantities. For Nicholas and Jenn, there were clear differences in their reasoning on the pre-interview cell-phone task and the post-interview business-cost task (Brady was not given the cell-phone task in his interview because an error was found in the task that was corrected for subsequent students). However, Brady and Jenn also showed that they were still working out how to coordinate the non-proportional and proportional relationships in $y=mx+b$, $b\neq0$ contexts.

**Low-level proportional reasoners.** Kendra, the lone student in this group, also exhibited marked improvement in her reasoning from pre- to post-interview,
which involved reasoning proportionally with quantities and attending more to changes in quantities (but not reasoning proportionally with changes in quantities). I interpreted these changes as a case of productive backward transfer.

**Pre-interview.** Kendra’s pre-interview reasoning could be described as predominantly non-proportional. For example, during the pre-interview water pump #1 task (see Figure 4.1a), she showed instances of *reasoning univariately*:

I think the fastest is . . . from 3 to 7.5 [points at 3 and 7.5 gallons in table] . . . Because it went from 3 and it added 4 . . . and a half . . . to 7.5. And from here to here it’s just 3 and a quarter [sweeps finger over 7.5 and 9.75 gallons in table] . . . From 7.5 to 9.75. And then that went 3 and 3 quarters [points at 9.75 and 13.5 gallons in table]. And so I think it’s going fastest right here [points at 3 and 7.5 gallons in table].

In this excerpt, Kendra did attend to changes in quantities in the pre-interview (e.g., she found the changes in volume from 3 to 7.5 gallons and from 7.5 to 9.75 gallons). However, she did not find the corresponding changes in time, form a composed unit, or reason proportionally.

On a task about making orange juice (based on the orange juice tasks designed by Harel, Behr, Lesh, & Post, 1991; see Figure 4.9), Kendra again *reasoned univariately*. However she also *reasoned additively across measures*:

If you’re going to water it so much it might take away the oranginess . . . four more of the water [writes 4 by Batch C], 3 more of the water [writes 3 by Batch B and a 1 by Batch A], I think this would be the orangiest [points to Batch A].
In this excerpt, Kendra subtracted the number of cans of orange concentrate from the number of glasses of water (e.g., 9 water glasses subtract by 6 orange concentrate cans). She concluded that Batch A was the “orangiest” because that batch had only one more glass of water than cans of orange juice concentrate while the other two batches had more.

Despite Kendra reasoning primarily in non-proportional ways during the pre-interview, she had an instance near the end of the cell-phone task, in which she began to reason proportionally with the changes in quantities. In particular, she suddenly appeared to form a composed unit with the first change in time, 6 hr (i.e., 6 – 0 hr) and the first change in cost, $3.60 (i.e., $9.60 – $6.00). Then, to find the cost of using the cell phone for 12 hr, she added $3.60 to the cost of using the cell phone for 6 hr (i.e., $9.60 + $3.60 = $13.20). Also, to find the cost of using the cell phone for 18 hr, she added $3.60 to the cost of using the phone for 12 hr (i.e., $13.20 + $3.60 = $16.80).
This reasoning was coded as *iterating a composed unit of changes in quantities*. This brief instance was the only clear example of Kendra reasoning proportionally during the pre-interview. Thus, it appeared that Kendra had some ability to reason proportionally but that she may have been unsure when to apply that reasoning.

**Post-interview.** During the post-interview, Kendra exhibited substantial changes in her thinking. Like the other students, Kendra exhibited greater attention to changes in quantities. Furthermore, she attended to changes in *both* quantities rather than just one. For example, during the post-interview water pump #2 task, Kendra recorded all changes in water volumes and all the corresponding changes in time (see Figures 4.10a and 4.10b). Similarly, she found all changes in quantities during the post-interview water pump #1 task and the business-cost tasks.

However, the key change in Kendra’s reasoning about linear contexts was that she engaged in more proportional reasoning. For example, on the water pump #1 and #2 tasks, Kendra partitioned composed units of water volume values and corresponding time values to determine that the water pumps were pumping at a constant rate. On the paint-making task (i.e., on the task about making pink paint by mixing red and white paint; see Figure 4.11), which was the parallel post-interview task to the pre-interview orange-juice making task, Kendra reasoned additively across measures as she had in the pre-interview. But, in the following excerpt she also multiplied a composed unit of the number of cans of white paint and corresponding number of cans of red paint:
Another pump was used to fill a different pool.
After 4 minutes, there were 10 gallons of water in the pool.
After 6 minutes, there were 15 gallons in the pool.
After 11 minutes, there were 27.5 gallons in the pool.
After 14 minutes, there were 35 gallons in the pool.

Figure 4.10. (a) Post-interview water pump #2 data; b) Kendra’s recordings of the changes in time and changes in water volumes.

Figure 4.11. Post-interview paint-making task.

3 [points at Batch A red paint] times 4 is 12 [points at Batch C red paint] and then right here 4 [points at Batch A white paint] times 4 is 16 [points at Batch C white paint] so . . . maybe it will be the same shade as this one [points at Batch A]?
In this excerpt, Kendra formed a composed unit of 3 red cans and 4 white cans. She then multiplied that composed unit by 4 to get 12 red cans and 16 white cans. Later in the interview, she also circled groups of 3 cans of red and 4 cans of white in Batch C as another way to show that Batch A and C were the same shade of pink.

On the business cost task, she confounded non-proportional and proportional relationships. For example, she incorrectly reasoned that by dividing $12875, which was the cost for 19 employees, by 19 she would “find out how much they were paying each employee.” She also incorrectly reasoned that a change in 8 employees changed the cost by $6000, instead of $5000. However, she reasoned correctly that if changing the number of employees by 8 would change the cost by $6000, then changing the number of employees by 5 would change the cost by “a little bit more than half” of $6000. These examples show that Kendra underwent marked productive changes in her reasoning from pre- to post-interviews, but that the level of her proportional reasoning in the post-interview was still below that of the mid-level group.

**High-level proportional reasoners.** Overall, the students in this group exhibited smaller changes in their reasoning than the mid- and low-level groups. One reason was that all three students maintained a fairly high level of proportional reasoning from pre- to post-interview. However, Peter, one of the three students in this group, showed a shift toward reasoning proportionally with changes in quantities that was similar to the mid-level group’s shift in reasoning. In particular, on the task in which reasoning with changes in quantities was necessary (i.e., the business-cost $y=mx+b$, $b\neq0$ task), Peter’s reasoning appeared to be significantly more productive
than during the pre-interview. I interpret this shift in reasoning as a case of productive backward transfer.

**Pre-interview.** Analysis of this group’s pre-interview reasoning revealed multiple instances of *iterating/partitioning a composed unit* and *consolidating iterating and partitioning*. Furthermore, George and Armando *iterated or partitioned composed units of changes in quantities* on several pre-interview tasks. For example, Armando found the rate for the cell-phone task by composing and partitioning a change in the cell-phone cost and corresponding change in the number of hours the cell phone was used (e.g., $9.60–6.00=3.60$, $6–0=6$ hr and $3.60/6$ hr=$.60/hr). He also used the rate to find the cost of using the cell phone for 30 hrs and the number of hours of cell-phone use that cost $21.00. Both instances were coded as *coordinating non-proportional and proportional relationships*. George also reasoned with changes on the cell-phone task to find the rate. Peter reasoned proportionally with quantities in similarly productive ways as George and Armando but did not reason with changes in quantities.

**Post-interview.** As stated above, out of the three students in the high-level group, the most significant changes in reasoning were exhibited by Peter. These changes in reasoning were particularly evident in Peter’s responses to the post-interview business-cost task (see Figure 4.5). On this task, he reasoned with the changes in numbers of employees and the corresponding changes in cost of running the business:

“It’s a difference of 12 to 14 [points at graph points 12, 8500 and 14, 9750] . . . and then that’s 2, a difference of 2 employees, so that’s 1250 a difference from this [points at 12, $8500$ and 14, $9750$] . . . oh, so
I’m supposed to multiply by 2 . . . So that’s 2500 [sweeps finger between points 8, 6000 and 12, 8500] and, so from here, all the way over here [sweeps finger between points for 8 and 14 employees] along here it’s going equal.

In this excerpt, Peter found the change in cost from $8500 to $9750 (i.e., $1250) and the corresponding change in number of employees from 12 to 14 employees (i.e., 2 employees). Then, he appeared to reason that the change in cost from 8 to 12 employees was twice as big as the change in cost from 12 to 14 (i.e., “I’m supposed to multiply by 2”). Because his calculation (i.e., $1250 \times 2$) matched the actual change in cost for going from 8 to 12 employees (i.e., $8500$-$6000$=$2500$), he concluded that the cost was increasing steadily. This reasoning was coded as iterating a composed unit of changes in quantities.

Later in the interview, he was asked to find the rate of change for the business-cost task. He replied, “This is in two, in two employees [sweeps finger between 12 employees, $8500$ and 14 employees, $9750$ on graph], so I’ll find it for every employee. So, $1250$ divided by $2 . . . so that’s $625 . . . per each employee.” In this excerpt, Peter partitioned a composed unit of changes in quantities. This reasoning was more sophisticated than Peter’s pre-interview reasoning, where he incorrectly reasoned as if there was a proportional relationship between the cell-phone cost and the number of hours the cell phone was used. In other words, Peter’s understanding of linearity appeared to have deepened from pre- to post-interview.

Peter also engaged in coordinating non-proportional and proportional relationships when he made the following statement:

2 and 6 employees, so that’s 4 because the difference from here is 4 [points to 8 and 12 employees]. So that is 4000, no that’s 2800 [points
at $2100$ and $4900$ which he previously determined (incorrectly) corresponded to $2$ and $4$ employees respectively]. So if we added this [points to $8$, $6000$] it should be but it’s not . . . So, that’s actually, this [points at $8$, $6000$] should be, if it was going at a steadily rate, it would be umm . . . $8800$, but it’s $8500$ [points at $12$, $8500$].

In this excerpt, Peter reasoned with the proportional relationship between the changes in the cost of running the business each month (i.e., the $2800$ change in cost for a change in employees of $4$ from $2$ to $6$ employees). Then, he coordinated that with the non-proportional relationship between the cost and the number of employees (i.e., the $6000$ cost for $8$ employees and the $8500$ cost for $12$ employees). Technically, the changes in cost values were incorrect because Peter approximated the costs from the graph. However, the subsequent reasoning was correct. Therefore, his reasoning was coded as *coordinating the non-proportional and proportional relationships*.

Gavin and Armando exhibited small changes in reasoning. One change was that both students provided more instances of *iterating and partitioning changes in quantities composed units* during the post-interview. George went from reasoning with changes in quantities on the water pump #2 task only (pre-interview), to reasoning with changes in quantities on the water pump #2 task and the business-cost task (post-interview). Armando went from reasoning with changes in quantities on the water pump #1 task only (pre-interview), to reasoning with changes in quantities on the water pump #2 task and the business-cost (post-interview).

Armando also had an instance of unproductive reasoning which appeared to be a temporary unproductive setback and which might be interpreted as unproductive backward transfer. This change will be discussed later in the section on unproductive changes. The lack of significant productive changes in George’s and Armando’s
understandings of linearity will be addressed in the discussion section.

In summary, the mid- and low-level groups showed substantial changes to their understanding of linearity. In fact, the mid-level group matched or in some cases exceeded the proportional reasoning of the high-level group in the post-interview. The low-level group also exhibited substantial changes but did not attain the level of proportional reasoning of the other two groups. The high-level group showed smaller productive changes to their understanding of linearity. The notable exception was Peter, who showed marked improvement in reasoning on the business-cost task when compared with the cell-phone task. Therefore, five of the students showed a significant deepening in their understanding of linearity which involved reasoning proportionally with changes in quantities.

**BTF 2: Mathematized Diagrams**

In this section, I provide evidence showing that six of the seven students that participated in this study drew *mathematized diagrams* to represent linear function contexts in the post interview, whereas in the pre-interview, the students’ diagrams were not as mathematized. According to Realistic Mathematics Education (RME) instructional theory, mathematizing is “the process of describing a context problem in mathematical terms” (Gravemeijer & Doorman, 1999) and occurs when “subject matter [is] taken from reality, and . . . organized according to mathematical patterns” (Gravemeijer, 1994). This is consistent with the way that mathematizing is being used in the dissertation study. In other words, it will be shown that five student’s diagrams of real-world linear function contexts were more organized according to mathematical
patterns in the post-interview than in the pre-interview.

There were two ways in which students’ diagrams became more mathematized from pre- to post-interview: (a) their post-interview linear function diagrams contained more quantitative precision (diSessa, 2002; diSessa et al., 1991), and (b) their post-interview linear function diagrams illustrated a greater degree of coordination of two varying quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). I interpreted this increased mathematization as evidence of a deepening of understanding of the relationship between the quantities involved in a linear function.

This finding is significant because mathematizing is seen as an important part in moving students from informal understandings about mathematics to more formal mathematical thinking (Gravemeijer & Doorman, 1999). Thus, diagrams about linear functions that are more mathematized provide evidence that students have made progress from informal toward more formal understandings of linear functions.

There were two remaining students, not included in the five students referred to above. One of these students never produced a pre-interview diagram but only a post-interview diagram. Therefore, it was not possible to determine if his post-interview diagram had changed. Nevertheless, his post-interview diagram was consistent with the more quantitatively precise diagrams that other five students produced. The other student’s diagrams showed no evidence of increased mathematization. In fact, there were very few changes in his diagrams from pre- to post-interview. Next, both ways in which students’ diagrams became more mathematized will be elaborated.
Quantitative Precision

As stated above, one of the ways in which students’ post-interview diagrams became more mathematized was through greater quantitative precision. diSessa (2002) defines quantitative precision, in the context of students designing and creating their own representations, as the degree to which the represented quantities can be “precisely read out” (p. 109). Loucas (2004) describes quantitative precision, in the context of learning science, as making an idea “sufficiently precise in order to maintain consistent meaning across different contexts” (p. 18). Putting the two definitions together, quantitative precision was taken to mean that the diagrams were consistent throughout and that they afforded the direct reading of quantities.

The following three features of the diagrams were identified as contributing to quantitative precision (see Table 4.4): (a) scaling of the diagram for one or both variables, (b) labeling of quantities and, (c) structuring the diagram spatially (Battista et al., 1998, Thompson, 2000). As was done for BTF 1, both a priori codes from the literature and inductive codes that emerged from the data were used to categorize student diagrams with respect to scaling, labeling and spatially structuring (Miles & Huberman, 1994). In particular, three inductive codes were used to categorize how the diagrams were scaled (see Table 4.4). The first code was tick marks used to scale the diagram. The second code, zero points of quantities denoted, was used either when a diagram contained an explicit tick mark (and label) for the zero point for one or more quantities or when a diagram contained precise scaling that resulted in an implied location for the zero point(s). The third code, more than given data included in scaling, was used when the scaling of diagrams included data between and/or
Two a priori codes and one inductive code were used to categorize student diagrams with respect to labeling. The first a priori code was *values of quantities* beyond the values of the given data.

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<th>Table 4.4. Coding of diagrams for quantitative precision.</th>
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<td><strong>Quantitative Precision Codes</strong></td>
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<td><strong>Scaling</strong></td>
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<td>Tick marks used to scale diagram</td>
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<td>Zero points of quantities denoted</td>
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<tr>
<td><strong>Labeling</strong></td>
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<td>Values of quantities labeled</td>
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<td>Units of quantities labeled</td>
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<td>Labels precisely located</td>
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<td>Spatial structuring used to represent multiple values of a quantity</td>
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<td><strong>Spatial structure</strong></td>
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<td><strong>POST-INTERVIEW quantitative precision in water-pump #2 diagrams</strong></td>
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<td>Quantitative Precision Codes</td>
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<td>Spatial structuring used to represent multiple values of a quantity</td>
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labeled (Lobato et al., in press). The second a priori code was units of quantities labeled (Lobato et al., in press). The third code, an inductive code called labels precisely located, was used for diagrams in which it was clear which tick mark each label referred to, or in the case of a label between tick marks, it was clear that the label referred to a change in a quantity.

An additional code, called spatial structure used to represent multiple values of a quantity, was used when student diagrams represented a set of values of a quantity in a way that made use of the same spatial structure. For example, several students drew pictures of a swimming pool with multiple levels of water on the same picture. This suggested that they were employing the same spatial structure to organize the entire set of water levels. In contrast, some students drew several separate diagrams of a pool, one for each level of water, with no apparent common spatial structure connecting the diagrams. This code was an adaptation of the a priori code spatial structuring, which Battista et al. (1998) define as “the mental operation of constructing an organization or form for an object or set of objects” (p. 503). Battista et al. originally developed this category as a way to account for how second-graders organized their drawings of two-dimensional rectangular arrays. In the dissertation study, spatial structuring is extended to number line and pictorial diagrams produced by middle-school students.

Evidence for this finding will be presented using the same groupings that were used for Finding 1, because there was a fair degree of alignment between the ways that students reasoned proportionally coming into the study and the changes that were present in their diagrams from pre- to post-interview. The low-level group will be
presented first because the diagrams of the student in this group demonstrated the most
dramatic changes. Note also that the evidence for this finding comes exclusively from
the pre- and post-water pump #2 task, which was the only task that explicitly asked
students to draw a picture (see Figure 4.12).

| (a) | After 4 minutes, there were 6 gallons of water in the pool.  
|     | After 6 minutes, there were 9 gallons in the pool.  
|     | After 11 minutes, there were 16.5 gallons in the pool.  
|     | After 14 minutes, there were 21 gallons in the pool.  

| (b) | After 4 minutes, there were 10 gallons of water in the pool.  
|     | After 6 minutes, there were 15 gallons in the pool.  
|     | After 11 minutes, there were 27.5 gallons in the pool.  
|     | After 14 minutes, there were 35 gallons in the pool.  

Figure 4.12. (a) Pre-interview water pump #2 task; b) Post-interview water pump #2 task.

Low-level proportional reasoners. Contrary to BTF 1, the changes in
Kendra’s diagram from pre- to post-interview were more dramatic than the changes in
the mid-level reasoners’ diagrams. She went from producing a diagram about a linear
context that contained no evidence of quantitative precision, to producing a diagram
that aligned with six of the seven quantitative precision codes. I interpreted the
changes in Kendra’s diagrams as an indication that she had deepened her
understanding of linear functions.

Pre-interview diagram. Kendra drew a series of four snapshots of a pool using
a top-down view (see Figure 4.13a). Each successive pool contained more water than
the previous snapshot. She referred to the water as “a puddle . . . and then a bigger
puddle.” Kendra’s diagrams lacked scaling of any kind and labeling of any quantities. Another feature was that the series of snapshots of the pool for a given diagram were not all drawn to the same scale but varied in size. For example, the first pool had a square shape while the other three were more rectangular. Also, the second, third and fourth pools got progressively bigger.

Kendra’s diagram contained an additional ambiguity: her final snapshot contained a rectangle with no puddle of water. It is unclear if the rectangle was supposed to represent a completely full pool or a completely empty pool.

**Post-interview diagram.** In the post-interview, Kendra drew a number-line diagram (see Figure 4.13b). There were several reasons why this diagram was more quantitatively precise than the pre-interview diagram. First, Kendra used tick marks to scale her diagram by 1 min intervals up to 16 min (although her tick marks were sometimes irregularly spaced). She denoted the zero point of the number line with a starting tick mark (but no label). Because she scaled by 1 min intervals, she included more than the given data in her scaling. However, it should also be noted that Kendra
had difficulty explaining how the scaling of the water volume fit with the scaling of the time. In particular, after stating that the rate was 2.5 gal per min, she appeared to re-appropriate the diagram’s scaling, because she reinterpreted the tick marks, which represented 1 min intervals, as 1 gal intervals (e.g., “I counted every 2.5, these are the little lines, every 2.5 and then right here and here” [counts tick marks as if they represent gallons]). Thus, Kendra appeared to still be working through how to represent two quantities with the same scale.

Second, in the post-interview, Kendra labeled the values of quantities and at least labeled some units of quantities (see Figure 4.13b). Specifically, she labeled every fourth tick mark for the time value and labeled the particular tick marks that corresponded to the given times in the data with the appropriate water volume. She also labeled the first two quantities of water volume with units (i.e., g). However, she also later mislabeled an additional tick mark between the second and third tick marks as 1 min as the result of her earlier difficulty with trying to re-appropriate the scaling of time as the scaling of water volume. Also, the labeling of her tick marks was somewhat imprecise because it was not always clear if a particular value belonged to a tick mark or to the interval between tick marks.

Third, Kendra’s post-interview diagram used spatial structure to represent multiple values of quantities. In particular, Kendra’s number-line diagram represented multiple times and the corresponding water volumes, all measured from the left-most tick mark on the same number line. In contrast, her four pre-interview snapshots of the pool, each one representing a different time and water volume, were of differing sizes which suggested that they did not share a common spatial structure.
Thus, despite some aspects of Kendra’s diagram being imprecise (e.g., tick marks irregularly spaced, some locations of labels imprecise, etc), Kendra’s post-interview diagram was significantly more quantitatively precise than her pre-interview diagram, as evidenced by the use of scaling, labeling and spatial structuring. There are two reasons why I interpreted this increase in quantitative precision as a deepening of understanding of linearity. First, by scaling the diagrams, denoting the zero point of quantities, labeling the values and the units of quantities, and locating some labels precisely, Kendra showed that she was treating the quantities involved in the linear function contexts as measures of time and of water volume instead of as numerical values. In other words, her post-interview diagram indicated that she was better able to incorporate the real-world quantities into her linear function reasoning. Second, by including more than the given data in her diagrams and by using spatial structuring on which to represent multiple values of quantities, Kendra treated the data set like part of an entire functional relationship between time and water volume rather than as isolated points. Therefore, I interpreted these changes in the low-level proportional reasoner’s diagrams as an indication of productive backward transfer.

**Mid-level proportional reasoners.** This group was made up of Brady, Jenn and Nicholas. Brady’s and Jenn’s pre-interview diagrams were significantly different from their post-interview diagrams. The diagrams went from aligning with only two of the quantitative precision codes to aligning with six of the seven codes. I interpreted the changes in Brady’s and Jenn’s diagrams as indicating a deepening of their understanding of the linear relationship between the quantities involved in the water pump task. Nicholas, who did not make a diagram during the pre-interview,
made a diagram in the post-interview that also aligned with six of the seven codes. Therefore, Nicholas’ post-interview diagram was consistent with the changes that Brady and Jenn exhibited.

**Pre-interview diagrams.** Brady and Jenn drew snapshot diagrams of the pre-interview water-pump #2 task. Brady drew side views of the pool while Jenn drew top-down views (see Figure 4.14a and 4.14b). Brady’s diagram showed the water level rise for three successive snapshots of the pool. Jenn’s diagrams showed a puddle of water growing outward toward the edges of the pool in two successive snapshots. Brady’s and Jenn’s diagrams contained labeling of quantity values and units. However, both students’ diagrams lacked any scaling. The lack of scaling meant that it would have been difficult to read water volume quantities other than the given quantities off the diagrams with any precision (e.g., on Brady’s diagrams, it would be difficult to determine with precision where an 8 gal water level would be).

Another feature of Brady and Jenn’s diagrams was that the series of snapshots of the pool varied in size. For example, Jenn’s first pool, which represented 6 gal in 4 min was significantly larger than the second pool which represented 9 gal in 6 min. Similarly, Brandon’s snapshots of the pool varied in scale.

**Post-interview diagrams.** In the post-interview, Brady and Jenn drew number-line diagrams instead snapshots, as did Nicholas. There were several reasons why Brady’s and Jenn’s post-interview water pump #2 diagrams were more quantitatively precise than their pre-interview diagrams (see Figures 4.15a and 4.15b). First, Brady and Jenn used tick marks to scale their diagrams. Brady drew two number lines, scaling one of them by 2 min from 0 to 14 and the other one by 1 min from 0 to 14.
Thus, Brady’s diagram included more tick marks than the given data that represented additional times and water volumes. Jenn scaled her number line with equally-spaced tick marks representing the given data only. Brady and Jenn denoted the zero points of quantities with tick marks and/or labels.

Second, Brady and Jenn used more labeling in their post-interview diagrams. In particular, they labeled one time value and one water volume value for each tick mark. Jenn also labeled each interval between tick marks with a change in time and the corresponding change in water volume and labeled the units for each change in time and change in water volume (i.e., min and gal). The students’ labels were precisely located, meaning that there was little ambiguity as to which label referred to which tick mark, and in Jenn’s case, whether a label referred to a tick mark or to the interval between tick marks.

Third, Brady’s and Jenn’s post-interview diagrams reflected greater spatial organization than their pre-interview diagrams. In other words, their number-line diagrams used a spatial structure to represent multiple values of quantities. For
example, Jenn’s post-interview diagram contained values of time and volume that were all measured from the left end of the number line. In contrast, her pre-interview snapshots did not seem to share the same spatial structure as evidenced by her diagram of the pool with 3 gal of water being smaller than her diagram of the pool with 2 gal of water. However, Jenn’s post-interview diagram seemed somewhat less spatially structured than Brady’s because Jenn’s evenly-spaced tick marks each represented one of the time/volume data points despite the given data consisting of uneven intervals of time and volume.

Nicholas’ post-interview diagrams—he drew two number lines—were
consistent with Brady’s and Jenn’s because (a) Nicholas scaled his number line
diagram with tick marks for each of the given data points, although he did not denote
the zero point (see Figure 4.15c), (b) he labeled intervals between tick marks for his
second number line, and (c) his labels were precisely located (i.e., there was little
ambiguity about what the labels referred to). However, like Jenn’s, Nicholas’ diagram
was somewhat less quantitatively precise than Brady’s because he spaced the tick
marks evenly despite those tick marks representing unevenly spaced data points.

To summarize, Brady’s, and Jenn’s diagrams increased in quantitative
precision from pre- to post-interviews, and Nicholas’ diagram was consistent with
their increased precision. As with the low-level group, I interpreted the changes in
Brady’s and Jenn’s diagrams from pre- to post-interview as evidence of a deepened
understanding of linearity because they were able to more precisely incorporate the
real-world quantities into their linear function reasoning (i.e., greater
mathematization), and they treated the data set like part of an entire functional
relationship. In other words, I took the changes in the mid-level group as an indication
of productive backward transfer.

**High-level proportional reasoners.** All three of these students’ pre-interview
diagrams were more quantitatively precise than the pre-interview diagrams from the
other two groups. Nevertheless, two of the students, Peter and Armando, constructed
diagrams that increased in quantitative precision by a limited amount from pre- to
post-interview, while the other student, George, constructed diagrams that decreased
in quantitative precision by a small amount.
Peter drew two kinds of diagrams. First he drew a set of snapshots of a swimming pool with different amounts of water (see Figure 4.16a). Peter’s snapshots were more quantitatively precise than Brady’s, Jenn’s and Kendra’s snapshot diagrams because he used tick marks to scale his diagrams. The scaling appeared to show 1 gal per tick mark because for the 4 min/6 gal swimming pool there were 6 tick marks, for the 6 min/9 gal pool there were 9 tick marks, etc. However, there were also aspects to Peter’s snapshots that reflected imprecision. For example, instead of drawing his three snapshots of the pool all the same size, he drew

Figure 4.16. (a) Peter’s pre-interview diagram #1; (b) pre-interview diagram #2; (c) Armando’s pre-interview diagram.
the water level for the 6 min/9 gal pool at virtually the same height as the water level for the 11 min/16.5 gal pool. Also, he did not label his tick marks.

After drawing the snapshots, Peter decided to draw a single diagram with multiple water levels represented (see Figure 4.16b). Peter again used tick marks to scale the diagram. By drawing just one pool with several levels of water, Peter used spatial structuring to represent multiple values of water volume. However, Peter did not label quantity values or units on this second diagram either.

Armando’s pre-interview diagram (see Figure 4.16c) was similar to Peter’s second pre-interview diagram except that Armando labeled time values and corresponding water volume values with units. For this diagram, Armando used the spatial structuring of one pool to represent multiple values of time and water volume. However, he did not locate the labels precisely because he placed the accumulated time and water volume labels between tick marks. He also included labels of distance values and meters units but did not explain where the distance measurements came from or what they represented. Armando did not denote the zero points of quantities.

George drew a bar graph in which he used tick marks to scale the diagram, included more than the given data, labeled values of quantities on the axes and labeled one of the axes with units (see Figure 4.17). The bar graph format provided a spatial structure on which to represent multiple values of time and water volume.

**Post-interview diagrams.** Peter’s post-interview diagram was more quantitatively precise than either of his pre-interview diagrams (see Figure 4.18a). One reason was that he labeled the values and units of quantities. Another reason was that he included more than the given data in the scaling of his diagram. Specifically,
he drew tick marks on the side of the pool and labeled them from 2 to 48 gal by twos. He also labeled each water level with a time value and minutes units. However, the time values were somewhat ambiguously located between successive water levels. Therefore, it was difficult to ascertain which water levels the time values applied to. Nevertheless, the increased scaling and labeling is evidence that Peter’s post-interview diagram was somewhat more quantitatively precise than his pre-interview diagram.

Armando drew a post-interview number-line diagram that shared several features with his pre-interview diagram. First, his diagram was scaled with tick marks. Second, it was labeled with values of quantities (i.e., from 0 to 40 gallons in 5-gallon intervals and from 2 to 16 minutes in 2-minute intervals). Third, it contained spatial structuring, with multiple water levels represented on the same number line.

There were also several features of Armando’s post-interview diagram that
reflected greater quantitative precision than the pre-interview diagram. First, Armando denoted the zero points of time and water volume with a tick mark and label at the left end of his number-line diagram. Second, he included more than the given data in the scaling of his diagram. Third, he improved on locating the labels precisely (i.e., the water volume labels were located directly under the tick mark that they belonged to and the change in minutes labels were located between tick marks). However, there remained several features that were imprecise. For example, the accumulated times were imprecisely located between tick marks. Also, Armando neglected to label the units for the water volume as he had done in the pre-interview diagram. Thus, his post-interview diagram overall was more quantitatively precise.
However, by neglecting units for water volume, at least one aspect was less precise.

George again drew a bar graph. His pre-interview bar graph included slightly more *labeling of quantities* (i.e., labeling of the time and gallon values with units) than his post-interview bar graph. Therefore, George’s post-interview diagram appeared to be slightly less quantitatively precise than his pre-interview diagram. Of course, a primary reason why George’s pre- and post-interview bar graphs were both imprecise was because the horizontal axis of each graph represented the continuous quantity time and so, to be consistent, the thickness of each bar should have represented an interval of time, even though it is likely that George intended each bar to represent an instant of time.

In summary, all three students provided evidence of higher levels of quantitative precision in their pre-interview diagrams than the mid- and low-level groups but showed smaller changes in their diagrams from pre- to post-interview. Peter and Armando showed a limited increase in quantitative precision, while George showed a small decrease in precision. Based on the reasons presented earlier, I interpreted the changes in Peter’s and Armando’s diagrams as evidence of productive backward transfer. I interpreted the small changes in George’s diagrams as possible evidence of unproductive backward transfer. Consequently, all but one student showed evidence of productive backward transfer in the form of increased quantitative precision, with the mid- and low-level groups showing the biggest effect.

**Coordination of Varying Quantities**

A second way in which students mathematized their diagrams was by increasing the level of coordination of varying quantities. According to Carlson et al.
(2002), covariational reasoning is defined as “the cognitive activities involved in coordinating [emphasis added] two varying quantities while attending to the ways in which they change in relation to each other” (p. 367). Analysis of the diagrams revealed that all students were coordinating the varying quantities of time and water volume to a greater degree in their post-interview diagrams than in their pre-interview diagrams.

To code the student diagrams, several of the levels from the Carlson et al. (2002) covariational framework were adapted (see Chapter 2 for details). The levels of the covariational framework were not used as is because student-created diagrams were being coded instead of mental images of covariation. The adapted codes were organized into hierarchical levels. However, they are not intended to define an exhaustive list of levels. Rather, they represent four levels of coordination of quantities that were seen in the students’ drawn diagrams.

Level 1 was called *one quantity shown changing* (see Table 4.5). This code was used to describe diagrams in which only the water volume was shown as changing. Level 2 was called *direction of two quantities shown changing in relation to each other*. This code was used for diagrams that showed both the time and the water volume changing and showed that as time increased, the water volume increased. Level 3 was called *amount of two quantities shown changing in relation to each other*. This code was used for diagrams in which changes in both quantities were represented and labeled with values. Level 4 was called *rate of change of one quantity shown with respect to the change in the other quantity*. This code was used for diagrams in which the rate at which one quantity changed as the other changed by a
fixed amount was shown across the diagram. Each successive code represents increased coordination of two quantities in a linear relationship in a diagram.

All students’ diagrams went up either one or two levels from pre- to post-interviews, except for Nicholas, who did not draw a pre-interview diagram. Because coordinating quantities involves identifying and reasoning with mathematical patterns (Carlson et al., 2002), I interpreted this increase in coordination as evidence that

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### Table 4.5. Coding of diagrams for coordination of quantities.

<table>
<thead>
<tr>
<th>Coordination of quantities codes</th>
<th>Students grouped by proportional reasoning level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
</tr>
<tr>
<td></td>
<td>George</td>
</tr>
<tr>
<td>One quantity shown changing (Level 1)</td>
<td></td>
</tr>
<tr>
<td>Direction of two quantities shown changing in relation to each other (Level 2)</td>
<td>✓</td>
</tr>
<tr>
<td>Amounts of two quantities shown changing in relation to each other (Level 3)</td>
<td></td>
</tr>
<tr>
<td>Rate of change of one quantity shown with respect to the change in the other quantity (Level 4)</td>
<td></td>
</tr>
</tbody>
</table>

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PRE-INTERVIEW coordination of quantities in water-pump #2 diagrams

POST-INTERVIEW coordination of quantities in water-pump #2 diagrams
students’ diagrams were more mathematized.

**Mid- and high-level proportional reasoners.** I present these two groups together because their pre-interview diagrams were coded identically with respect to the coordination of quantities, and their post-interview diagrams showed increases in the level of coordination. More specifically, five of the six mid- and high-level proportional reasoners showed evidence in their post-interview diagrams of increased coordination of quantities.

**Pre-interview diagrams.** Two of three students in the mid-level group—Jenn and Brady—and all three students in the high-level reasoners’ group—Peter, Armando and George—produced pre-interview diagrams that were coded as *direction of two quantities shown changing in relation to each other* (Level 2). This was because all five students produced diagrams that clearly showed the water level increase as the time increased, but none of the diagrams highlighted the amounts of changes in time or water level (e.g., see Figures 4.14 and 4.16).

The reader might wonder whether students might actually be coordinating the changes in quantities but without highlighting them or recording them in the diagram. After all, it would be clear to an adult from Armando’s picture, that there was a change in time and water volume of 1 min and 2.5 gal from 5 min/7.5 gal to 6 min/9 gal (see Figure 4.16c). However, prior research has shown that when middle-school students do not annotate a quantity in a diagram, then they are likely not focusing on that quantity while drawing the diagram (Lobato et al., in press). Thus, it was likely that these students were not attending to the changes in quantities as they constructed their pre-interview diagrams.
Post-interview diagrams. Three of the students from the mid- and high-level groups—Armando, Brady and Nicholas—drew post-interview diagrams that were coded as rate of change of one quantity shown with respect to the change in the other quantity (Level 4). In other words, these three students represented the rate of change over the entire length of the diagram. For example, Armando labeled each tick mark on his number line with a multiple of 5 from 0 to 40, drew arches between successive tick marks and labeled each arch 2 min (see Figure 4.18b). He explained this as, “every two minutes, it’s going by 5 . . . it’s 2 minutes and 5 gallons.” In a similar way, Brady and Nicholas highlighted equally-spaced tick marks with arches and labeled each tick mark clearly and systematically with 1 minute and 2.5 gallon labels (see Figures 4.15a and 4.15c). By showing the rate of change across most or all of the number lines, these diagrams reflect more sophisticated coordination of quantities than that shown in the pre-interview diagrams where only the direction of change of one quantity in relation to the other was explicitly shown.

Peter and Jenn drew post-interview diagrams that were coded as amounts of two quantities shown changing in relation to each other (Level 3). Jenn and Peter used labeling and brackets to show the amounts of two quantities changing in relation to each other. For example, Peter drew one pool from the side view, scaled the pool along the side with tick marks to represent the gallons of water and drew horizontal lines across the pool to represent multiple levels of water (see Figure 4.18a). Between successive levels of water he recorded the changes in water volume and the accumulated times. On the left side of the diagram, Peter used brackets and arrows to show that from 4 to 6 min, 5 gal would have been pumped into the pool (i.e., he wrote
“2 min of 5 gal”). On the right side of the diagram, Peter used brackets and arrows to show that the amount of water pumped into the pool in the first 4 min and half of the amount of water pumped into the pool from 4 to 6 min would equal the amount of water pumped into the pool from 6 to 11 min. Thus, Peter showed in his diagram how the water volume changed in relation to how the time changed. Jenn also did this in her diagram.

George used brackets on his bar graph to highlight the difference in height between successive bars. However, he did not connect the change in water volume to the change in time on his diagram. Therefore, his post-interview diagram was coded like his pre-interview diagram as direction of two quantities shown changing in relation to each other (Level 2). I interpreted the highlighting of the changes in water volume as a small shift toward more coordination of quantities.

To summarize, four of the students in the mid- and high-level proportional reasoners’ groups shifted one or two levels in the direction of more sophisticated coordination of quantities according to the coding scheme that was used. Moreover, Nicholas’ post-interview diagram was consistent with the shifts that the other students exhibited. George also exhibited a small shift toward greater coordination of quantities, but the shift did not result in his post-interview diagram being coded at a higher level than his pre-interview diagram.

**Low-level proportional reasoners.** Kendra, the lone student in this group, produced diagrams that showed evidence of lower levels of coordination of quantities than the students in the other two groups. However, she too exhibited a shift from pre-
to post-interviews in the direction of increased coordination of the quantities in a linear relationship. I interpreted this as productive backward transfer.

**Pre-interview diagram.** As described earlier, Kendra drew four snapshots of the pool (see Figure 4.13a). Each of the first three snapshots showed a rectangular pool with a successively larger puddle of water. The fourth snapshot was a rectangular pool that was either completely full or completely empty. With these four snapshots, Kendra showed that the water volume was changing. However, the diagrams did not show explicitly that the time was also changing. Therefore, her diagram was coded as *one quantity shown changing* (Level 1).

**Post-interview diagram.** Kendra drew a number-line diagram (see Figure 4.13b). She scaled the number line with tick marks and labeled every fourth tick mark with a multiple of 4. She then drew dots on the particular tick marks that matched the times given in the data and drew arches between dots. She also labeled the dots with the appropriate water volume (in gallons) that corresponded to the given time. In doing so, Kendra showed that as the time increased, the water volume increased. Furthermore, by drawing arches between dots, she showed that she was beginning to attend to the changes in time and water volume. However, she did not label her diagram as systematically as the other students; thus, it was difficult to see exactly what the amounts of change in each quantity were. Her diagram was coded as *direction of two quantities shown changing in relation to each other* (Level 2). Thus, Kendra’s diagram in the post-interview showed a shift toward increased coordination of quantities despite being a level or two behind the other students.

In summary, all students’ diagrams (except Nicholas) showed an advance
toward higher levels of coordination in the post-interview diagrams when compared to
the pre-interview diagrams. Taken together with the evidence about quantitative
precision, six of the seven students’ post-interview diagrams about linear function
contexts were more mathematized. This evidence supports the claim that students’
understanding of linearity had deepened by the post-interview (i.e., that productive
backward transfer had occurred).

**BTF 3: Conceiving of Division with a Partitive Model**

In this section, I present evidence showing that most students went from
providing a procedural (Clark, Moore, & Carlson, 2008) or rudimentary explanation of
what division did to providing a more sophisticated conceptually-based explanation
that reflected a partitive model of division (Fischbein, Deri, Nello, & Marino, 1985).
During the pre-interview, most students talked about division in procedural terms. In
contrast, all students gave post-interview explanations of division that appeared to be
grounded in understanding (BTF 3). Specifically, they talked about the meaning of
division and explained division in terms of a real world context. Furthermore, their
explanations involved grouping and a partitive model of division.

This finding is significant because understanding division is foundational for
reasoning proportionally and hence for reasoning productively about linear functions
(Lobato & Ellis, 2010; Nabors, 2003). Therefore, an understanding of division that
shifts from being procedurally-based toward being conceptually-based could translate
into an improved understanding of linearity. This is what appears to have happened
for these students because the conceptually-based explanations of division provided in
the post-interviews involved division that supported productive reasoning about linearity. In other words, students’ improved understanding of division appeared to play an important role in students’ improved reasoning about linear functions from pre- to post-interview.

A priori codes were used to categorize students’ responses about division (see Table 4.6). The code procedure-based explanation (Clark, et al., 2008; Simon, 1993) was further subdivided into two codes: (a) a numerical explanation was used to code explanations of division that involved stating “numerical calculations . . . to obtain answers to a problem” (Clark, et al., 2008, p. 297), and (b) an algorithmic explanation was used to code explanations that focused on algorithmic procedures (Tirosh & Graeber, 1989). The code halving-based explanation was used to code explanations of division that focused on cutting in half or dividing to reduce values of quantities (Bell, Swan, & Taylor, 1981; Pothier & Sawada, 1983). Although this explanation is somewhat conceptually-based, thinking about division as halving is a rudimentary conception of division (Pothier & Sawada, 1983). The code partitive explanation was used to categorize conceptually-based explanations of division that involved a partitive model (Fischbein et al., 1985). According to this model, dividing means partitioning a dividend equally into a set number of groups, where the divisor defines the number of groups. For example, dividing 4 m by 3, according to the partitive model of division, means partitioning 4 m into 3 equal groups, with 1. m in each group.

As can be seen in Table 4.6, all but one student used division on one or more pre-interview linear tasks, and all students used division on two or more post-
interview tasks. This is not surprising because as reported earlier, students engaged in more proportional reasoning from pre- to post-interview which, in many cases, involved division. What was surprising was the differences in the kinds of

<table>
<thead>
<tr>
<th>Students grouped by proportional reasoning level</th>
<th>PRE-INTERVIEW Tasks</th>
<th>Water pump #1</th>
<th>Water pump #2</th>
<th>Orange Juice</th>
<th>Cell- phone</th>
<th>Explanation of Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>High level</td>
<td>George</td>
<td>÷ used</td>
<td>÷ used</td>
<td>÷ used</td>
<td></td>
<td>Halving-based</td>
</tr>
<tr>
<td></td>
<td>Armando</td>
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<td>÷ used</td>
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<td></td>
<td>Numerical</td>
</tr>
<tr>
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<td>Peter</td>
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<td>÷ used</td>
<td></td>
<td></td>
<td>Algorithmic</td>
</tr>
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<td>÷ used</td>
<td>÷ used</td>
<td></td>
<td>Halving-based</td>
</tr>
<tr>
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<td>Brady</td>
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<td></td>
<td></td>
<td>No explanation</td>
</tr>
<tr>
<td></td>
<td>Jenn</td>
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<td></td>
<td></td>
<td>No explanation</td>
</tr>
<tr>
<td>Low level</td>
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<td></td>
<td>÷ used</td>
<td></td>
<td></td>
<td>Numerical</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Students grouped by proportional reasoning level</th>
<th>POST-INTERVIEW Tasks</th>
<th>Water pump #1</th>
<th>Water pump #2</th>
<th>Paint making</th>
<th>Business-cost</th>
<th>Explanation of Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>High level</td>
<td>George</td>
<td>÷ used</td>
<td></td>
<td>÷ used</td>
<td></td>
<td>Partitive/ Procedural</td>
</tr>
<tr>
<td></td>
<td>Armando</td>
<td>÷ used</td>
<td>÷ used</td>
<td></td>
<td></td>
<td>Partitive</td>
</tr>
<tr>
<td></td>
<td>Peter</td>
<td>÷ used</td>
<td></td>
<td></td>
<td>÷ used</td>
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<tr>
<td>Mid-level</td>
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<td>÷ used</td>
<td>÷ used</td>
<td>÷ used</td>
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<td>Partitive</td>
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<tr>
<td></td>
<td>Brady</td>
<td>÷ used</td>
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<td></td>
<td>÷ used</td>
<td>Partitive</td>
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<td></td>
<td>Jenn</td>
<td>÷ used</td>
<td>÷ used</td>
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<td>Low level</td>
<td>Kendra</td>
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<td>÷ used</td>
<td></td>
<td>÷ used</td>
<td>Partitive/ Procedural</td>
</tr>
</tbody>
</table>
explanations of division that students provided from pre- to post-interview.

I used the student groupings with respect to their levels of proportional reasoning coming into the study to present the evidence for this finding. The students’ explanations of division from the pre-interview will be presented first, followed by evidence from the post-interview. Post-interview diagrams that students drew to support their explanations will also be presented.

**Pre-interview Explanations of Division**

All but one student used division on one or more of the linear tasks during the pre-interview. When students were asked to explain their use of division, two students gave a numerical explanation, one student gave an algorithmic explanation and two students gave a halving-based explanation (see Table 4.6). The remaining two students did not give explanations of division. While the evidence will show that students’ explanations were either procedurally-based or conceptually rudimentary, this does not necessarily mean that students had no understanding of division beyond the explanations that they gave. However, the evidence does suggest that student attention was more focused on division as a procedure than on a conceptual model of the meaning of division.

**High-level proportional reasoners.** Armando, Peter and George provided procedurally-based explanations about division in the pre-interview. For example, after dividing the first values in the water pump #2 data set by 2 and then multiplying by 3 to get the second values in the data set (i.e., 4 min/6 gal ÷ 2 × 3 = 6 min/9 gal; see Figure 4.12a for data set), Armando provided a *procedurally-based numerical explanation* of division:
I just used these [points at the long division statements; see Figure 4.19] to find out the first one [points at 4 and 6 minutes]. So I divided this by 2 [points at the 4 divided by 2 long division statement], divided this one by 2 too [point at the 6 divided by 2 long division statement] . . . They’re just numbers . . . to see . . . to multiply these two [points at the 2 and the 3 quotients in the long division statements] to get . . . 6 minutes . . . so I did the same with this one [points at the long division statement 6 divided by 2].

![Figure 4.19. Armando’s pre-interview water pump #2 written division statements.](image)

In this excerpt, Armando explained division as a process of using numbers to get other numbers. As stated above, it is possible that Armando did possess more understanding than his responses suggested. However, he appeared to focus primarily on the procedure of division rather than on the meaning of division.

Peter provided a procedurally-based algorithmic explanation. This occurred when he divided 6 gal by 4 min: “If you change one side, you can change the other side, and it’ll still be equal [points to 4 min and 6 gal in problem sentences], if you do the same thing . . . it gives you a lower proportion, lower ratio.” During this excerpt, Peter was likely thinking about solving a proportional statement because he seemed to be referring to two sides of an equation. He may also have been thinking about reducing ratios (i.e., “a lower proportion, lower ratio”). In either case, Peter’s explanation did not address the conceptual meaning of dividing 6 gal by 4 min, but focused on a procedure.
George did not provide an explanation of division for any of the linear tasks in the pre-interview, but he did provide a \textit{procedurally-based numerical explanation} on a task about a clown walking according to a quadratic distance-time function. On that task, George found the speed of walking 6 m in 2 s to be 3 m/s and explained, “I’m trying to figure out how many meters per second, and so dividing it by 2 will get rid of the 2 and make it a 1.” George’s explanation that he divided by 2 to “get rid of the 2,” suggested a focus on division as a numerical calculation rather than on the conceptual meaning of division.

\textbf{Mid-level proportional reasoners.} No student in this group provided an explicit explanation of division. Brady and Jenn used division very little or not at all, while Nicholas used division but provided no explicit explanation. However, Nicholas’ talk appeared to be \textit{procedurally-based}, focusing primarily on \textit{dividing to make smaller}. On numerous instances in the pre-interview, Nicholas divided by 2. He usually referred to dividing by 2 as \textit{cutting in half}. For example, when he divided 4 min and 10 gal by 2 he described it as, “So you can’t get 4 into 10 equally, so you just \textit{cut the 4 in half} [emphasis added] and make it 2 and then it would go into 10, but then you have to cut the three in half as well.” Later, when he divided 1.5 gal in 2 min by 2, explained it as, “it’s equal all the way through because one and a half \textit{cut in half} [emphasis added] is seventy-five.” In all, Nicholas referred to division as \textit{cutting in half} seven times during the pre-interview.

Nicholas’ emphasis on cutting in half when dividing appears to be similar to what Pothier and Sawada (1983) call a “halving mechanism” (p. 311). This mechanism is thought to form when young children begin to learn about sharing in
social settings (Kieren, 1980, cited in Pothier & Sawada, 1983). Therefore, even though Nicholas may have understood division at a deeper level than indicated, he appeared to be focused on a fairly rudimentary conception of division.

**Low-level proportional reasoners.** When Kendra was asked how fast the water pump #2 was pumping, she divided 4 minutes by 3 (the 3 came from noticing that from 4 to 6 minutes, 3 gallons were added to the pool) and explained, “I want to figure out per minute how many gallons is.” When asked, “So how do you know which number to pick to do that division with?” she said, “I didn’t think about that; I just picked the first number.” In this explanation, Kendra appeared to be focused on the calculational aspects of division. Her explanation was coded as a procedurally-based numerical explanation.

**Post-interview Explanations of Division**

In the post-interview, every student used division in at least two of the four linear tasks, which as stated above, is likely a reflection of students engaging in proportional reasoning to a greater degree. More importantly, every student provided an explanation that aligned with a partitive understanding of division, although some students’ explanations were sometimes procedurally-based and sometimes conceptually-based (see Table 4.6). As stated above, a partitive model of division is when division is conceived of as sharing a dividend equally among a set number of groups, where the divisor represents the number of groups and the quotient represents the number within each group. The same groupings of students will be used that were used to present the pre-interview evidence.
**High-level proportional reasoners.** During the post-interview, the focus of Armando’s, Peter’s and George’s explanations shifted away from procedures toward a conceptual model of the meaning of division. Specifically, each student provided an explanation that reflected a partitive model of division.

Armando provided an example of a conceptually-based explanation after he divided 20 gal over 8 min by 2 twice to find out that the post-interview water pump #2 was pumping 5 gal into the pool every 2 min. He explained:

> I put 20 gallons over 8 minutes [writes “20/8”]. Then I kept dividing so it’s 10 over 4 [writes “10/4”], and it’s 5 over 2 [writes “5/2”] . . . it breaks it into groups . . . [it] separates it into 4 groups of 5 gallons in 2 minutes [draws four circles and labels each with a 5g and a 2m; see Figure 4.20].

In this excerpt, Armando described dividing as forming 4 equal groupings with 5 gal and 2 min in each group. This explanation was coded as a *partitive explanation* because the number of groups was set (i.e., 4 groups) and the dividend was shared equally among the groups (i.e., 20 gal and 8 min shared so each group has 5 gal and 2 min). Peter provided a similar explanation.

George provided an explanation that appeared to have both a procedural and a conceptual basis. In the following excerpt from the water pump #2 task, George provided a halving-based explanation of division in the context of reducing 5 gal in 2 min to 2.5 gal per min:

> If you divide by 2 over 2 [writes “÷2/2” beside the ratio “5 G/2 min”], because the 2 minutes, you want to make that into 1, as small as you can. And so if you divide that by 2 [points to “2 min” in “5 G/2 min” ratio] that will give you 1, and you divide 5 by 2, you get 2.5.
However, when pressed further, he provided a more conceptually-based explanation:

“Umm, you’re putting this into groups of [points at 5 G/2 min] two different groups of something.” When asked “what goes in each group,” he said, “2.5 gallons and 1 minute.” This explanation was coded as a partitive explanation because the number of groups was set (i.e., 2 groups) and the dividend was partitioned so that an equal amount went to each group (i.e., 5 gal and 2 min partitioned into 2 groups with 2.5 gal and 1 min in each group). Thus, George provided both procedurally-based and conceptually-based explanations.

**Mid-level proportional reasoners.** In the post-interview, the mid-level proportional reasoners used division often when reasoning about linear contexts and provided explanations about division that aligned with a partitive model of division. Brady provided an example of a conceptually-based explanation when asked to explain why he had divided a change in volume of 5 gal by the corresponding change in time of 8 min:

These are five gallons and these are 8 minutes . . . if you make groups . . . of, what groups can I make of . . . if you put . . .625, if you put each one in a group. It will equal 5, I think because . . . because like you make 8 . . . 1, 2, 3, 4, 5, 6, 7, 8 [draws 8 circles; see Figure 4.4]. If you put .625 in each one [writes .625 in each circle] . . . and now what you could do, I think you could like add them or you could just, instead
of adding them you could multiply them by 8. So then if you multiply .625 times 8 [uses calculator] you’ll get 5 gallons.

This explanation was coded as a partitive explanation because Brady spoke about making groups, where the number of groups was set (i.e., 8 groups) and the dividend was partitioned so that each group had an equal amount (i.e., 5 gal partitioned into 8 groups with .625 gal in each group). Nicholas and Jenn provided similar explanations, although Jenn, at times, relied on the more procedural halving-based explanation. In sum, Brady and Nicholas provided only conceptually-based explanations, while Jenn provided both conceptually-based and procedurally-based explanations.

**Low-level proportional reasoners.** Kendra provided explanations of division that suggested that her attention was more directed toward the meaning of division but, like George and Jenn, there was still some attention directed toward division as a procedure. For example, Kendra divided 35 gal by 14 min to find the rate of pumping to be 2.5 gal/min. Her initial explanation sounded procedurally-based: “[I] use the divisor and the dividend to get a quotient . . . in the class we divided a lot to get something per something else.” This explanation was coded as an algorithmic explanation.

However, Kendra also provided an explanation of dividing 35 gal by 14 min that appeared to be conceptually based:

It’s separating it and then putting it in 14 groups . . . you count 35 [draws 35 circles] . . . So, you put them in 14 groups, but it’s going to look kind of messy because it’s 2.5 so . . . 2 point like cutting through [draws circle around two and one-half circles; see Figure 4.21]. And then . . . that’s two and then the other half [draws another circle around two and one-half circles].
In this excerpt and accompanying diagram, Kendra’s explanation appeared to have shifted toward a *partitive explanation*. In other words, she spoke of partitioning 35 into a set number of groups (i.e., 14 groups), with an equal amount in each group (i.e., 2.5 per group). Notice, however, that her explanation was devoid of references to the quantities involved in the division (i.e., gallons and minutes). This suggested that her attention was no longer primarily on division as a numerical calculation but was now split between the numerical calculation of division and the meaning of division.

To summarize, the four students who provided procedurally-based explanations of division in the pre-interview, shifted to conceptually-based explanations in the post-interview which aligned with a partitive model of division. Furthermore, the other students—those that provided no explanation of division in the pre-interview—also provided conceptually-based explanations in the post-interview that aligned with a partitive model of division. It was shown here and in BTF 1 that students used division to reason proportionally with quantities and changes in quantities in linear contexts in more productive ways in the post-interview. Therefore, the changes in explanations not only indicated that students were more focused on meaning; they also seem to reflect a deepened understanding of linearity.
Classroom Evidence Connecting Findings to Backward Transfer

The reader may wonder if something in the interviews themselves contributed to the changes that were observed from pre- to post-interview. For example, perhaps the students learned something in the interviews that contributed to their more productive proportional reasoning. Or, perhaps the persistent probing of the interviewers during the post-interview contributed to the emergence of more conceptually-based explanations in the post-interview. In order to address these questions, evidence will be presented from the analysis of the classroom data to show conceptual connections between the reasoning students’ exhibited on the post-interview and similar reasoning in class. These conceptual links support the claim that the changes observed from pre- to post-interview were a case of productive backward transfer (i.e., that some event or events during quadratics instruction contributed to a deepening of students’ understanding of linearity).

Classroom Evidence Connected to BTF 1

In this section, classroom evidence in support of the claim that BTF 1 was a case of productive backward transfer will be presented. In particular, an episode from Lesson 8 of the quadratic functions instruction shows that the increase in attention toward changes in quantities and the increase in reasoning with changes in quantities exhibited by students in the post-interview had roots in students’ classroom reasoning.

During Lesson 8, Brady and the other students were presented with quadratic distance-time data for a remote-control car (see Figure 4.22). They were asked to draw a diagram to show the speed. Brady created a diagram in which he highlighted
the changes in distance and changes in time (see Figure 4.23). Specifically, Brady highlighted each change in distance in the given data with a bracket and a label (i.e., 0 to 16 yd and 16 to 64 yd). He also highlighted each corresponding change in time with a bracket and a label (0 to 4 s and 4 to 8 seconds). When he explained his diagram, he reasoned with the changes in distance and changes in time to find the speed:

And here [points to beginning of 4-8 s section] to here [points at end of 4-8 s section] is 48 yards, and this is 4 seconds in total . . . And I divided 48 by 4 and I got 12 yards per second . . . and the change in time [points to the 4-8 s label] and change in yar--the distance [points at 16 to 64 yd label].

In this excerpt, Brady referred to a 4 s change in time from 4 to 8 s and the corresponding 48 yd change in distance from 16 to 64 yd. He appeared to form a
composed unit of 48 yd and 4 s, which he partitioned by dividing by 4 to get “12 yards per second.” This evidence is reminiscent of his reasoning during the post-interview water pump tasks. For example, on those tasks he found changes in water volume and the changes in time and then reasoned that the change in water volume divided by the change in time equaled the rate of pumping.

Other evidence from the instructional intervention was the inscriptions that Brady made when he was shown a second table for the same remote-control car with additional distances at 2 s and 6 s (see Figure 4.24). In these inscriptions, he recorded changes in distance and changes in time in fraction form, in the same way that he recorded changes in water volumes and changes in time during the post-interview water pump tasks and the same way that he recorded the changes in monthly cost and corresponding changes in employees for the business-cost task. This evidence links the instructional intervention to the post-interview and thus, supports the claim that the change in reasoning that Brady and the other students exhibited in the post-interview can be traced back to the instructional intervention on quadratics.

**Classroom Evidence Connected to BTF 2**

In this section, evidence in support of the claim that BTF 2 was a case of productive backward transfer will be presented. In particular, several diagrams that students constructed during the instruction on quadratic functions will be described that show that the *increased mathematization* (i.e., *increased quantitative precision* and *increased coordination of quantities*) of students’ diagrams from pre- to post-interview had roots in the kinds of diagrams they drew in the instructional intervention.
There were many examples of students’ diagrams of quadratic motion created in class that suggest that the increased *quantitative precision* of students’ diagrams for linear functions had origins in the instructional intervention. For example, Jenn drew a diagram of a quadratic distance-time function (see Figure 4.25) that had all features of quantitative precision that were coded for in students’ interview diagrams. In particular, she *used tick marks to scale* a number line, *denoted the zero points* for the time in minutes and the distance in miles with a tick mark and labels, represented *more values of times and distance than were in the given data*, *labeled the values and the units* of time and distance, *located her labels precisely* and *spatially structured the number line to represent multiple values of time and distance*. Therefore, this diagram contained a higher level of quantitative precision than students’ pre-interview diagrams. This level of quantitative precision was common among students’ instructional intervention diagrams.

There were also many examples of diagrams that students generated in class that suggest that the increased *coordination of quantities* in diagrams originated in the instructional intervention. In one example, Kendra drew a diagram of a quadratic distance-time relationship that clearly *showed the amounts of time and distances*
changing in relation to each other (see Figure 4.26a). She accomplished this by drawing “bumps” or arches between tick marks, together with labels that recorded the changes in quantities. In another example, Nicholas used arches and labels to clearly highlight the rates of change of time with respect to the change in distance over the diagram (see Figure 4.26b).

Thus, diagrams from the instructional intervention showed comparable quantitative precision and similar levels of coordination of varying quantities as seen in the post-interview diagrams. This link between the instructional intervention and the post-interview supports the claim that the quadratics instructional intervention had an influence on students’ diagrams about linear function contexts. In other words, it suggested that the change in the diagrams was a case of productive backward transfer.

**Classroom Evidence Connected to BTF 3**

In this section, evidence in support of the claim that BTF 3 was a case of productive backward transfer will be presented. In particular, an episode from Lesson 5 will be described. The event shows that the changes in how students described
division had origins in the ways that they talked about division in the instructional intervention.

Lesson 5 was designed to help students make sense of speed in preparation for an exploration of quadratic functions in a distance-time context. During a whole-class discussion, several students began to use division to explain why traveling 10 cm in 4 s was the same speed as 30 cm in 12 s. When the teacher asked them to explain what division meant, students provided a *procedurally-based numerical explanation*. For example, Jenn described dividing 30 cm by 4 s as reducing the fraction 30/4:

Because to get the bottom, the denominator of 30 over 12, you would divide the whole fraction by 12. So when you divide the bottom by 12

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**Figure 4.26.** (a) Kendra’s diagram highlighting amounts of changes in quantities; (b) Nicholas’ diagram highlighting rates of change.
it’s going to give you 1 second and when you divide the top by 12 it’s going to give you 2.5 centimeters over 1 second.

Kendra talked about division as something that combines numbers: “It like combines them, so 10 centimeters divided by 4 seconds would equal 2.5 centimeters per second.” Peter explained division of 10 by 4 as moving backward, presumably from 4 s to 1 second: “if she divides it by 4, you go four back and . . . that would mean in 1 second it gets 2.5 centimeters. Peter also said, “I think because dividing centimeters by the seconds will just remove it . . . give you a lower fraction.”

The teacher pressed students for a conceptually-based explanation but the students continued to provide explanations like those above. Therefore, the teacher introduced a new problem. The whole class discussion of this problem seemed to mark the beginning of students focusing on the meaning of division. The problem was about dividing 6 cookies by 3. The teacher described it as follows:

So let me take just for a second, a very simple problem like 6 cookies divided by 3 [writes $6 \text{ cookies} \div 3$ on the board]. So here we have a very simple division problem, everyone knows the answer. Can anybody think of a real-world situation that this could apply to?

George responded, “You have 6 cookies and 3 people want an even cookie.” The teacher wrote George’s question on the board. Peter said, “6 cookies and the jar holds 3 cookies, so how many jars you need.” The teacher changed Peter’s response slightly, writing on the board, $6 \text{ cookies and how many groups of 3}$. Note that the first real-world situation is a partitive interpretation of the division problem (i.e., sharing cookies equally among groups, where the number of groups is set), while the second situation is a quotative interpretation of the division problem (i.e., dividing the cookies into groups, where the size of the group is set).
Next, the teacher asked for students to come up to the board and draw “6 cookies and 3 people want an even cookie.” George came to the board and drew 6 circles to represent 6 cookies, 3 happy faces to represent 3 people and three lines connecting each happy face with 2 cookies (see Figure 4.27a). In other words, he interpreted the problem using a partitive model of division because the number of groups was set (i.e., 3 people). The teacher also asked for a student to come up to the board and draw the quotative interpretation of the problem (see Figure 4.27b). However, the quotative model did not come up again during the instructional intervention.

As stated above, this cookie problem marked the beginning of students focusing on the meaning of division, and more specifically on the partitive model of division. For example, the next day, the teacher showed the students a PowerPoint slide on which he had captured the diagram that a student had drawn the previous day. The diagram represented a fish swimming 6 ft in 2 s and swimming 18 ft in 6 s (see
The teacher had annotated the diagram with two large arrows highlighting 6 ft in 2 s and 18 ft in 6 seconds. The teacher asked students to use division or multiplication to show that the speeds were the same. Nicholas said, “It’s 6 groups of 3 feet per second because it’s six bumps and each of them and each bump represents 3 feet in 1 second.” Then the teacher asked Jenn if she “could show how 18 feet in 6 seconds is the same as 6 feet in 2 seconds, not 3 feet in 1 second.” Jenn said, “3 divided the 6 seconds into 3 groups . . . you divided the seconds by 3, getting 2 seconds in each and 6 feet in each.” In these excerpts, both students referred to division using a partitive model of division, where the number of groups is set.

Based on these classroom examples and others like it, I concluded that the change in students explanations of division from procedurally-based explanations in the pre-interview to conceptually-based *partitive explanations* in the post-interview had origins in the discussion about division during the quadratics instruction. In other words, these links between the instruction and the post-interview support the claim.

**Figure 4.28.** Teacher’s PowerPoint slide of fish swimming 6 ft in 2 s and 18 ft in 6 s.
that the changes in the explanations of division were a case of backward transfer.

At this point the reader may wonder if this finding is nothing more than the teacher telling the students about division, and the students demonstrating their learning in the post-interview. To address this concern, four reasons are offered for why this finding is a case of productive backward transfer. First, the contexts in which the discussions about division occurred in the intervention were different than the contexts in which the pre- and post-interview tasks were set. Recall that the context in which division was initially discussed in the instructional intervention did not explicitly involve either linear or quadratic functions (i.e., 6 cookies and divided by 3). Subsequent discussions about division occurred in the context of quadratic distance-time functions. In contrast, the interview tasks were set in linear function contexts, none of which were distance/time contexts.

Second, the primary instructional goal behind discussing division was to help students make sense of changing speeds. For example, during instruction students were shown the same quadratic distance-time data several times, each time with more data points so that the intervals of time grew smaller and smaller (e.g., students were shown the same data with 4 s intervals, 2 s intervals and 1 s intervals). Students used division to find the speeds between intervals and to compare how the speeds were changing. It was this instructional goal that guided the design of the learning activities in which students developed their understanding of division. Yet, when students were shown linear function data in the post-interview in which the rates of change were constant, they reasoned with division more productively than they had in the pre-interviews.
Third, students were asked different kinds of questions during the instructional intervention discussions about division than they were asked in the interviews. During instruction, they were asked to show with division that one speed was the same as another speed or that two speeds over two different intervals of time were different. In the interviews, they were asked to make sense of linear data presented in sentence form, tabular form, picture form or graphical form. Only when they used division, were they asked to explain why they used division.

Fourth, evidence showed that despite the fact that the class had a lengthy discussion about division using a simple problem about sharing cookies, the focus on the partitive model of division continued to develop significantly in the context of subsequent lessons on quadratic functions. This evidence will be presented in Chapter 6. Based on these four reasons, I interpret this finding as a case of productive backward transfer.

**Findings of Unproductive Backward Transfer**

Along with the findings of productive backward transfer, there were also isolated instances of unproductive effects. Specifically, there were three instances where students appeared to do worse on the linear function post-interview task than on corresponding pre-interview task. Two of these instances, which have already been mentioned above, were related to BTF 1 and 2. A third instance occurred when a student was explaining the rate of change of one of the water pumps.

The first instance of possible unproductive backward transfer is captured in Table 4.1. Notice that the code *confounds non-proportional and proportional*
relationship was used only for George in the pre-interview in the $y=mx+b, b\neq 0$ task, but for George, Armando, Brady, Jenn, and Kendra in the corresponding post-interview task. One might conclude that this indicates an unproductive backward transfer effect for Armando, Brady, Jenn and Kendra. However, my interpretation is that this change in reasoning only indicates an unproductive backward transfer effect for Armando.

The reasoning behind this interpretation is that the change in the other four students’ reasoning was a byproduct of reasoning in more sophisticated ways in the post-interview. In particular, to confound the non-proportional and proportional relationships in a $y=mx+b, b\neq 0$ context, students have to be reasoning proportionally with the changes in quantities. Otherwise there is nothing for them to confound. Since Brady, Jenn and Kendra did not reason proportionally with the changes in quantities in the pre-interview but did so in the post-interview, it indicates that they were reasoning in a more sophisticated manner but that there were aspects of this new way of reasoning that they had yet to sort out.

Armando, on the other hand, did reason proportionally with the changes in quantities on the pre-interview $y=mx+b, b\neq 0$ task and even coordinated the non-proportional and proportional relationships. Therefore, the instance in the post-interview in which he confounded these relationships was taken as a case of unproductive backward transfer. However, it did not seem to be a particularly strong effect because, as soon as the interviewer directed Armando’s attention to the non-zero $y$-intercept, Armando immediately changed his response and began to coordinate the non-proportional and proportional relationships.
The second instance of possible unproductive backward transfer is captured in Table 4.4. Notice that George, Armando and Brady labeled their diagrams with units in the pre-interview but not the post-interview. In George’s case this was the only change in his diagram, which suggested that, according to the coding scheme, his post-interview diagram was slightly less quantitatively precise than his pre-interview diagram. For the other two students, this unproductive change was offset by other changes that increased the quantitative precision. Nevertheless, the instructional intervention may have unproductively influenced these students to be more inconsistent in their labeling of their diagrams.

The third instance of possible unproductive backward transfer occurred when Nicholas was working on the water pump #2 task. Nicholas reported that the pump was “going equally fast . . . 2.5 gallons per minute every minute” (see Figure 4.29). The units that Nicholas used to report the rate of pumping puzzled the interviewer (I) who asked Nicholas (N) to explain:

I: And why do you need this [points at every minute]?
N: Because it’s part of the acceleration in which the pool is filling the, or the pump is filling the pool.
I: What does acceleration mean?
N: It’s . . . it’s the rate of change . . . the change of the rate of change in respect to time.
I: OK and is this pump accelerating?
N: It’s going at the same, it’s going at constant speed.
I: It’s going at constant speed? So what’s the it’s acceleration?
N: Mmm . . . 0.
I: 0? What’s your thinking behind that?
N: If it’s not speeding up at all, it’s just, there’s no acceleration.

In this excerpt, Nicholas contradicted himself because he reported the pumping rate as if it was acceleration (i.e., a ratio composed of a change in speed and the
Nicholas’ way of talking about acceleration in the post-interview was also observed in the instructional intervention. For example, during Lesson 15, the class was talking about a constant acceleration of a rocket of 8 mi per min every min. Nicholas provided the following explanation:

The ‘per minute’ in 8 miles per minute is part of the change in speed and then ‘every minute’ is the change in time, so it has to include both of them . . . [‘every minute’] it’s part of acceleration, it’s not just the change in speed, acceleration is change in time as well.

In this excerpt, Nicholas talked about the acceleration in a quadratic context, similarly to how he talked about the speed in a (post-interview) linear context. Consequently, it appeared that this way of talking about acceleration in a quadratic context unproductively influenced Nicholas’ reasoning on the post-interview linear task.

These three instances of backward transfer show that despite explicitly attempting to support productive backward transfer, several small unproductive influences also were produced. In keeping with the design-based approach that was used for this study, these findings of unproductive backward transfer could inform future revisions of the instructional intervention that could further promote productive
backward transfer.

**Discussion**

In this section, I discuss (a) the relationships between the nature of one’s prior knowledge and the occurrence of productive backward transfer and (b) the pedagogical implications from the backward transfer findings.

**Prior Knowledge and Backward Transfer**

In the introduction, a finding from linguistics research was presented that showed that average-level second language learners demonstrated greater backward transfer effects from learning a second language than high- and low-level second language learners (Chen, 2006). It was conjectured that this trend would also manifest itself in the dissertation study. The study findings support this conjecture to some degree, when comparing the mid- and high-level proportional reasoners’ groups. However, the conjecture is not well-supported when comparing mid- and low-level groups or comparing high- and low-level groups. Furthermore, for each of the three findings, the groups sorted themselves differently with respect to which group exhibited the most dramatic changes in reasoning.

For BTF 1 (i.e., attending to and reasoning proportionally with changes in quantities), the mid-level group did appear to exhibit the most substantial changes. The student in the low-level group also experienced substantial and important changes, but her level of proportional reasoning in the post-interview did not match that of the other two groups. The high-level group exhibited smaller changes. For BTF 2 (i.e., drawing more mathematized diagrams), the low-level reasoner exhibited the most dramatic change in the quantitative precision of her constructed diagrams,
with the mid-level group not far behind, and the high-level group, once again, making smaller increases in quantitative precision. With respect to the coordination of quantities on the diagrams, the mid- and high-level groups had the most substantial changes. Finally, for BTF 3 (i.e., providing partitive explanations of division), all students appeared to exhibit a similar change from pre- to post-interview, moving from procedurally-based to conceptually-based explanations of division in linear function contexts.

What can be learned from the differential ways in which students’ linear function reasoning changed from pre- to post-interview? One observation is that mid-level proportional reasoners can experience backward transfer effects that are not only more dramatic than the backward transfer effects that high-level proportional reasoners experience but that can result in the mid-level reasoners outperforming the high-level reasoners on some problems. This goes against the hypothesis that high-level proportional reasoners would not experience dramatic backward transfer effects because their understanding requires less refinement than mid-level proportional reasoners. This observation is based on the finding that the mid-level group outperformed the high-level group on the post-interview $y=mx+b, b\neq 0$ task. This could mean that perhaps some aspects of high-level proportional reasoners’ understandings are resistant to significant productive changes as new knowledge is constructed. For example, perhaps high-level reasoners like George and Armando had well-established procedures by which they reasoned linearly but that well-established procedural understanding is more resistant to backward transfer effects than conceptual understanding.
Furthermore, low-level proportional reasoners can experience dramatic backward transfer effects. This goes against the hypothesis that the low-level proportional reasoners would experience low levels of backward transfer because their understanding of linearity was too limited to be affected. This observation is based on the finding that the low-level proportional reasoner exhibited the most dramatic changes of the three groups in terms of increased levels of quantitative precision on diagrams. This could mean that perhaps some aspects of low-level proportional reasoners’ understandings are sensitive to influences from new knowledge being constructed. Each of these observations suggests future research opportunities.

**Implications for Instruction**

Within the mathematics education literature, three recommendations for how to approach teaching mathematics are to engage students in (a) quantitative reasoning, (b) covariational reasoning, and (c) sense-making. The overwhelming implication that emerges from the three findings presented above is that these three ways of approaching mathematics education have payoffs not just for the content at hand but also for deepening students understanding of prior foundational knowledge.

As stated earlier, researchers have argued that *quantitative reasoning* (i.e., being able to conceive of, reason with and manipulate quantities) is foundational for reasoning algebraically (Carraher, Schliemann, & Schwartz, 2008; Dougherty, 2008; Smith & Thompson, 2008; Thompson, 1994c). An implication from BTF 1 is that engaging students in quantitative reasoning in one context can also lead to students engaging in reasoning with the same quantities in other contexts, including contexts that involve their prior knowledge. In particular, the results from this study suggest
that engaging students in reasoning with changes in quantities in quadratic function contexts may lead to productive increases in reasoning with changes in quantities in linear contexts.

*Covariational reasoning* (i.e., coordinating the values of one variable and the corresponding values of another variable; Carlson et al., 2002) has been promoted as a critical component for developing a full understanding of functions (Carlson et al., 2002; Confrey & Smith, 1994, 1995; Lobato et al., 2003; Saldanha & Thompson, 1998). Therefore, covariational reasoning was strongly promoted in the instructional intervention of the dissertation study. In other words, activities were designed to direct students’ attention equally to changes in both quantities involved in a quadratic function. An implication from BTF 2 was that engaging students in activities that direct their attention to both quantities in a new context can also lead students to attend to both quantities in a context that they have had previous experience with. In particular, the results from this study suggest that engaging students in activities that promote covariational reasoning in quadratic function contexts appears to also promote higher levels of covariational reasoning in linear function contexts.

*Sense-making* has long been touted as a fundamental component of learning mathematics (Cobb, Yackel, & Wood, 1992; Schoenfeld, 1992). Within the context of this study, engaging students in sense-making about the mathematics of quadratic functions was a major goal. As an example, recall the discussion about division in the context of finding speeds for a quadratic distance-time function. The instructional goal was for students to make sense of division so that they would have a better understanding of the speeds that they found when dividing distance by time in
quadratic distance-time contexts. An implication from BTF 3 is that engaging students in sense-making in a new context can lead students to engage in sense-making in contexts with which they have had previous experience. In particular, the results from this study suggest that engaging students in sense-making about the meaning of division in quadratic function contexts appears to also promote thinking about the meaning of division in linear function contexts. These findings are a further endorsement for engaging students in quantitative reasoning, covariational reasoning and sense-making.

Finally, there were two content-specific implications from the results presented in this chapter. First, quadratic functions situations were shown to serve as good contexts with which to promote greater levels of attending to and reasoning with changes in quantities. A likely reason for this is that quadratic functions have the property that their rates of change vary over the domain of the function, instead of remaining fixed, as is the case with linear functions. It is this property that likely draws attention to changes in quantities.

Second, quadratic functions were shown to be useful contexts in which to promote the meaning of division. In particular, what seemed especially productive for developing an understanding of division was when students used division to explore different sized time intervals for the same quadratic data. For example, when the students explored the remote-control car task presented in 4 s intervals in the instructional intervention, they divided the distance traveled in each 4 s interval by the 4 s interval. Later, when they explored the same remote-control car for data presented in 2 s intervals, they again divided the distance traveled by the time interval. Because
the intervals in the second set of data were shorter than for the first set of data, the students obtained different answers (i.e., for a quadratic distance-time function, the average rate of change over a 4 s interval is different than the average rate of change over each 2 s interval that makes up the 4 s interval). Thus, students encountered opportunities to develop meaning for division in the process of trying to make sense of quadratic functions, which in turn influenced how they thought about division in linear function contexts as well.

In summary, the three findings of productive backward transfer suggest several future research directions with respect to the differential ways that student reasoning changed from pre- to post-interview. Furthermore, the findings add further support for why engaging students in sense-making, quantitative reasoning and covariational reasoning is fundamentally important for teaching and learning mathematics and suggest that quadratic functions can play an important role in the development of students’ understanding of linearity.
CHAPTER 5:
RESULTS ON FORWARD TRANSFER AND CONNECTIONS TO BACKWARD TRANSFER

In this chapter, I present findings from analysis of pre- and post-interview data that address the second research question. Recall that the second question was the following:

In what ways are backward transfer (influences by a person’s newly constructed knowledge on prior knowledge) and forward transfer (influences by a person’s newly constructed knowledge on how they come to understand novel situations) related?

In this chapter, I present evidence supporting the claim that there were two findings of forward transfer, as determined by an actor-oriented transfer (AOT) approach (Lobato et al., 2011a). First, all students coordinated or were anticipating to coordinate the changes in the dependent variable with the changes in the independent variable on the post-interview quadratic function transfer task (Forward Transfer Finding 1 [FTF 1]). Second, four of the students interpreted the changes in the dependent variable of the quadratic function transfer task as increasing at a constant rate (FTF 2). I also present evidence supporting the claim that there were two ways in which the forward transfer effects reported in this chapter and the backward transfer effects reported in Chapter 4 were connected. First FTF 1 and BTF 1 both involved all students attending more to changes in quantities and some students engaging more in coordinating two quantities in a functional relationship. Second, FTF 2 and the second part of BTF 1 both involved students exhibiting an increased (or developing) ability to
coordinate multiple levels of quantities.

The findings of forward transfer and the connections to backward transfer reported in this chapter are significant because they validate the assumption underlying this dissertation study that backward transfer is a part of transfer, as it is defined by the AOT approach. Moreover, these results imply that the conceptualization of transfer, according to the AOT approach, could be expanded to include backward transfer.

**Quadratic Function Transfer Task**

Evidence of forward transfer, as reported in this chapter, comes from students’ reasoning on the growing area task (see Figure 5.1). The same task was used in both pre- and post-interviews. The task involves a square displayed on a computer screen that can be enlarged by clicking and dragging one of the square’s corners. Students were not shown an actual computer but a picture of a computer with four overlapping squares displayed on the screen. Each square in the picture was highlighted with a different color (i.e., the $1 \times 1$ square was grey, the $3 \times 3$ square was green, the $5 \times 5$ square was orange and the $7 \times 7$ square was pink). Students were also given lengths and corresponding areas for the four sizes of the square, as well as a cardboard version of each square.

This task was divided into two parts. In the first part, the interviewer asked the students if the square was growing at a constant rate. For example, one interviewer asked Jenn the following: “So my question for you is, if I drag the square from being 1, to 3, to 5, to 7 centimeters in length, does the area grow at a constant rate or does it
go faster or slower or something else?” The other interviewer asked Kamilah, “Alright so you already said the area is getting bigger each time that you click on it. How is it growing, is it growing at a constant rate, is it growing in a way that the rate is increasing or decreasing?”

In the second part of the task, the interviewer showed students three observations from fictitious students about what they noticed in the growing area data. The first fictitious student noticed that “the green square is 8 cm² more than the grey square, the orange square is 16 cm² more than the green square and the pink square is 24 cm² more than the orange square.” The second fictitious student noticed that “areas increase by 8 cm², 16 cm² and 24 cm²,” and “at the same time, the lengths go up by 2 cm for each square.” The third fictitious student noticed the following:
If you subtract 8 cm\(^2\) growth of area per 2 cm increase in length from 16 cm\(^2\) growth of area per 2 cm increase in length, you get 8 and if you subtract 16 cm\(^2\) per 2 cm from 24 cm\(^2\) per 2 cm you also get 8.

The intention of showing these observations from fictitious students was to provide additional information that would allow the researcher to confirm or refute hypotheses generated during the clinical interview regarding the student’s thinking (Ginsburg, 1997). The first fictitious observation was designed to allow the interviewer to see if students could notice that the changes in area were increasing at a constant rate, once their attention was drawn to changes in area. The second observation was designed to see if students would be able to compose changes in area and length to form average rates of change, once their attention was directed to changes in length. Finally, the third observation was designed to see what meaning students attributed to the change in the change of area and whether or not coordinating this with the change in length was part of their conceptualization of the rate of change of a growth rate.

There were three reasons why this task was chosen as a quadratic function transfer task. First, the relationship between the length of a square and the area of a square is a quadratic function relationship (i.e., \(y = x^2\)). Second, this task involved a new real-world context that students had not explored in the instructional intervention, namely an area-length function. In contrast, all quadratic contexts in the instructional intervention were distance-time functions. Third, the growing area task involved finding rates for non-time-based quantities. It is likely that the middle school students in this study would not have had many experiences thinking about rates of change of one quantity with respect to another if time was not involved.

In designing this task, it was anticipated that the students might struggle with
the following features of the problem. First, middle-school students may find working with non-time-based functions unfamiliar, and thus, more difficult. Second, students could find it difficult to compose area with length, because the resulting unit (e.g., cm$^2$/cm) would likely seem unusual to a middle school student. Third, because the lengths were spaced 2 cm apart, it was anticipated that some students would struggle to find unit rates (“per 1 cm”) and unit rates of rates (“per 1 cm per 1 cm”). Therefore, the growing area task represented a significantly different and potentially more difficult task from the tasks the students had explored in the instructional intervention and in prior mathematics learning experiences.

Next, the findings of forward transfer are presented. The evidence is organized according to the ways that students sorted themselves with respect to each finding. To make the influence from the instructional intervention more apparent, the presentation of post-interview reasoning is followed by the evidence from classroom instead of from the pre-interview, which is presented last. This organization differs from that used in Chapter 4 because of the differing role that the pre-interview plays in a case for forward transfer as opposed to backward transfer. To make a case for forward transfer, from an AOT perspective, one’s primary goal is to demonstrate how the learner appears to see the transfer task (in the post-interview) as an instance of something he or she has already encountered (i.e., during the instructional intervention). This establishes the influence of prior experiences on learners’ activity in novel situations, or more broadly, the generalization of learning experiences. The role of the pre-interview is to ensure that the reasoning on the post-interview was not already in place prior to instruction. In contrast, to make a case for backward transfer,
one needs to show a deepening of the prior knowledge as an influence of new learning experiences. Thus, a primary goal is to establish changes in reasoning from pre- to post-interviews and then link this with reasoning from the instructional environment.

**FTF 1: Coordinating Changes in Area and Length**

I claim that all students coordinated or anticipated to coordinate changes in area quantities with changes in length quantities on part 1 of the post-interview quadratic function transfer task. There was some variation in students’ reasoning within this finding (see Figure 5.2). Specifically, students sorted themselves into three groups with respect to how they reasoned in the post-interview quadratic transfer task. Jenn, Brady and George formed Group 1. Peter and Nicholas formed Group 2. And Armando and Kendra formed Group 3.

Four codes were needed to capture the ways that the three groups coordinated quantities on the growing area task. Some codes were adapted a priori codes and others were inductive codes (à la a mixed methods approach; Miles & Huberman, 1994). Three codes, each associated with one of the three groups, were a priori codes from the mathematics literature adapted for the dissertation study. The fourth code, which was needed to capture an additional aspect of the reasoning from Group 3, emerged from the data using open coding (Strauss & Corbin, 1990; Strauss, 1987). Specifically, Group 1 was associated with *coordinating the changes in area and changes in length using a “covariational approach to functions”* (adapted a priori code; Confrey & Smith, 1995, p.67). Group 2 was associated with *anticipating to coordinate changes in area and length using a correspondence approach to functions*
Group 3 was associated with coordinating accumulated areas with accumulated lengths using a covariational approach (adapted a priori code; Confrey & Smith, 1995) and coordinating quantities and changes in quantities associated with the area and the length (code induced from the data).

The distinction between a correspondence approach and a covariational approach to functions originates with Confrey and Smith (1995), who explain that a correspondence approach emphasizes the directionality (or dependency) of the functional relationship between two quantities, while a covariational approach places more emphasis on the predictable ways that one quantity in a functional relationship changes in relation to the predictable ways that the other quantity changes. Confrey and Smith (1995) found that a covariational view provides greater access to thinking...
about rate of change and increments of change.

Two additional codes were needed to categorize student reasoning in the pre-interview. The codes were a priori codes from the mathematics literature, adapted for the dissertation study. The codes were reasoning univariately with accumulated areas and lengths and reasoning univariately with changes in area and changes in length, which were both adaptations of reasoning univariately (Harel, Behr, Lesh, & Post, 1994; Lobato & Ellis, 2010).

The six codes for coordinating quantities are represented in Figure 5.2. Evidence will show that students’ post-interview reasonings were closely related because, despite the differences, all students coordinated or anticipated to coordinate changes in area with changes in another quantity. Furthermore, pre-interview evidence will help to account for some of the post-interview differences in reasoning. Note that George was not linked to any of the pre-interview reasoning because he only attempted the quadratic transfer task in the post-interview.

As stated above, the interview data used for this finding came from the growing area quadratic function transfer task. However, only part 1 of this task was used for this first finding because this finding reports on what quantities students reasoned with of their own accord (before they were exposed to the reasoning of the fictitious students).

Coordinating Quantities with a Covariational Approach (Group 1)

The students in Group 1, Jenn, Brady and George, coordinated the changes in area with the changes in length using a covariational approach to functions.
illustrate this group’s reasoning, Jenn’s reasoning during the first part of the post-
interview quadratic function transfer task will be compared to her reasoning during the
instructional intervention.

**Post-interview.** Jenn coordinated changes in area with changes in length on
the growing area task. For example, when asked how the square was growing, she said:

Well like I said, it doesn’t look like it’s going at a constant rate . . . the length is going at a constant rate, but if you were to just look at the area it wouldn’t be . . . usually you find a pattern like this one, 2 centimeters for the length [points at 1 cm, 3 cm, 5 cm and 7 cm lengths].

In this excerpt, Jenn identified a pattern in the lengths (i.e., the changes in length were each 2 cm). After this excerpt, she found all the changes in area (i.e., 8 cm², 16 cm² and 24 cm²), recorded them in three fractions with 2 cm as each numerator (see Figure 5.3) and said, “the change of distance goes up 8 centimeters squared each time [points at space between 8 cm² and 16 cm² fractions] . . . it’d be growing 8 centimeters squared for every 2 centimeters.” In this statement, Jenn coordinated the 8, 16 and 24 cm² changes in area with the 2 cm changes in length by saying that the change in area increased by 8 cm² each time the length increased by 2 cm. However, absent from her coordination of quantities was an explicit mention of directionality in the relationship between change in area and length (i.e., she did not say something like, “when the length increases by 2 cm, then the change in area increase by 8 cm²”). These excerpts provide evidence that Jenn coordinated the changes in both quantities using a covariational approach to functions. Brady and George also provided instances of coordinating changes in area and length using a covariational approach.
Instructional intervention. During the instructional unit, Jenn began to coordinate the changes in distance with the changes in time in a manner consistent with how she reasoned in the post-interview. For example, during Lesson 14, Jenn described her diagram for the quadratic rocket task (see Figure 5.4) by explicitly coordinating changes in distance with changes in time:

Like right here it’s kind of hard to see but at 1 minute, it’s gonna be at 4 miles. So it’s gonna be 4 miles change of speed so the change of time is 1 minute, with a speed of 4 miles per minute. And right here . . . it’s 2 minutes, it be 16 miles [points at 16 mi and 2 s accumulated quantities], so the change of distance would be 12 miles and the change of time is 1 minute [points at 12 mi change in distance label and 1 min change in time labels] with a speed of 12 miles per minute [points 12 mi/min speed label]. And then in 3 minutes it be at 36 miles [points at accumulated labels], so the change of distance is 20 miles and the change of time of 1 minute [points at change in quantities labels] which is the same as a speed [points at 20 mi/min speed label], and then in 4 minutes it be at 64 miles [points at accumulated labels] with a change of distance of 28 miles and a change of time 1 minute [points at changes in quantities labels] and speed would be 28 miles per minute [points at speed labels].

In this excerpt, Jenn coordinated the changes in distance, change in time and change in speed. When asked by Kendra, “Is there a pattern here . . . for the miles per minute, the 12, 20, 28, 36, 44, 52?” Jenn said:

Yeah, adding 8 to all of them . . . a change in speed is 8 miles every 1 minute I think . . . 8 miles because when you add it, 8 miles to this you [points at 4 mi change in distance label], 8 miles to 4, you get 12 miles and so on.
In this excerpt, Jenn coordinated the 12, 20, 28, 36, 44, and 52 mi/min speeds with the 1 min changes in time to conclude that the changes in speed was “8 miles every 1 minute” (this is not entirely correct since the change in speed is 8 mi/min every 1 min not 8 mi every 1 min). However, in neither excerpt did Jenn explicitly mention directionality in the relationships between the related quantities. This is consistent with a covariational approach to functions. Brady and George also coordinated changes in distance with changes in time and speeds with changes in time during the instructional intervention.

Using a covariational approach to coordinate changes in distance with changes in time or speeds with changes in time during instruction is similar to using a covariational approach to coordinate changes in area with changes in length on the post-interview growing area transfer task. Therefore, the evidence suggests that there was a connection between the way that students in Group 1 reasoned during the instructional intervention and the way that they reasoned during the post-interview. My interpretation was that this was a case of forward transfer.
Anticipating to Coordinate Quantities with a Correspondence Approach (Group 2)

The students in Group 2, Nicholas and Peter, were *anticipating to coordinate the changes in area and changes in another quantity using a correspondence approach to functions* (see Figure 5.2). However, it appeared that these students were uncertain what quantity to coordinate with the changes in area because they weren’t sure what quantity was in a dependency relation with the changes in area. To illustrate this category, Nicholas’ reasoning during the first part of the post-interview quadratic function transfer task will be compared to his reasoning during the instructional intervention.

**Post-interview.** Nicholas indicated that he was anticipating to coordinate changes in area with another quantity but had difficulty doing so. For example, first Nicholas reasoned with the changes in length when asked about how the square was growing:

2 centimeters for each length . . . there are 2 here and 2 here [puts grey square on top of green square and the points to green square’s extra 2 cm length and width] and then again [puts green square on orange square and points to orange square’s extra 2 cm length and width ] and again [puts orange square on pink square and points to pink square].

Here, Nicholas focused on the constant 2 cm changes in length. Next, Nicholas switched to reasoning with the changes in area:

The area of 1 to the area of 9 changes 8 centimeters squared and then 9 to 25, I put 16 and 25 to 49, it was 24 . . . So it adds the 1,2,3,4,5,6,7,8, to this one [points at 8 extra cm² on the green square that stick out behind the grey square]. And it adds the 16 [points at orange square sticking out behind green square] and it adds 24 [points at the pink square sticking out behind orange square].
In this excerpt, he reasoned that the changes in area were 8, 16 and 24 cm$^2$ as the square grew from 1 to 9 to 25 to 49 cm$^2$. After identifying the changes in area, he made a statement in which he ran into trouble:

The area of 1 to the area of 9 changes 8 centimeters squared [points from grey to green square] and then 9 to 25, I put 16 [points green to orange square], and 25 to 49, it was 24 [points from orange to pink square]. So it’s going 8 centimeters squared every . . . there’s no change in time [emphasis added].

In this excerpt, Nicholas identified the pattern in the changes in area (i.e., “it’s going 8 centimeters squared”). However, he appeared to struggle with what quantity to coordinate with the changes in area. The teacher then asked, “What do you mean there’s no change in time? What are you thinking?” Nicholas responded, “I was gonna like I guess I could just say every click it goes up 8 centimeters squared.” The teacher asked, “The click. And what’s happening when you click?” to which Nicholas responded, “It adds an extra 8 every time. So it’s 8 centimeters squared per click every click.” In these statements, Nicholas appeared to coordinate the changes in area with clicking the computer mouse to make the square grow. His reasoning was that clicks add “an extra 8 every time.”

Thus, even though he was previously able to isolate changes in length, Nicholas did not coordinate those changes with the changes in area. Instead, he attempted to coordinate the changes in area with another quantity, clicks. His use of clicks appeared to be an effort to capture a dependency relation (i.e., the growing area depends on the clicks of the mouse). In other words, he appeared to focus on the area growing with each “click of the corner” of the square. This reflects a correspondence approach to functions.
Later in the interview, Nicholas changed from reporting the rate of growth as “8 centimeters squared per click every click” to “8 centimeters squared *every square* [emphasis added]” and then also briefly mentioned “8 centimeters squared *every 2 centimeters squared in length* [emphasis added].” These examples of Nicholas’ reasoning suggest that he was at the initial stages of coordinating changes in area with changes in length.

Peter also appeared to be anticipating to coordinate changes in area with another quantity. Instead of coordinating changes in area with changes in length, Peter said “8 cm² more . . . every time you drag it” [gestures dragging the corner of the computer-drawn square with the computer pointer to make the square grow]. The use of the statement “every time you drag it” suggests that Peter was also trying to describe a dependency relation between changes in area and another quantity. Therefore, he was also coded using a correspondence approach to functions.

**Instructional intervention.** Nicholas’ reasoning on the growing area task suggests that he associated the rate of growth of the square with the coordination of the changes in area and the changes in a second quantity (which shifts from “every time” to “every click” to “every square” and finally to “every 2 cm² in length”). This reasoning appeared to be connected to his reasoning about quadratic distance-time functions during instruction, in which he coordinated changes in distance with changes in time. Furthermore, during both the growing area task and the instructional intervention quadratic distance-time tasks, he appeared to be operating with a correspondence approach to functions.

For example, when Nicholas was working on the rocket task (Lesson 14; see
Figure 5.5a for the data), he responded to the following prompt: “Draw the first 7 minutes of rocket #1’s journey, using the pattern you notice when you drew the first 3 minutes. Label all distances and times, changes in distances and times, speeds and changes in speed.” Nicholas made a new table with times and distances for 7 min of the rocket’s journey. He also recorded the changes in distance and the changes in the changes in distance beside the distance column (see Figure 5.5b). When the teacher asked him what the numbers beside the distance column represented, he said:

I think that, I was looking at the change in time and change in distance, and for every minute it goes up by this is 12 [points the 12 mi change in distance recorded in table between the 4th and 16th miles] and then this is 20 [points at the 20 mi change in distance]. So I was just like adding 1 minute and an extra 8 [points at the 8 mi change in the change in distance recorded in table] to th--so it was like in that minute it was 12 and if you add 8, it would become 20 so it’d be this [points at the 20 mi change in distance] and then it’d be 28 from 36 to 64.

During this episode, Nicholas coordinated changes in distance with changes in time when he noted that the distance changed by 12 mi, from the 4th mi to the 16th mi, as the time changed by 1 min from the 1st min to the 2nd min (see Figure 5.5b). His statement “for every minute it goes up,” suggests that he thought the changes in area depended on the changes in time and may explain why he did not find it difficult to know what to coordinate with the changes in distance.

Later, Nicholas provided additional evidence that he used a correspondence approach to functions, when he said, “Like for every . . . like the time, every time it goes up a second or a minute, the speed changes accelerates.” In this excerpt, he appeared to emphasize that speed depended on time. Peter also began to coordinate changes in distance and time using a correspondence approach to functions during
Coordinating changes in distance with changes in time using a correspondence approach during instruction is similar to anticipating to coordinate changes in area with another quantity using a correspondence approach on the growing area task. Therefore, the evidence suggests that there was a connection between the way that students in Group 2 reasoned during the instructional intervention and the way that they reasoned during the post-interview. My interpretation was that this was a case of forward transfer.

**Coordinating Associated Quantities (Group 3)**

The students in Group 3, Armando and Kendra, coordinated accumulated areas with accumulated lengths and coordinated quantities and changes in quantities associated with the area and the length. To illustrate, Armando’s reasoning during the post-interview transfer task will be compared to his reasoning during the instructional intervention.

![Figure 5.5. (a) Lesson 14 rocket task data; (b) Nicholas’ extended table of distance (mi) and time (min) data for rocket task.](image)
Post-interview. Armando did not attempt to coordinate changes in quantities right away. Instead, he began by coordinating accumulations of area with accumulations of length:

It gets faster . . . because at 1 centimeter, the area is 1 centimeter squared [writes 1 cm 1 cm$^2$]. At 3 centimeters, it’s at 9 centimeters squared [writes 3 cm 9 cm$^2$]. But if you divide these two, 9 divided by 3, it’s 3, so it’ll be 1 centimeter 3 centimeters squared [uses calculator and writes 1 cm 3 cm$^2$].

In this excerpt, Armando appeared to try to reason proportionally, dividing the 3 cm accumulated length and the 9 cm$^2$ accumulated area for the second (green) square by 3, perhaps to compare it to the 1 cm accumulated length and 1 cm$^2$ accumulated area for the first (grey) square. Thus, he appeared to coordinate the accumulations of length and the accumulations of area.

Next, Armando attempted to coordinate the change in area and length of the left-most column of 1 cm$^2$ squares for the green, orange and pink squares. Specifically, he made a table and recorded the lengths and areas for the left-most column of the grey, green, orange and pink squares (see Figure 5.6). Then, he reported the growth of the square as “2 more centimeters squared, for every 1 centimeter.” When he said this, he pointed to the top 2 cm$^2$ of the left-most column of the orange square and at the bottom left 1 cm length of the first column of the orange square (see Figure 5.7). He then elaborated:

Every centimeter, it’s adding 2 centimeters squared. So, right here for every 1 centimeter [points at length of the column] it’s adding 2 more [points at top 2 cm$^2$ of orange square]. So off--to get this one [points at orange square] it adds two more and it adds two more [points at pink square].
In this excerpt, Armando appeared to coordinate quantities. However, the quantities he was reasoning with were not the changes in area and length of entire squares. Instead, he reasoned with the changes in area and the length for the left-most 1 cm$^2$ column of each square. It did not seem like Armando noticed the changes in area for the entire square. In fact, contrary to Groups 1 and 2, Armando never mentioned the 8, 16 and 24 cm$^2$ changes in area of his own accord. Interestingly, Armando did not seem to have trouble coordinating area quantities with length quantities, in the same way that Nicholas and Peter did, as evidenced by the statement, “every centimeter, it’s adding 2 centimeters squared.” Therefore, I inferred that Armando was using a covariational approach to functions when he was coordinating quantities. Kendra also fits this group because she mixed the independent and dependent variables together as she tried to coordinate area and length quantities.
Instructional intervention. During the instructional intervention, Armando and Kendra coordinated changes in distance and time for quadratic function data but not at the same consistent level as the students in Groups 1 and 2. For example, despite recording the correct changes in distance and changes in time for each of the 7 seconds in his Lesson 14 rocket task diagram, Armando divided the accumulated distances by the accumulated times to get the incorrect speeds for each 1 s interval (i.e., 4, 8, 12, 16, 20, 24 and 28 mi/min). He then concluded, “Yeah, and this is showing that it is adding to each one by 4.” In other words, on this task, Armando coordinated accumulated distances and times instead of coordinating the changes in distance and time.

Then, when explicitly reminded by the teacher to use changes in distance and time to find the speeds, he found the correct speeds for each 1 s interval (i.e., 4, 12, 20, 28, 36, 44, and 52 mi/min) and concluded, “[these are] the real speeds and this shows that it’s adding 8 in each one.” Thus, when his attention was directed to the changes
in distance and time, he appeared to coordinate changes in distance and time.

There were also instances in which he coordinated changes in distance and time without the teacher’s prompting. For example, during the Lesson 9 remote-control car task, he recorded the speeds as the changes in distance over the corresponding changes in time of his own accord (see Figure 5.8). Kendra likewise sometimes coordinated the changes in distance and time and sometimes coordinated the accumulated distances and times during the intervention.

Coordinating the accumulated distances with the accumulated times and the changes in distance with changes in time during the intervention is similar to coordinating accumulated areas with accumulated lengths and coordinating area-related quantities with length-related quantities during the growing area task because, in both cases, Group 3 seemed to be focusing on coordinating quantities while still working out what to coordinate. Therefore, the evidence suggests that there was a connection between the way that students in Group 3 reasoned during the instructional intervention and the way that they reasoned during the post-interview. My interpretation is that this was a case of forward transfer.

In summary, all seven students showed evidence of an influence from the quadratics instruction when they reasoned about the post-interview quadratic function transfer task. In other words, they all appeared to experience forward transfer. More specifically, all students coordinated changes in area and length to varying degrees. The pre-interview evidence I present next will show that the degree of success students had with coordinating quantities was related to the differential abilities that they came into the study with.
Pre-interview Reasoning with Quantities

During part 1 of the pre-interview growing area task, all students reasoned differently than they did during the same task on the post-interview. Jenn engaged in reasoning univariately with changes in area and changes in length, Peter, Brady and Nicholas engaged in coordinating accumulated area and accumulated length using a covariational approach to functions and Armando and Kendra engaged in reasoning univariately with accumulated areas and lengths and coordinating accumulated areas and accumulated lengths using a covariational approach to functions. In other words, Jenn came into the study with the greatest focus on changes in area and length, Peter, Brady and Nicholas came into the study with a lesser focus than Jenn on changes in area and length but a greater focus than Jenn on coordinating quantities, and Armando and Kendra came into the study with a lesser focus than Jenn on changes in area and length and a lesser focus than Peter, Brady and Nicholas on coordinating quantities.

An example from Jenn’s pre-interview, in which she reasoned with the changes in the length of the growing square only, was when she said the following:

Figure 5.8. Armando’s Lesson 9 diagram highlighting changes in distance, changes in time and speeds.
The length of the square is going by... the odds, so it’d be going up by 2. And then for the area... the area can only be followed, it doesn’t really have a pattern, just follows the length.

In this excerpt, Jenn did not reason with the changes in area, just the changes in length.

An example of when Jenn reasoned with the changes in area only was when she said the following:

It’s going at different rates, because this one’s adding 16 [writes $+16$ under the 9 cm$^2$ and 25 cm$^2$ accumulated area labels for the square; see Figure 5.9]... this one’s adding 24, right? [writes $+24$ under 25 cm$^2$ and 49 cm$^2$], so they’re going at, this one went by 16, then 24 it’s changing at different rates, but going higher.

In this excerpt, Jenn switched from reasoning with the 2 cm changes in length to reasoning with the changes in area. Jenn did not provide evidence of coordinating the changes in area and length. Therefore, I concluded that her pre-interview reasoning with quantities was different from her post-interview reasoning.

To illustrate the reasoning that Peter, Brady and Nicholas used during the pre-interview, consider the following statement by Peter about the growing area:

Let’s see, 1 to 9, 3 to no, well, 1 to 1, 3 to 9, 5 to 25 and 7 to 49 [writes 1:1, 3:9, 5:25, 7:49], so... it constantly gets more, it doesn’t go, like at the same pace, like it doesn’t go at the... it’s faster.

In this excerpt, Peter appeared to coordinate the accumulated lengths and areas of the squares without indicating a dependency relation between area and length. In another example, Brady made an x, y table with the entries (1, 1), (3, 9), (5, 25) and (7, 49). He also wrote down the formula $y=x^2$. Then, he said:

Because right here, this is the formula right? [points at formula] So like $y$ equals 1 squared, so that’d be 1 times 1 is 1 [points at (1, 1) in table], then 3 times 3 is 9 [points to (3, 9) in table] and that’s how it shows everything, like how is it growing in, being a steady square and all that.
In this excerpt, Brady appeared to coordinate the accumulated lengths and the accumulated areas. Note also that Brady did not explicitly refer to a dependency in the relationship between area and length (i.e., he used a covariational approach to functions). Nicholas provided similar reasoning. I concluded that Peter’s, Nicholas’ and Brady’s reasoning was different from Jenn’s reasoning, and that their coordinating of quantities was different from the coordinating of quantities they displayed in the post-interviews.

To illustrate the reasoning that Armando and Kendra used during the pre-interview, consider the following statement by Armando about the growing area:

It’s going fast . . . because if it was going constant, instead of 1 through 9, then 9 to 25, it’d be 9 through 18, and instead of 25 through . . . 49, it’d be from 18 to 36.

In this excerpt, Armando appeared to notice that the accumulated areas (i.e., 1 cm², 9 cm², 25 cm² and 49 cm²) were not all multiples of 9 and concluded that the growth of the area was not constant. Thus, he reasoned univariately with accumulated areas. Kendra reasoned in a similar manner.
Later in the interview, Armando appeared to coordinate accumulated areas with accumulated lengths:

"Wait . . . 2 centimeters more . . . a constant speed . . . because if you divide 3 by . . . 9 it’s 3, 5 divided by [25] is 5, 7 divided by 49 is 7 . . . and 1 plus 2 is 3, plus 2 is 5, plus 2 is 7."

In this excerpt, Armando appeared to divide each accumulated area by the corresponding length of the square (i.e., 9 cm$^2$÷3 cm, 25 cm$^2$÷5 cm, and 49 cm$^2$÷7 cm) and subsequently noticed a pattern in the quotients. Although it is not clear what Armando thought the quotients represented, this example shows him coordinating the accumulated areas and lengths. Furthermore, Armando did not explicitly indicate a dependency in the relationship between area and length and so I inferred that he used a covariational approach to functions. Kendra provided instances of similar reasoning. Because Armando and Kendra did not mention changes in quantities related to the area and length and because they reasoned univariately with accumulated quantities and also coordinated accumulations of area and length, I concluded that (a) Armando’s and Kendra’s reasoning was different from Jenn’s, Peter’s, Brady’s and Nicholas’ reasoning, and (b) their reasoning with quantities was different from the reasoning they displayed in the post-interviews.

In summary, evidence showed that all three groups were reasoning with quantities differently during the post-interview than the pre-interview on the same quadratic function transfer task. This further supports the claim that there was a forward transfer effect involving students coordinating quantities on the post-interview quadratic function transfer task. Also, the differential ways that students coordinated quantities in the pre-interviews aligned with the differences in the forward transfer
effects that students exhibited in the post-interview and thus help account for those
differences.

**FTF 2: Interpreting the Constant Changes in Area**

I provide evidence for a forward transfer effect involving how students interpreted the changes in the area during part 2 of the growing area task in the post-interview (FTF 2). As with FTF 1, there was some variation in students’ interpretations within this finding, which is summarized in Table 5.1. As can be seen in the table, students sorted themselves into two groups according to their post-interview interpretation of the changes in area. Jenn, Nicholas, Peter and George formed Group 1. Brady and Armando formed Group 2. George’s pre-interview interpretation is not identified in Table 5.1, because he did not attempt the quadratic transfer task in the pre-interview. Kendra’s post-interview interpretation is not identified in the table, because time ran out before she could complete the task.

The Group 1 interpretation was that the *changes in area were increasing at a constant rate*. The Group 2 interpretation was that the *changes in area were constant*. A third interpretation was also observed, but only in the pre-interview, namely that the *changes in area were increasing*.

In the *changes in area are increasing* code, students note that a given set of changes in area are increasing but they either do not try to describe the way the changes in area are increasing, or they identify a pattern that does not involve changes in changes in area. Students in the *changes in area are constant* code calculate the changes in the changes in area for a given set of changes in area, and note that these
differences are constant. However, they incorrectly interpret this as meaning that the area grows at a constant rate. Finally, the students in the changes in area are increasing at a constant rate code, similarly calculate the changes in changes in area, but they interpret this as indicating that the changes in area are increasing at a constant rate.

The data used for this finding was part 2 of the growing area quadratic function transfer task. Recall that this part of the task involved showing students the observations of fictitious students and asking them to reason about those observations. With these fictitious observations, students’ attention was directed to the 8, 16 and 24 cm² changes in area and the corresponding 2 cm changes in length as the square grew from a 1×1 cm square to a 7×7 cm square in 2 cm increments. Students’ interpretations of the changes in area were coded using the three codes described above (i.e., as increasing, as constant or as increasing at a constant rate). Next,

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<th>Changes are increasing</th>
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<tbody>
<tr>
<td>Jenn (pre)</td>
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<td>Group 1</td>
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<td>Nicholas (pre)</td>
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<td>Brady (pre)</td>
<td>Brady (post)</td>
<td>Group 2</td>
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<td>Armando (pre)</td>
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<td>Kendra (pre)</td>
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evidence from the post-interviews will be presented by group to illustrate the two interpretations.

**Changes in Area Increase at a Constant Rate**

(Any Group 1)

The students in Group 1, Jenn, Nicholas, Peter and George, interpreted the changes in area as increasing at a constant rate (see Table 5.1). To illustrate this group, evidence from George’s and Peter’s reasoning during the second part of the post-interview quadratic function transfer task will be compared to their reasoning during the instructional intervention.

**Interpreting changes in area during the post-interview.** During the second part of the post-interview transfer task, George was shown the first fictitious student’s observation about the changes in area. The observation was that the area of the green square was 8 cm² more than the area of the grey square, the area of the orange square was 16 cm² more than the area of the green square, and the area of the pink square was 24 cm² more than the area of the orange square. His response to the observation was:

Every square it . . . the space left, there is 8 more. So it first starts at 8
[sweeps fingers over the part of the green square that sticks out from under the grey square] . . . then 16 [sweeps fingers over the part of the orange square that sticks out from under the green square] . . . then 24 left [sweeps fingers over the part of the pink square that sticks out from under the orange square].

Based on his gestures over the part of each square that stuck out from under the previous square and based on his use of the phrase “the space left,” George appeared to interpret the 8, 16 and 24 cm² (i.e., the changes in area) as the amount of area that each successive square was bigger than the previous square. Furthermore, he appeared to interpret the 8 cm² changes from 8 to 16 cm² and from 16 to 24 cm² as the
amount that each “space left” was more than the previous “space left” (i.e., he said “every square the space left, there’s 8 more). Therefore, in part 2, George was coded as making a changes in area are increasing at a constant rate interpretation.

In the following response to the same fictitious observation, Peter made a similar interpretation:

It’s growing by 8 centimeters squared, each time . . . well, more, like more each time, like this [puts grey square on green square], it grew 8 more [outlines an imaginary square on the desk] . . . because this [points to bottom-left corner of green square] was one of this [points to grey square] and this is 8 [sweeps finger over remaining squares on green square]. And then this [places green square over orange square] it grew 16 more, so it added 8 more, and that’s 16 [sweeps fingers over squares in orange square that are not covered by green square]. And then this [places orange square over the pink square], over here . . . it grew 24 [sweeps finger over squares in pink square that are not covered by orange square] because it added 8 more to 16.

In this excerpt, Peter made a similar gesture as George had made over the part of each square that stuck out from under the previous square. He also similarly reported that the parts of green, orange and pink squares that stuck out from under the previous square were 8, 16 and 24 cm² respectively (although he did not use units). Finally, like George, he concluded that the amount that each square grew “added 8 more” to the amount the previous square grew by. Therefore, Peter was also coded as making a changes in area are increasing at a constant rate interpretation. Jenn and Nicholas made similar interpretations about the changes in area.

Interpreting changes in distance in the instructional intervention. There were connections between Group 1’s interpretations of the changes in area on the post-interview quadratic function transfer task and what occurred during the instruction. Evidence of these connections was found in student reasoning during Lessons 14 and
More specifically, during the rocket task and the subsequent class discussion, several students began to refer to the changes in distance for the rocket’s quadratic distance-time function as increasing at a constant rate (see figure 5.5a for the data).

For example, consider George’s explanation of his rocket task diagram that he made to the class during Lesson 15 (see Figure 5.10). Recall that the task prompt was, “Draw the first 7 minutes of rocket #1’s journey, using the pattern you noticed when you drew the first 3 minutes. Label all distances and times, changes in distances and times, speeds and changes in speed.” George gave the following explanation: “So it’s like 20 over 1 and 28 over 1, and it add, if you add 8, then it goes up by 8 . . . so from 12 mile in 1 minute to 20 miles in 1 minute.” In this excerpt, George gave three speeds from his diagram, 12, 20 and 28 mi/min, and then explained that from one speed to the next, “it goes up by 8.”

Figure 5.10. George’s diagram for Lesson 14 rocket activity.

Peter also provided an instance of interpreting changes in speed during the instructional intervention that seemed connected to the interpretation he made about changes in area during the post-interview. In Lesson 16, when students were considering a speed-time table to be 24 mi/min after 1 min, 48 mi/min after 2 min, 72 mi/min after 3 min and 96 mi/min after 4 min (see Figure 5.11a for data set), Peter said that the acceleration of the rocket was 24 miles per minute every minute. Then he
explained his thinking:

It adds up every 24, by 24 each minute . . . so in the first minute it’s 24, in the second minute it adds up by 24 more so that’s 48, and the third it adds up by 24 so that’s 72 and for the fourth it adds up by 24 to get 96 miles per minute.

In this excerpt, Peter appeared to be speaking about the changes in speed, and not the speed, when he said “adding up by 24 more” because he called 24 mi/min/min the acceleration. This was similar to the statement he made about “add[ing] 8 more” to each of the changes in area on the post-interview quadratic function transfer task.

Jenn and Nicholas made similar statements during Lesson 14 and 15 of the instructional intervention.

The evidence presented above suggests that there was a connection between the reasoning that the students in Group 1 exhibited in the instructional intervention and the interpretation that they made during the post-interview quadratic function transfer task. Specifically, this group interpreted the speeds during Lesson 15 as changing at a constant rate which likely means that they also interpret the changes in distance as changing at a constant rate. This then is similar to interpreting the changes in area for the growing area task as increasing at a constant rate. Thus, this evidence
provides support for the claim that these students’ post-interview reasoning was a case of forward transfer.

**Changes in Area are Constant (Group 2)**

The students in Group 2, Armando and Brady, interpreted the changes in area in the post-interview quadratic function transfer task as being constant. As stated above, this interpretation appeared to be intermediate between the other two interpretations that were exhibited by the students in this study. To illustrate that this interpretation was connected to the instructional intervention, evidence from Armando’s reasoning during the second part of the transfer task will be compared to his reasoning during the instructional intervention.

**Interpreting changes in area during the post-interview.** The interviewer told Armando about the fictitious student who said that the area of the green square was 8 cm² more than the area of the grey square, the area of the orange square was 16 cm² more than the area of the green square, and the area of the pink square was 24 cm² more than the area of the orange square. Armando responded:

> Yeah, they’re going by 8 so for everyo--one it’s adding 8, for every other square it’s getting 8 bigger, so they’re multiples of 8 so 8 times 2 is 16 and 8 times 3 is 24. So they’re growing at the same rate . . . I think it’s going at a constant rate . . . 8 centimeters for every centimeter squared.

In this excerpt, Armando incorrectly interpreted the constant 8 cm² increase in the changes in area as a constant rate. Brady also incorrectly maintained that the rate was constant. I interpreted this as evidence that Armando and Brady interpreted the changes in area as constant.
Interpreting changes in distance in the instructional intervention. As with Group 1, there were connections between Group 2’s interpretations of the changes in area on the post-interview quadratic function transfer task and what occurred during the instructional intervention. The evidence from student reasoning during Lessons 15 suggests that these students sometimes interpreted changes in speed as a constant speed and other times as a constant increase in speed. For example during Lesson 15, students were responding to the following prompt:

When Maria looked at a movie of Clown walking according to a quadratic distance-time function, she said: ‘the speed changes 10 cm/second when the time changes by 5 seconds.’ Explain what Maria means by this and what this tells you about the acceleration. You could make up some speeds to help you explain.

Armando responded to this prompt with the following written statement:

In the first second, Maria saw that Clown stopped at 10 cm. In the next second, it stopped at 20 cm. Then it kept going through 30 cm, 40 cm, and 50 cm for the end in 25 seconds equaling a speed of 2 cm/s (5 s=10 cm, 10 s=20 cm, 15 s=30 cm, 20 s=40 cm, 25 s=50 cm).

In this written statement, Armando incorrectly interpreted the 10 cm/s change in speed as a constant speed. This is similar to interpreting the changes in area as constant.

However, at other times in the instructional intervention, Armando appeared to interpret the constant changes in speed as constantly increasing. For example, at another point in Lesson 15, students were shown a table of data for a rocket that was accelerating at a rate of 32 mi/min/min (see Figure 5.11). The teacher asked, “What do you think that table is showing?” Armando said, “That for every 4 minutes, the speed is getting bigger by 32.” In this instance, Armando correctly interpreted the speed as increasing by a constant 32 mi/min every 4 min. Similarly, Brady provided
instances of incorrectly interpreting constant changes in speed as constant speed and then other instances of correctly interpreting constant changes in speed as speed increasing at a constant rate.

This evidence suggests that there was a connection between the reasoning that Group 2 students exhibited in the instructional intervention and the interpretations they made during the post-interview quadratic function transfer task. In particular, incorrectly interpreting constant changes in speed in the instructional intervention is similar to incorrectly interpreting constant changes in the growth rate of area as indicating constant change in area on the quadratic function transfer task. Thus, this evidence supports the claim that Group 2’s post-interview interpretations were the result of forward transfer. Next, evidence from the pre-interview quadratic transfer task is presented which further supports this claim.

**Pre-interview Reasoning about Changes in Area**

During part 2 of the pre-interview quadratic function transfer task, all students’ interpretations about the changes in area—except Kendra’s and George’s—changed from pre- to post-interview (see Table 5.1). Jenn and Nicholas changed from a changes are constant to a changes are increasing at a constant rate interpretation, Peter changed from a changes are increasing to a changes are increasing at a constant rate interpretation and Brady and Armando changed from a changes are increasing to a changes are constant interpretation.

In the pre-interview, Jenn, Kendra and Nicholas interpreted the constant changes in changes in area as indicating that the area grew at a constant rate. For example, when Jenn was told that the changes in the area between successive squares
were 8 cm$^2$, 16 cm$^2$ and 24 cm$^2$, she said, “it’s showing that it’s going up by 8 centimeters . . . the area would be growing at a rate of 8 centimeters.” The interviewer tried to get clarification by asking, “Are you saying the rate is 8 or are you saying the rate is increasing by 8 or are those the same thing?” Jenn responded with “the same thing.” Here, Jenn appeared to be saying that the area was growing at a constant rate. Or perhaps she was unable to distinguish the difference between a constant rate and a constant increase in rate. I coded this interpretation as changes are constant.

Nicholas exhibited similar reasoning. Thus, Jenn’s and Nicholas’ thinking about the changes in area appears to be different than their thinking during the post-interview.

Kendra’s pre-interview reasoning aligned with Jenn’s and Nicholas’ because when she was told about the 8 cm$^2$, 16 cm$^2$ and the 24 cm$^2$ changes in area, she said, “Oh, yeah, because it’s going at a constant [emphasis added] well like speed I guess also because 8 time 2 is 16 and 8 times 3 is 24.” In other words, Kendra interpreted the changes in area pattern as comparable to a constant speed.

Peter, Brady and Armando interpreted the changes in area, in the pre-interview quadratic function transfer task, as increasing. For example, Peter correctly noticed that the area of the orange square (25 cm$^2$) was 8 cm$^2$ times 2 more than the area of the green square (9 cm$^2$). He also concluded that the area of the pink square (49 cm$^2$) was 8 cm$^2$ times 2 times 2 more than the area of the orange square (25 cm$^2$), which is incorrect. In other words, he did not notice that the changes in area were increasing by 8 but thought they were increasing by factors of 2. Thus, I coded this as changes are increasing. This contrasts with his reasoning on the post-interview, where he not only noted that the changes in the changes in the area were constant but he was able to
correctly interpret that pattern as indicating that the changes in area were increasing by a constant rate.

In another pre-interview example, when Brady was told that a fictitious student had noticed that “the green square is 8 cm² more than the grey square, the orange square is 16 cm² more than the green square and the pink square is 24 cm² more than the orange square,” he said:

She’s trying to tell me that this one is bigger [points at green square] by how much like, you subtract 9 by 1 and it’s 8. And like this is 25 [points at orange square], you subtract it by 9 and that’s 16. And 24 [points at pink square] minus 25 [points at orange square], I mean 49 minus 25 is 24. Like he’s trying to tell me like by how much it’s bigger.

In this excerpt, Brady explained the increasing changes in area as “by how much it’s bigger.” In other words, he appeared to conceive of the changes in area as getting bigger. However, he did not provide evidence that he noticed that the change in area was increasing at a constant rate. Armando exhibited similar reasoning. Thus, Peter’s, Brady’s and Armando’s interpretations of the changes in area appear to be different than during the post-interview. This pre-interview evidence further supports the claim that these findings are cases of forward transfer.

**Discussion**

Recall from Chapter 2, that the reason for identifying forward transfer effects in the dissertation study was to address the second research question (i.e., in what ways are backward transfer and forward transfer related?). In this discussion section, the results from comparing the forward transfer findings presented in this chapter and the backward transfer findings presented in Chapter 4 will be presented. In all, two
relationships between forward and backward transfer were identified. After the relationships are presented, an important difference between forward and backward transfer will also be briefly discussed.

**Relationships between Forward and Backward Transfer**

There appear to be at least two relationships between the findings of forward transfer and the findings of backward transfer (see Figure 5.12 for a summary of the relationships). First, FTF 1 (i.e., *coordinating or anticipating to coordinate the changes in the dependent variable with the changes in the independent variable of a quadratic function*) and the first part of BTF 1 (i.e., *iterating and partitioning changes in quantities composed units*) are related because they both involved (a) all students attending more to *changes in quantities*, and (b) some students being more capable of coordinating two quantities involved in a functional relationship. Second, FTF 2 (i.e., *interpreting the changes in the dependent variable of a quadratic function as increasing, constant or as increasing at a constant rate*) and the second part of BTF 1 (i.e., coordinating non-proportional and proportional relationships in \( y=mx+b, \ b\neq0 \) contexts) are related because they both involved most students exhibiting an increased ability to coordinate multiple levels of quantities. BTF 3 and BTF 4 (i.e., the meaning of division and the mathematization of diagrams) will not be discussed in relation to the forward transfer findings because students were not asked about division nor were they asked to draw diagrams during the quadratics function transfer task.

**Forward and backward transfer involves attending to changes in quantities and coordinating quantities in a functional relationship.** I will show
that FTF 1 and the first part of BTF 1 involved students attending more to changes in quantities. FTF 1 defined three groups of students who attended more to changes in quantities: (a) students who went from reasoning univariately with changes in area and length to coordinating changes in area with changes in length; (b) students who went from coordinating accumulated areas with accumulated lengths to anticipating to coordinate changes in area with changes in another quantity, and (c) students who went from reasoning univariately and coordinating accumulated quantities to coordinating quantities and changes in quantities associated with area and length. Thus, all three groups of students increased their level of attention to changes in quantities.

The first part of BTF defined three groups of proportional reasoners (i.e., the high-, mid- and low-level proportional reasoners). Evidence presented in Chapter 4 showed that each group attended to and/or reasoned more with changes in quantities on linear function tasks during the post-interview compared to the pre-interview.
Therefore, FTF 1 and BTF 1 are related because both findings involved students attending more to changes in quantities.

Furthermore, FTF 1 and the first part of BTF 1 are related for two students because both the findings involved these students engaging more in coordinating two quantities involved in a functional relationship. As described earlier in this chapter when presenting FTF 1, Jenn, Armando and Kendra engaged in univariate reasoning in the pre-interview and in coordinating quantities in the post-interview. Evidence provided in Chapter 4 showed that Jenn, Brady and Kendra went from reasoning univariately on linear function tasks in the pre-interview to engaging more in coordinating quantities involved in linear functional relationships in the post-interview. Therefore, for Jenn and Kendra, FTF 1 and BTF 1 appeared to be related because for both students, both findings involved engaging more in coordinating quantities.

**Forward and backward transfer involves coordinating multiple levels of quantities.** I will show that FTF 2 and the second part of BTF 1 were related for five students because they both involved students exhibiting an increased (or developing) ability to coordinate multiple levels of quantities. I use *coordinating multiple levels of quantities* to describe reasoning across levels of hierarchically-organized quantities, where each quantity is derived from the previous quantity, starting with a given quantity (Hackenberg, 2010; Steffe, 1994). In particular, I call the given data for a variable, a first-order quantity, the changes in that variable, a second-order quantity and the changes in the changes in that quantity, a third-order quantity. For example, in the post-interview linear function business-cost task, I took the business cost (for
given numbers of employees) as a first-order quantity and the changes in business cost (for given changes in the number of employees) as a second-order quantity. The changes in the changes in the business cost (for given changes in the number of employees) would then be a third-order quantity. In order to make the relationship between FTF 2 and the second part of BTF 1 salient, I will next reinterpret the codes for the two findings in terms of coordinating multiple levels of quantities.

For FTF 2, students’ reasoning was coded according to their interpretations about the changes in area for the growing square. Students interpretations were coded as either (a) the changes in area are increasing, (b) the changes in area are constant, or (c) the changes in area are increasing at a constant rate. These interpretations could be reinterpreted in terms of coordinating multiple levels of quantities. For example, the changes are increasing interpretation could be thought of as coordinating one’s thinking about the given areas for the growing square (a first-order quantity) and one’s thinking about the changes in the area of the square (a second-order quantity), without considering the changes in the changes in the area of the square (a third-order quantity). In other words, some students noticed that the second-order quantities were increasing (i.e., the changes in area were increasing) without noticing that the third-order quantity was constant (i.e., that the changes in the changes in area were constant). Therefore, an alternate code for changes are increasing could be coordinating the first- and second-order quantities only.

Similarly, the changes are constant interpretation could also be reinterpreted. Specifically, this interpretation could be thought of as mistakenly taking the constant changes in the changes in the area (a third-order quantity) as the constant changes in
the area (a second-order quantity). Therefore, an alternate code for changes are constant could be confounding the second- and third order quantities.

The changes are increasing at a constant rate interpretation could also be reinterpreted in terms of coordinating multiple levels of quantities. Specifically, this interpretation could be thought of as coordinating one’s thinking about the area (a first-order quantity), the changes in the area (a second-order quantity) and the changes in the changes in the area (a third-order quantity). Therefore, an alternate code for changes are increasing at a constant rate could be coordinating the first-, second- and third-order quantities.

For the second part of BTF 1, students’ reasoning was coded according to the ways that they reasoned about the non-proportional and proportional relationships that are inherent in $y=mx+b$, $b \neq 0$ contexts. Students’ reasoning was coded as either coordinating or confounding the non-proportional and proportional relationships (or both). However, these ways of reasoning could also have been reinterpreted as coordinating or confounding multiple levels of quantities. To illustrate this reinterpretation of the codes, consider Nicholas’ reasoning during the post-interview business-cost task (see Figure 4.5). Nicholas reasoned with 19 employees and $12875$, which were part of the first-order (non-proportional) relationship between the number of employees and the cost of running the business. He also reasoned with 5 employees and $3125$, which are part of the second-order (proportional) relationship between the change in employees and the change in cost of running the business. The result of this reasoning led Nicholas to determine another pair of quantities that were part of the first-order (non-proportional) relationship (i.e., 14 employees and $9750$).
Thus, Nicholas coordinated first- and second-order quantities in order to find another pair of first-order quantities.

Using this reinterpretation of the codes to compare students with respect to each finding, I concluded that five students exhibited an increased ability to coordinate multiple levels of quantities across these two findings. Specifically, for FTF 2, Peter, Jenn and Nicholas came to see the changes in area as increasing at a constant rate. In other words, they coordinated the first-, second- and third-order quantities. For the second part of BTF 1, Peter, Jenn and Nicholas were better able to coordinate levels of quantities because in the post-interview, they coordinated non-proportional first-order quantities and proportional second-order quantities. Therefore, for Peter, Nicholas and Jenn, FTF 2 and the second part of BTF 1 both reflect their increased ability to coordinate multiple levels of quantities.

Additionally, Brady, Armando and Kendra confounded the second- and third-order quantities on the post-interview growing area task (FTF 1) and confounded first- and second-order quantities on the post-interview $y=mx+b, b\neq 0$ task (BTF 1). The reader may conclude that this regularity across findings suggests unproductive instead of productive transfer. However, Brady and Kendra went from not coordinating to confounding first- and second-order quantities on the linear tasks and from coordinating first- and second-order quantities to confounding second- and third-order quantities on the growing area task. Therefore, these instances of confounding quantities reflected initial attempts to coordinate quantities in new ways and I interpreted them as evidence of a developing ability to coordinate quantities. However, the instance in which Armando went from coordinating to confounding
levels of quantities in the $y=mx+b$, $b \neq 0$ context appears to be a case of unproductive transfer (see discussion in Chapter 4). In summary, for Peter, Nicholas, Jenn, Brady and Kendra, the FTF 1 and the second part of BTF 1 findings appear to be related because both findings reflect an increased (or developing) ability to coordinate multiple levels of quantities. George did not fit the regularities observed in the other students because he coordinated first-, second- and third-order quantities on the growing area task, like Jenn, Nicholas and Peter, but confounded first- and second-order quantities on the post-interview $y=mx+b$ tasks, like Brady, Armando and Kendra.

In summary, there appeared to be a relationship between the forward and backward transfer effects for all students that involved students directing greater attention to changes in quantities. There also appeared to be a relationship for some students between forward and backward transfer that involved students engaging more in coordinating two quantities in a functional relationship. Finally, there appeared to be a relationship between forward and backward transfer for most students that involved students exhibiting an increased (or developing) ability to coordinate multiple levels of quantities.

A Difference between Forward and Backward Transfer

The identified relationships between forward and backward transfer support the claim that the backward transfer effects were, in fact, transfer effects as defined by the AOT approach and not some other phenomenon. However, there is also an important difference between the forward and backward transfer effects that is
reflected in the differential ways that the Chapter 4 and Chapter 5 results were
presented (i.e., juxtaposing the pre- and post-interviews for Chapter 4 and juxtaposing
the intervention and post-interviews in Chapter 5). As explained earlier, to establish
that backward transfer was produced, it was most important to compare how students
were thinking during the post-interview with how they were thinking during the pre-
interview to see how students thinking had changed. The main purpose of considering
the instructional intervention was to show that there was a link between the instruction
and the changed thinking, so that the possibility that the changed thinking had been the
result of spontaneous learning during the post-interview could be ruled out.

In contrast, to establish that forward transfer had been produced, it was most
important to compare how students were thinking during instruction with how they
were thinking during the post-interview transfer tasks to determine how the instruction
had influenced their thinking. The main purpose of considering the pre-interviews
was to confirm that similarities in how students were thinking during instruction and
during the transfer tasks were not present prior to the instructional intervention. From
a research standpoint, this difference could have implications for where researchers
direct more of their attention, depending on what they are most interested in
examining.

In summary, there were relationships between forward and backward transfer
that suggested that both are part of the same phenomenon. However, there also at
least one important difference between these two kinds of transfer that has
methodological implications for research. This suggests that backward transfer may
also have other unique properties that distinguish it from forward transfer.
CHAPTER 6:
RESULTS ABOUT NOTICING AS A BACKWARD TRANSFER PROCESS

In this chapter, I present findings from analysis of the classroom data collected during the instructional intervention. This analysis addresses the third dissertation research question, stated as follows.

Question 3: What are the transfer processes by which classroom instruction leads to productive backward transfer?

As explained in Chapter 1, a previous study found a connection between what students noticed mathematically in the classroom and the kinds of forward transfer effects that they later exhibited on novel transfer tasks (Lobato et al., 2011a). More specifically, Lobato et al. showed a conceptual connection between the particular mathematical features that students noticed during instruction—called the centers of focus (CoF)—and observed changes in student reasoning on novel tasks from pre- to post-interview. Similarly, one of the goals of the dissertation study was to identify and examine shifts in centers of focus. However, in contrast to the Lobato et al. (2011a) study, rather than establishing conceptual connections between shifts in centers of focus and forward transfer, the analysis for this study examined whether or not there were connections between shifts in centers of focus and backward transfer.

Recall that centers of focus are defined, in the context of mathematics education, as the “mathematical features, regularities, or conceptual objects” (Lobato et al., 2011a, p. 9) that individuals notice within a given domain of scrutiny. The
centers of focus comprise the first dimension of Lobato et al.’s (2011a) focusing framework. It should be noted that, in general, the word noticing will be used to represent what individuals attend to, and the word focusing will be used to represent how the teacher and the social situation direct individuals’ attention. However, the term center of focus, which refers to individuals, is used because of the obvious difficulty with ending the phrase centers of noticing with a verb.

In this chapter, I present evidence to support the claim that there were three shifts in centers of focus during the instructional intervention involving quantities in quadratic-function relationships. First, there was a shift from primarily noticing accumulated quantities (e.g., distances and times) or noticing changes in one quantity (e.g., noticing the changes in distance only) to noticing the changes in both quantities (CoF 1; see Table 6.1). Second, there was a shift from noticing numeric patterns to noticing quantitative patterns (CoF 2). Third, there was a shift from noticing that division is an arithmetic operation to noticing that division partitions composed units made up of continuous quantities into equal sections (CoF 3).

As explained in Chapter 3, the focusing framework (Lobato et al., 2011a) was used in the analysis of the centers of focus to account for social aspects of noticing. The rationale for considering these aspects of noticing comes from Goodwin (1994), who argued that “the ability to see a meaningful event is not a transparent, psychological process, but is instead a socially situated activity accomplished through the deployment of a range of historically constituted discursive practices” (p. 606).

To account for the socially situated aspects of how the new centers of focus emerged, the following three additional dimensions of Lobato et al.’s (2011a) focusing
framework were employed: (a) the focusing interactions (i.e., the set of discursive practices that help shape what gets noticed mathematically within a perceptual or conceptual domain of scrutiny, (b) the features of mathematical tasks (i.e., the affordances and constraints associated with a particular task or tasks that help to shape what become the centers of focus mathematically), and (d) the nature of the mathematical activity (i.e., the classroom participatory structures [e.g., roles, expectations for participation, etc.] that influence what emerges as the centers of focus).

Together with the evidence about centers of focus, I present evidence to support the claim that particular focusing interactions, the features of the mathematical

<table>
<thead>
<tr>
<th>Initial Center of Focus</th>
<th>New Center of Focus</th>
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<tbody>
<tr>
<td></td>
<td>CoF 1: Noticing both changes in distance and changes in time</td>
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<tr>
<td>Noticing accumulated distance and accumulated time</td>
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<tr>
<td>Noticing changes in distance</td>
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<tr>
<td>Noticing patterns not directly associated with the changes in the dependent variable</td>
<td>CoF 2: Noticing quantitative patterns in the changes in the dependent variable or in the average rate of change</td>
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<tr>
<td>Noticing division as an arithmetic calculation</td>
<td>CoF 3: Noticing that division partitions a composed units made up of continuous quantities into equal sections</td>
</tr>
<tr>
<td>Noticing that division shares a set of discrete objects between equal groups</td>
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Table 6.1. Summary of shifts in centers of focus.
tasks and participatory structures (i.e., the nature of the mathematical activity) helped to bring about each shift in noticing. Specifically, I provide evidence that what influenced the emergence of CoF 1 were (a) the discursive practices of naming, quantitative dialogue, producing graphic representations and highlighting, (b) the features of the mathematical tasks involving specifically targeting changes in quantities and varying the relationships between changing quantities, and (c) the participatory structures of presenting diagrams, noticing features of others’ diagrams and asking *Can you explain why?* questions about diagrams (see Table 6.2). I provide evidence that what helped bring about CoF 2 were (a) the discursive practices of naming and quantitative dialogue, (b) the features of mathematical tasks involving gradually narrowed the spacing of the time intervals, necessitating the generation of additional data points and having a complex relationship between distance and time, and (c) the participatory structures of presenting student work and noticing features of others’ diagrams. Finally, I provide evidence that what helped bring about CoF 3 were (a) the discursive practices of quantitative dialogue and highlighting working together, (b) the features of the mathematical tasks involving exploring division of continuous quantities and exploring division of composed units, and (c) the participatory structures of presenting diagrams and noticing features of others’ diagrams.

Each of the next three sections of this chapter tackles one of the three noticing findings. In the first section, evidence will be presented to establish CoF 1 (along with what students initially noticed). This will be following by evidence for the particular focusing interactions, features of the mathematical tasks and participatory structures
that appeared to contribute to the emergence of CoF 1. The second and third sections, which are devoted to CoF 2 and 3 respectively, will be organized in a similar fashion. After all three noticing findings have been presented, the conceptual connections between the noticing findings and the backward transfer effects, as well as some observations about noticing as a transfer process and about backward transfer, will be discussed.
CoF 1: Establishment and Origins

In this section, I present evidence supporting the claim that a new center of focus emerged during the instructional intervention with respect to the quantities changes in distance and changes in time. In particular, three students went from noticing accumulated distances and times and four went from noticing changes in distance only to all seven students noticing the changes in both the distance and the time (CoF 1).

The evidence presented will also show three unique features of the emergence of CoF 1 that distinguish it from the emergence of CoF 2 and 3. First, evidence will show that all students noticed the changes in distance and time (CoF 1) the first time the teacher introduced these quantities. This is in contrast to CoF 2 and 3, which appeared more difficult for students to see. Second, CoF 1 appeared to emerge for different students at different times (see Figure 6.1). This is in contrast to CoF 3, which will be described later, where the majority of the students followed a similar schedule for when the center of focus emerged. Each of these interesting features provided insights into how centers of focus get established, which will be discussed later.

To count as evidence that students noticed changes in distance and time in this study, they had to explicitly record changes in distance and time in student-created diagrams or report changes in distance and time verbally. Reports of speeds without explicit reports of changes in distance and or time were not counted as evidence of CoF 1, the reason being that students could conceivably think about speed without associating it directly with distance and time. Reporting the changes in distance and
time after being directly asked by the teacher to do so was also not counted as evidence of CoF 1. Only when students attended to changes in distance and time of their own accord was it counted as evidence that this center of focus had been established.

Evidence of what students were focused on prior to CoF 1 being established will be presented first. Then, evidence will be presented to show that CoF 1 was established. Finally, the focusing interactions, the features of the mathematical tasks and the nature of the mathematical activity that contributed to the emergence of this new center of focus will be discussed.

**Initial Foci Prior to Shift**

As shown in Figure 6.1, two centers of focus appeared to be initially at play with respect to the quantities students noticed in distance-time data. One group of students primarily noticed accumulated distances and times. A second group of

![Figure 6.1. Shifts in centers of focus that led to the establishment of CoF 1 and the lessons in which they occurred.](image-url)
students primarily noticed changes in distance.

**Focus on accumulated distances and times.** Initially, George’s, Brady’s and Peter’s center of focus was on accumulated distances and times when shown distance-time data (Lessons 1 and 2). For example, during Lesson 2, students were shown an 8-second animated movie of a swimming fish. The fish swam according to a quadratic distance-time function so that after 0 s it was at 0 ft, after 1 s it was at 2 ft, after 2 s it was at 8 ft, after 3 s it was at 18 ft, after 4 s it was at 32 ft, after 5 s it was at 50 ft, after 6 s it was at 72 ft, after 7 s it was at 98 ft and after 8 s it was at 128 ft (see Figure 6.2 for a screen shot of the movie). Students were not given any other information about the way the fish swam. They were asked, “Do you think the blue fish is swimming equally fast over time or is he going faster or slower at times? Explain how you know.” By repeatedly playing the movie, all three students correctly identified the accumulated distances\(^{11}\) that the fish had swum after each second.

Then, George and Brady created fractions with accumulated times in the numerators and the corresponding accumulated distances in the denominators (i.e., 1 s/2 ft, 2 s/8 ft, 3 s/18 ft, 4 s/32 ft, 5 s/50 ft, 6 s/72 ft, and 7 s/98 ft). They reduced each fraction (i.e., 1/2, 1/4, 1/6, 1/8, 1/10, 1/12 and 1/14) and concluded that “every second the blue fish goes faster.” Peter wrote his own response:

It’s going faster over time constantly, it doesn’t speed up all at once and it doesn’t speed up much each time it does. It goes from two to

---

\(^{11}\) The term *accumulated distance* will be used to refer to the total distance traveled from some starting location denoted as the *zero point*. Similarly, accumulated time will be used to refer to the total time elapsed after some starting time denoted as the *zero time*. 
Thus, in forming these correct conclusions, George, Brady and Peter appeared to primarily notice the accumulated distances and times and did not appear to notice the changes in distance (e.g., they did not notice the 6 ft change from 2 to 8 ft, the 10 ft change from 8 to 18 ft, and so on).

Despite their conclusions being correct, it is likely that George and Brady did not realize that the time/distance fractions they created were becoming increasingly inaccurate representations of the actual quadratic function for each successive fraction (i.e., the fraction $1 \text{ s} / 2 \text{ ft}$ closely represented the actual ratio of time to distance traveled by the blue fish in the 1st second while the fraction $7 \text{ sec} / 98 \text{ ft}$ did not closely represent the ratio of time to distance the blue fish travelled in the 7th second).

**Focus on changes in distance.** Initially, Nicholas’, Kendra’s, Armando’s and Jenn’s focus was on changes in distance when shown distance-time data (Lesson 1 and 2). For example, on the same task described above about the swimming blue fish, Nicholas recorded, in a table, the fish’s changes in distance paired with the corresponding accumulated times, not the corresponding changes in time (see Figure 6.3).
Similarly, in their written responses to the blue fish task prompt, Kendra and Armando recorded changes in distance paired with accumulated times. For example, Kendra wrote:

. . . in the 2nd second it goes 6 ft, in the 3rd second it goes 18 ft, which is 10 feet. Then in the 4th second it goes 32 ft, which is 14 ft. In the 5th second it goes from 32 ft to 50, which is 18 ft.

In this response, Kendra noticed the 6, 10, 14 and 18 ft changes in distance for the 2nd, 3rd, 4th and 5th seconds respectively.

The reader might conclude that, in referring to the 2nd, 3rd, 4th and 5th seconds, Kendra noticed the changes in time. For an adult, it is obvious that the 2nd, 3rd, 4th and 5th seconds imply 1-second time intervals. However, prior research has shown that on these kinds of quadratic distance-time function tasks, this is not obvious for middle school students (Lobato et al., in press; Lobato et al., in press). Because Kendra did not explicitly record or report the 1-second changes in time with each corresponding change in distance, she was coded as having a primary focus on changes in distance. Armando provided a similar written response to the blue fish task prompt. Jenn, the remaining student, primarily noticed changes in distance on a task prior to the blue fish task. However, her center of focus appeared to already have

![Figure 6.3. Nicholas’ table for blue fish activity recording changes in distance and accumulated times.](image)
shifted to noticing changes in distance and changes in time by the blue fish task.

**CoF 1: Noticing Changes in Distance and Time**

As shown in Figure 6.1, all students exhibited a shift in their centers of focus with respect to the quantities they noticed in distance-time data. Furthermore, the shift involved students converging on CoF 1, which was defined as noticing both the changes in distance and the changes in time. Because four of the seven students exhibited a shift in their center of focus in Lesson 8, most of the evidence supporting the claim that CoF 1 was established comes from this group.

**Students who were previously focused on accumulated quantities.** Of the three students that were previously focused on accumulated quantities, George and Brady had begun to notice changes in distance or changes in time by Lesson 6, without the teacher pointing these quantities out. For example, George’s Lesson 6 diagram for a piece-wise distance-time function included changes in distance labels but no changes in time labels and Brady’s Lesson 6 diagram included changes in time interval labels (i.e., 0-3 s, 3-5 s, 5-11 s) but no changes in distance labels. However, Peter’s Lesson 6 diagram did not yet contain changes in distance or changes in time labels.

By Lesson 8, CoF 1 appeared to be established for all three students. One way that they exhibited CoF 1 was by recording changes in distance and time in their diagrams of quadratic distance-time data. In almost all cases, George, Brady and Peter did this from Lesson 8 on. For example, they were presented with tabular quadratic distance-time data representing the motion of a remote-control car, where the data was spaced in 4-second intervals (see Figure 4.21). They were asked to produce a diagram
showing the speeds of the car. George recorded all eight 1-second changes in time and all eight corresponding changes in distance in his diagram (see Figure 6.4a), while Peter and Brady recorded both 4-second changes in time and the two corresponding changes in distance (Figure 6.4b).

A second way that these three students exhibited CoF 1 was by referring to changes in distance and time when they talked and reasoned about their diagrams. For example, Brady referred to particular changes in distance and time when explaining his Lesson 8 diagram (see Figure 4.22):

I did 0 to 4 [points at 0 and 4 s in the table and then at 0 to 4 seconds label in his diagram], and right here 4 to 8 seconds [points at 4 to 8 seconds label in his diagram]. So in total it’s 8 seconds [sweeps hand across diagram] . . . And then, right here, the first . . . 4 seconds [points at 0-4 s section in his diagram], it went 4 yards per second. So that’s 4 times 4 is 16 yards. It went 16 yards in total. And here [points at 4-8 s section in his diagram] it was 48 yards in total. Like here to here [points at 16-64 yd section] is 48 yards, and this is 4 seconds in total, 1, 2, 3, 4 [points at the four right loops in diagram that partition the distance from 16-64 yd into four equal sections].

In this excerpt, Brady noticed the 16 yd and 48 yd changes in distance that correspond to the 0-4 s and the 4-8 s changes in time.

**Students who were previously focused on changes in distance.** Of the four students in this group, CoF 1 was already established for Jenn and Nicholas by Lesson 6. For Armando and Kendra it was not. For example, Armando’s Lesson 6 diagram contained changes in distance labels but no changes in time labels and Kendra’s Lesson 6 diagram contained no changes in distance or changes in time labels.
By Lesson 8, CoF 1 appeared to be well-established for Nicholas, Jenn and Armando. In fact, Jenn and Nicholas had already begun recording changes in distance and time in their Lesson 6 diagrams, while Armando recorded changes in distance and time for the first time during the Lesson 8 remote-control car task described above (see Figure 6.5a). These three students also exhibited CoF 1 by referring to changes in distance and time when they talked and reasoned about their diagrams. For example, during Lesson 6, students were asked to draw a diagram from an animated movie showing a fish swimming 15 ft in 3 s, followed by 2 ft in 2 s, followed by 18 ft in 6 s, for a total distance of 35 ft in a total time of 11 s. Jenn explained her drawing (see Figure 6.5b) as follows:

I have the change in distance and the change in time, from each point [points to change in distance and change in time labels] that takes one second. And I have the times where from 1 second it’s right here it’s gonna be at 5 feet and in 2 seconds it’s gonna be at 10 feet and so on.
[points at accumulated time and distance labels]. And how this is 5 feet apart [points at 5 ft change in distance labels]. And these are smaller because they are 1 foot apart [points at 1 ft change of distance labels], and these are in the middle because they’re 3 feet apart [points at 3 ft change in distance labels].

In this excerpt, Jenn referred to all eleven 1-second changes in time and the corresponding eleven changes in distance.

Kendra’s shift in focus lagged behind the other students. However, by Lesson 11, she too exhibited evidence in her drawings and in her responses, of attending to changes in distance and time.

There were a limited number of instances after Lesson 8 in which it appeared that particular individual students were not focused on changes in distance and time. One instance occurred during Lesson 10, when Armando only recorded changes in distance on his diagram (but not changes in time), and then he and Brady used accumulated distances and times to determine speeds. The other instance occurred
during Lesson 16, when Peter did not record changes in distance or time on his diagram. However, his diagram was not completed and so it is possible that he would have added them had he been given more time.

**Focusing Interactions**

Because this center of focus emerged gradually for students, there were several focusing interactions (i.e., social practices that focus attention) that shaped CoF 1. In particular, evidence will show that there were at least four important kinds of focusing interactions, (a) naming, (b) quantitative dialogue, (c) producing graphic representations, and (d) highlighting.

**Naming.** The first kind of focusing interaction was naming (Lobato & Rhodehamel, 2010), which is similar to Goodwin’s (1994) coding. Naming is defined as “the act of using a category of meaning from mathematical practice to classify and label some mathematical characteristic or property” (Lobato et al., 2011a, p. 35). Evidence will show that there were two aspects to the role that naming played in shaping CoF 1.

The first aspect of naming was that the teacher explicitly named the quantities *changes in distance* and *changes in time* during Lesson 3, and then emphasized these names repeatedly throughout the instructional intervention. Not only did he use the names extensively, he also asked students to use them. For example, when Peter identified one of the changes in distance as “from 6 to 12,” the teacher said:

OK, so see if you can use the words that I’m using. I said what’s the change in distance, or sorry, can you see a change in distance of 6 when there’s a change in time of 1? So see if you can use the word ‘change’ in how or what you’re saying.
Similarly, when George said, “So this 12 right here [points at 12 on number line]. So 12 subtracted by 0 is 12 and then you have 2 seconds [points at clock],” the teacher responded with, “OK, so could you use the words ‘change’ like Peter did? What I’m looking for is where’s the distance change by 6 when the time changes by 1?”

In later lessons, the teacher continued to encourage students to use the terms change of distance and change of time. For example, during Lesson 4, when George identified 2.5 cm in 1 s on a diagram that he was making to show that moving 10 cm in 4 s was the same fastness as moving 30 cm in 12 s, the teacher asked him to recall what the 2.5 cm was called. When George did not know, the teacher said, “We were talking about that being the change in distance . . . these are the changes in distance.” During Lesson 6, Brady had drawn a circle around the labels and tick marks representing 10 cm and 20 cm on a number line, the teacher asked, “What about what we call these, the circled part, what do we call that?” Brady responded, “I think it’s called the change in distance.”

A hypothesis for why the names changes in distance and changes in time may have influenced what was noticed is that the names may have evoked for students a strong image of making comparisons. The reasoning behind this hypothesis is that virtually all students made comparisons between two distances and/or two times when initially asked, “Can anybody see a place where the change in distance is 6 when the change in time is 1?” For example, for their first attempt at identifying changes in distance and time, Peter and Jenn pointed from one distance to another on a number line. Similarly, for his first attempt at identifying changes, Armando said “The distance changes from 6 to 18 by 12 meters and 2 seconds.” This hypothesis is
consistent with other findings that have shown that mathematical terms can create powerful images (e.g., Siebert & Gaskin, 2006).

Another hypothesis regarding the role that naming played a role in shaping CoF 1 is closely related to Goodwin’s (1994) argument coding schemes. Goodwin (1994) describes coding schemes as a “systematic practice used to transform the world into the categories and events that are relevant to the work of the profession” (Cicourel 1964, 1968; in Goodwin, 1994, p. 608). He provides an illustrative example in which the archeologist profession utilizes the Munsell color chart to transform patches of dirt into color categories. Although the work the students and teacher were engaged in during the instructional intervention was not the kind of profession Goodwin would have had in mind, naming of quantities became a shared classroom practice that became relevant to making sense of quadratic function data and that transformed the problem contexts that were being considered into quantity categories.

Specifically, much of the students’ efforts revolved around using a classification system to categorizing quantities. The system initially had three categories of quantities: (a) accumulated distances and times, (b) changes in distance and time, and (c) speeds over an interval. Later, three categories were added: (a) speeds at a point, (b) changes in speed, and (c) acceleration. One of the principal ways by which students made sense of quadratic distance-time data was by categorizing the data according to the quantities that they represented. Goodwin (1994) describes this as “organiz[ing] their perception” (p. 609).

Categorizing numerical values as changes in distance and time was a key step in categorizing other numerical values, such as the speeds over an interval. Therefore,
in doing this categorizing work, the students were naturally oriented toward changes in
distance and time. In other words, by situating the categorization of changes in
distance and time within the larger practice of categorizing quantities, not only were
changes in distance and time relevant in their own right but also an important aspect of
a larger effort.

An example of naming changes in distance and changes in time being part of a
larger effort to categorize quantities came in an exchange between the teacher (T) and
Kendra (K), who was working on her Lesson 6 drawing of an animated movie
showing a fish swimming 15 ft in 3 s, followed by 2 ft in 2 s, followed by 18 ft in 6 s,
for a total distance of 35 ft in a total time of 11 s (see Figure 6.6 for a screen shot):

T: Nice, so you’ve got the change in distances the changes in time
and what are these here?
K: That’s the amount of feet well the changes in distance per time,
I mean per second.
T: Nice, great. What are those called, change in distance per
change in time?
K: Umm.
T: Here let me give you a hint, which is faster this or this or this
[points to each section of Kendra’s diagram].
K: That one [points at last section].
T: So it’s a measure of how fast it’s going . . .
K: Speed?
T: Speed, yeah, good.

In this transcript, the teacher named the changes in distance per changes in time. Later
in the lesson, he came back and said to Kendra, “Is there a way that you could fit this
in and this somewhere and this and this could those fit in there, in your picture [points
to speeds Kendra had written on a whiteboard]? Is there a way that you could label
those?” Finally, Kendra presented her diagram to the class, the teacher then said:
The one reason that I really thought we should look at Kendra’s is because Kendra put in the speeds. And I think that putting in the speeds is really important besides putting in the changes in distance and the changes in time, putting in the speeds shows how fast the fish is going. So can you point at the speeds? [Kendra points at speeds in her diagram] Yeah, so those are the speeds in there. Kendra has put in the speed. Some of you put in the changes in distance and the changes in time and it also would help to put in the speeds as well.

Therefore, in encouraging students to label speeds in their diagrams, the teacher was situating the naming of changes in distance and time within a larger effort to classify other quantities like speed.

**Quantitative dialogue.** The second kind of focusing interaction that influenced the emergence of this center of focus was *quantitative dialogue* (Lobato et al., 2011a). Quantitative dialogue is defined as “verbal communication that directs attention to quantities as measureable attributes of objects or situations” and that encourages students to “link numeric statements to the corresponding quantities in a given context” (p. 36). Evidence will show that the teacher used quantitative dialogue to focus student attention on quantities in general and on changes in distance and time in particular.

In the instructional intervention, the teacher consistently encouraged students
to link the appropriate measurable attributes with each numerical value. For example,

Jenn gave the response about how to produce distance-time pairs that would be the
same speed as a rabbit traveling 10 cm in 4 s:

I noticed that equivalent fractions, 20 over 8, simplified is 10 over 4. Which is rabbit’s time . . . and distance. And then we did a lot of them multiplying by 2, like if the distance was 20, we multiplied it by 2 and then if the time was 8, we multiplied it by 2, and 40 as the distance and 16 as the time. And then we added zeros, like if the distance was 5 and the time was 2, the distance would be 50 and the time would be 20.

In response the teacher said:

OK, now before you go down Jenn, what I was mentioning to this group over here, and I think it’s important for both of our groups to think about a lot or to try to keep in mind all the time is that we’re talking about quantities. We’re talking about things that we measure about rabbit. And when we use just numbers, we’re kind of forgetting about the quantities. So I’m wondering Jenn, could you repeat what you said because you did it so well and this time see if you can refer always to the quantities. So for instance, you talked about the distance is 20, say the distance is 20 feet times 2 and times 8 seconds times 2. See if you can go through the whole explanation again, this time try to focus on quantities, rather than just the numbers. Try again.

This excerpt was an example of quantitative dialogue because the teacher directed
Jenn and the rest of the students toward thinking about the measurable attributes of the
numerical values. Jenn then provided the following revised response [emphasis added]:

So for equivalent fractions, if your distance is 20 centimeters and umm it is 8 seconds, simplified it’s gonna give you 10 centimeters as distance and 4 seconds as time. So that’s what rabbit has. And when you mult—when you do the multiplying by 2 laps, that’s the distance is 20 centimeters and you times it by 2 and umm the time was 8 seconds and you times it by 2, the distance would be 40 centimeters and the time would be 16 seconds. And if you added zeros, the distance was 5 centimeters and the time was 2 centimeters, I mean that the time was 2 seconds, then when you added zeros the distance would be 50 centimeters and the time would be 20 seconds.
During this revised response, Jenn appeared to be significantly more attentive to quantities than when she made the first response.

In a second example, George was working on a diagram that illustrated why moving 10 cm in 4 s was the same fastness as moving 30 cm in 12 s (see Figure 6.7). The teacher asked him what the numerical value labels 2.5, which were written between tick marks, represented. George said they represented “2.5 centimeters over 1 second.” Then, the teacher (T) and George (G) had the following exchange, in which the teacher pressed George to link the numerical value with a quantity:

T: So what about these ones? The two-point-fives? [points at change in distance labels]
G: These are how much he ran in a second.
T: What do we call those again?
G: Umm, rate?
T: Like he’s at 5 and then he’s at 7.5 [points at accumulated distance labels]. So what’s the 2.5?
G: Umm, the fastness? The . . . Oh, no not the fastness.
T: So he changed from here to here.
G: Yeah.
T: So what is that 2.5?
G: What is that called?
T: We were talking about that being the change in distance.
G: Right right.

In this exchange between George and the teacher, the teacher pressed George to explain several numerical values in his diagram. Thus, this exchange involved quantitative dialogue.

The use of quantitative dialogue with respect to changes in distance and time began to be appropriated by students. For example, during Kendra’s presentation of her work in Lesson 7, Nicholas asked her, “Can you explain why you put the change in time and distance?” In another example, during Lesson 15, Jenn asked George
about numerical values on his diagram: “Can you explain why you used those numbers for the 4, 12, 20 and all those . . . the 4 miles and 12 miles and the 20 miles, why did you choose those numbers?” With these kinds of questions, students directed each other’s attention to the meaning of quantities.

**Producing graphic representations.** The third kind of focusing interaction was *producing graphic representations* (Goodwin, 1994). Goodwin, who defines graphic representations more broadly than is typically associated with mathematics (i.e., as diagrams, still shots from video recordings, maps, graphs, photographs, etc), contends that they are often central to the discourse of a profession. Although the tasks the teacher and students were engaged in during the instructional intervention were likely not the professions Goodwin had in mind, evidence will show that graphic representations were central to the classroom discourse about quadratic functions.

Much of the discussion about changes in distance and time occurred around the following two types of graphic representations of moving objects: (a) computer-generated simulations (Lessons 1-3, 6 and 11), and (b) student-generated diagrams (Lessons 4-16). Thus, at least one of the two types of graphic representation was
important to each lesson. Note that neither of these two types of representations involved Cartesian graphs.

Graphic representations likely played an important role in shaping CoF 1 because, according to Goodwin (1994), they “organize phenomena in ways that spoken language cannot—for example, by collecting records of a range of disparate events onto a single visible surface” (p. 611). Both the computer-generated and the student-generated graphic representations displayed multiple events in the form of multiple snap-shots of the moving object on one screen or diagram. For the computer simulations, the students set the program (using the “drop marks” function) so the moving objects left “marks” as they traveled along a number line across the screen (e.g., the small triangles in the screen capture shown Figure 6.6 are the “marks” that are “dropped” every second as an object moves across the screen). When the computer simulation had finished playing, the computer screen contained all the drop marks left by the moving object next to a number line. This graphic then served as a context for discourse that often centered on changes in distance and time.

For example, during Lesson 6, Kendra (K), Nicholas (N) and Jenn (J) were working with the orange fish computer simulation. They were instructed to write down everything they noticed about the motion of the fish. In the following exchange their attention was drawn to the changes in distance and time as they considered the drop marks that the fish had left behind as it traveled across the screen (see Figure 6.6):

K: Is that a 5 or a 4 [points to space between 2nd and 3rd drop marks on computer screen]?
N: It’s 5.
In this excerpt, Kendra and Nicholas noticed the change in distance between the second and third drop mark, with Kendra asking if it was 5 or 4 ft and Nicholas responding that it was 5 ft. Then, Kendra and Jenn noticed the changes in distance from 15 to 17 ft and the corresponding change in time of 2 s.

The student-generated graphic representations also provided a context in which changes in distance and time were the center of the discourse. For example, during Lesson 7, an enlargement of Armando’s diagram was displayed for the class (see Figure 6.8) and the teacher had the following exchange with Armando (A), Brady (B) and Nicholas (N):

T: So this is Armando’s picture and I didn’t take the whole picture, I just took the part from 17 to I think it’s 35.
A: 30, 35.
T: 35? So I just took that part. And then on top of it I put some arrows again. So I’m looking from the very beginning to the very end. And Brady what does that arrow represent?
B: The . . . the whole distance and the seconds too.
T: The whole distance? OK and so what is that whole distance in that section?
B: The change in distance of 18 feet and 6 seconds.
T: And can you see 18 feet in there somewhere?
B: Umm . . .
T: This starts at 17 [points to beginning of picture] and it goes up to 35 [points to end of picture].
B: . . . I don’t see 18.
T: OK. Could someone help him? Is there an 18 up there? This says a change in distance of 18 [points to 18 ft change of distance label written on arrow] . . . Nicholas?
N: From the 17 feet to 35 feet there’s 18 feet within them.
T: Does that make sense Brady?
B: Yeah.
T: OK, what about do you see the 6 seconds anywhere?
B: Yeah, umm from 18 uhh to 24, last one 20 and 26.
T: So 20 to 26 would be umm those are those are feet, you might not be able to see that but most of them say feet on there so from 20 to 26 feet would be 6 feet. Could someone help Brady where do you see the 6 seconds in there somewhere? Peter do you see the 6 seconds in there somewhere?
P: . . . Oh yeah because it has 6 bumps.
T: Because it has the 6 bumps. What Armando was drawing was each of these represents a bump. So there’s 6 bumps, 6 seconds.

In this excerpt, the teacher used a student-produced graphic representation to focus a discussion around changes in distance and time. Brady indicated that he did not see where the 18 ft change in distance and the 6 s change in time were represented. However, Nicholas and Peter were able to explain where each change was. These kinds of discussions around graphic representations, which occurred often during the intervention, likely contributed to the emergence of CoF 1.

**Highlighting.** The fourth kind of focusing interaction was *highlighting* (Goodwin, 1994). Goodwin defines highlighting as “methods used to divide a domain
of scrutiny into a figure and a ground, so that events relevant to the activity of the moment stand out” (p. 610). Goodwin considers a wide variety of methods as examples of highlighting, including using colored markers, handwritten annotations and sticky-notes to highlight documents, using traces in the dirt to highlight post molds on archeological digs, using gestures to highlight features of a visual field, and using cropping and enlarging to highlight features of photographs.

Highlighting in the instructional intervention was closely associated with producing graphic representations because the teacher and students highlighted the student-generated diagrams to foreground changes in distance and time, using gestures, diagram enlargements and written annotations. An example of gestural highlighting was when Jenn was explaining the diagram that she had produced during the Lesson 6 swimming fish activity and she swept her finger back and forth between positions of the ball to highlight particular changes in distance and time (see Figure 6.9).

The teacher also engaged in highlighting the student-generated diagrams. As seen in Figure 6.8, he enlarged portions of student diagrams to make changes in distance and time prominent. He would also gesture over diagrams that were projected for the class to see, highlighting changes in distance and time. Both kinds of highlighting likely influenced the emergence of CoF 1.
In summary, there were four kinds of focusing interactions that helped shape a focus on changes in distance and time: naming, quantitative dialogue, producing graphic representations, and highlighting. As explained above, perhaps such a diverse set of focusing interactions was involved because the center of focus emerged gradually, with students developing the practice of attending to changes in distance and time rather than it emerging suddenly as an insight.

**Mathematical Tasks**

There were two features of the mathematical tasks that likely contributed most to the emergence of CoF 1. First, during Lesson 3 and Lesson 8, students worked on tasks that were specifically about changes in distance and time. Second, during Lesson 6 and Lesson 8, the tasks involved students drawing diagrams of varying the relationships between changes in quantities.

**Tasks specifically targeting changes in distance and time.** During Lesson 3, students watched a computer simulation of a rabbit running 24 ft in 4 s at a constant...
speed. Then, the students were asked to respond to the following three prompts: (a) Find 5 places where rabbit’s distance changes by 6 ft when the time changes by 1 s, (b) Find 3 places where his distance changes by 12 ft when the time changes by 2 s, and (c) Find 3 places where his distance changes by 3 ft when the time changes by .5 s. Each of these prompts was designed to try to promote the noticing of changes in both quantities. For example, the first prompt asked for five changes in distance of 6 ft so that students would need to look for an additional 1 s change in time besides the four obvious changes in time from 0-1 s, from 1-2 s, from 2-3 s and from 3-4 s (e.g., Brady’s written response was “1.5-.5 sec : 3 ft-9 ft”). The second and third prompts promoted noticing changes in both quantities because they involved non-unit changes in time (e.g., Armando’s written response for the third prompt was “9 through 12 F, 1.5-2 s; 12 through 15 F, 2-2.5 s; 15 through 18 F, 2.5-3 s”). Thus, it was necessary for students to explicitly attend to changes in time in order to identify changes in distance.

During Lesson 8, the teacher and students played a game involving the identification of changes in distance and time for the computer-simulation of the swimming orange fish. A screen-shot from the simulation of the fish swimming the first 15 ft in 3 s, the next 2 ft in 2 s, and the last 18 ft in 6 s, was shown to students. The screen-shot showed the fish’s drop marks for each second (see Figure 6.6). The students were divided into teams, and each team was asked to find a particular change in distance and the corresponding change in time, where the change in time was greater than 1 second (e.g., a change in distance of 11 ft for a change in time of 3 s). Each group scored a point for correctly identifying their change in distance and time.
The class played four rounds of this game. For the first question the teacher said:

So this group here they get, a change of 6 feet, sorry a change of distance of 6 feet for a change in time of 2 seconds. So your group work on that [points to one of the teams] and then I’ll come back to you for an answer. And then your group [points to other team], find a change of distance of 5 feet for a change of time of three seconds.

In response to their question, the second team, composed of George (G), Armando (A), Peter (P) and Brady had the following exchange:

A: I got it.
G: Got it.
P: What did he say?
G: Change in time of 5 feet in 3 seconds.
A: From 15 to 20.
G: Yeah, 15 to 20 which equals 5 feet and from the 1st second, 2nd second, third second . . .
A: It’s 3 seconds.
G: So from 3 seconds to 6 seconds . . . equals 3 seconds [writes 3 s-6 s=3 s]. So 5 feet and 3 seconds.
A: From 3 to 6?
G: There.
A: Oh, from 3 to 6 seconds.
G: [explains to Peter] So it’s from 15 to 20 that’s 5 feet right and so from 3 to 6 that’s 3 f--that’s 3 seconds, and so that will be 5 feet in 3 seconds.
P: Yeah, from 15 to 20 [writes 15-20 ft]. I have it too.

In this excerpt, George, Armando and Peter appeared to notice changes in distance and changes in time. Two features of this task that likely promoted attending to changes in both quantities were (a) that the changes in time was greater than 1 s and (b) that the speed of the fish was variable. Thus, it was necessary for students to attend to the number of drop marks for the change in time and to the distances between drop marks for the changes in distance.
Tasks with varying relationships between changes in quantities. During Lessons 6 and 8 (i.e., the lessons in which many students’ center of focus shifted to CoF 1), students worked on diagrams involving variable speeds. The Lesson 6 diagram involved the swimming fish which had three speeds (i.e., 5 ft/s, 1 ft/s and 6 ft/s; see Figure 6.5b). The Lesson 8 diagram involved the remote-control car which had a constantly increasing speed (i.e., a quadratic distance-time function; see Figure 6.5a). The variable speeds for both problems meant that for given changes in time, the changes in distance varied. This changing relationship between changes in distance and changes in time likely promoted students noticing changes in both quantities as they drew diagrams.

For example, when Jenn (J) drew her Lesson 8 diagram (see Figure 6.10), she had the following exchange with the teacher (T):

T: Let’s see what you’ve got here [to Jenn], so you’ve got 12, 12 yards from 4 to 16 . . .
J: 16 and then 16 yards from 0 to 16 . . .
T: And you’ve got 48 across here [points to the 48 yd change in distance from 16 yd to 64 yd], and how did you break this up, 20 and 28 [points to the 20 and 28 yd change in distance labels that make up the 48 yd change in distance] oh it’s to 36 [points at 16 and 36 yd accumulated time labels] I get it, and this is 2 seconds, 2 seconds [points at 2 s change in time labels below the 20 and 28 yd change in distance labels], so what’s the speed in this?
J: Well 2, right here it’s 28 yards apart and 2 seconds apart because umm from 36 to 64 [points at 28 yd change in distance from 36 to 64 yd].
T: Could you say what the speed was, this is like a speed 28 yards in 2 seconds. Could you say what the speed was each second in that time?
J: Umm 14 each second.
Jenn’s diagram and subsequent explanation indicate that through drawing the diagram of a changing relationship between changes in distance and changes in time (e.g., a 20 yd change in distance with a 2 s change in time followed by a 28 yd change in distance with a 2 s change in time) she noticed both changes in distance and changes in time. Furthermore, she noticed changes in distance and time nested within changes in distance and time (e.g., 20 yd and 28 yd changes in distance nested within a 48 yd change in distance and two 2 s changes in time nested within a 4 s change in time).

Based on this example and others like it, it seemed like the variable relationship between changes in distance and time promoted a high level of noticing changes in distance and time. In Jenn’s case, it was as if she considered each pair of changes in distance and changes in time individually as she drew each part of her diagram. Therefore, the tasks in which students drew diagrams of variable speed contexts likely contributed to the emergence of CoF 1.
Nature of the Mathematical Activity

Three classroom practices (Cobb & Yackel, 1996) were established during the instructional intervention that likely contributed to the emergence of CoF 1: (a) the practice of presenting student work to the class, (b) the practice of noticing features of other students’ diagrams, and (c) the practice of students asking each other Can you explain why? questions. All three practices were centered around the presentation of student work.

The practice of presenting student work. It was a common practice for all students to present their work to the class. Usually the work consisted of a diagram of a distance-time function set in a physical context. For example, Brady presented his diagram for the remote-control car (see Figure 4.22):

I showed the change in time from right here it went 0 to 4 seconds [points at section from 0 to 16 yd] and then 4 to 8 seconds [points at section from 16 to 64 yd]. On the bottom I showed the change in time from 0 to 16 yards and then from 16 to 64 yards [points at change in time labels]. And then I showed umm by how many s--yards was it going like per second. So the first 4 seconds it was going 4 yards per second and the last 4 seconds it was going 12 yards per second . . . In the beginning it was going 4 yards per second [points at 0-4 s section], so these is 2 yards [points at tick marks] so it was going like 4 yards every second and they’re small because it was going slow and right here they’re big because they were going 12 yards per second [points at 4-8 s section].

As stated above, these graphic representations served as the context for important discussions about quadratic function. However, the practice of presenting appeared to serve the additional purposes of providing students with examples of what other students were noticing as they were creating their diagrams. This conclusion was based on the fact that by Lesson 8 the diagrams were fairly consistent from one
student to the next in terms of quality and in terms of what was included and how it was represented, this despite the teacher never drawing a diagram himself. The continued practice of students presenting diagrams likely contributed to the gradual increase in the number of quantities represented in the student diagrams as the instructional intervention went on.

**The practice of noticing features of others’ diagrams.** Along with the practice of presenting diagrams, the practice of noticing features of others’ diagrams was also prominent throughout the instructional intervention. Specifically, a routine was established in which, before the student said anything, the teacher would call on another student to say what they noticed about the presenting student’s diagram. For example, before George presented his orange fish diagram, the teacher (T) asked Jenn (J) to explain what she saw in George’s diagram:

T: So let’s see if someone else can make sense of George’s picture before he does. Umm Jenn can you explain what you see of his drawing as far as orange fish?
J: Umm, it says like in 1 second it’s gonna be 5 feet, that’s the change of distance. And then it shows like where it stops, and it continues like for the other ones, like it shows how small it makes for each distance.
T: OK, anything else?
J: It has the change in time.
T: OK, Alright, so George why don’t you go through it, umm that was a good explanation Jenn.

In this excerpt, Jenn noticed that George had recorded the changes in distance and the changes in time. This routine likely helped the entire class notice more features of the student-generated diagrams because it created an expectation that students should closely inspect each other’s diagram. Closely inspecting each other’s diagrams also likely raised students’ level of awareness of their own diagrams with respect to what
they included in their diagrams and how they represented the various features of their diagrams.

The practice of students asking each other *Can you explain why?* questions. The practice of asking *Can you explain why?* questions occurred immediately after a student presented their diagram. Specifically, once a student had finished explaining their diagram to the class, the teacher would call on another student to ask a *Can you explain why?* question. For example, after Kendra (K) presented her drawing for the remote-control car, the teacher (T) asked Peter (P) to ask a *Can you explain why?* question:

T: OK. So let’s again do that, let’s again do that idea or that guide, that umm technique where one of us asks an explaining question so you can explain. So Peter can you ask Kendra an explaining question, something that you want to know about or something that you maybe think you do understand about her drawing that she could explain?

[long pause]

T: Could even be something that you think you understand about her drawing, but just something to to you could ask her to explain.

P: Can you explain why you made the loops?

K: Huh?

P: Can you explain why you made those loops?

K: They are, represent the change in distance.

T: OK, does that answer your question?

P: [nods]

Interestingly, students appeared to take the asking of *Can you explain why?* questions quite seriously because, as Peter did above, they often paused to thoughtfully considered the student’s diagram, before asking a question. This practice likely promoted both students noticing more about other students’ diagrams and noticing more about their own diagrams. Because the diagrams generally increased in the
consistency with which both changes in distance and time were recorded over the
course of the instructional intervention, it suggests that these classroom practices
influenced the emergence of CoF 1.

CoF 2: Establishment and Origins

In this section, I present evidence supporting the claim that a new center of
focus emerged during the instruction with respect to noticing patterns in the changes in
the dependent variable of distance-time functions. In particular, students went from
noticing numeric patterns not directly associated with the changes in the distance to
noticing numeric and/or quantitative patterns in the changes in the distance (CoF 2:
see Figure 6.11). Furthermore, I present evidence that (a) the focusing interactions of
naming and quantitative dialogue, (b) the features of the mathematical tasks involving
gradually narrowing the size of the independent variable spacing, necessitating the
generation of additional data points and having a complex quadratic relationship
between variables, and (c) the participatory structures of presenting student work and
of noticing features of others’ diagrams influenced the emergence of CoF 2.

Evidence will show that CoF 2, like CoF 1, did not emerge for students all at
once. However, unlike CoF 1 and CoF 3, there was some variability in the nature of
CoF 2. In other words, not all students shifted to the same center of focus. CoF 2
represented several related foci such as quantitative patterns in elapsed distances and
quantitative patterns in speeds.
The 5 codes shown in Figure 6.11 fall within three types of patterns: quantitative, partially-quantitative, and numeric. To illustrate, consider the following examples. If the changes in distance each second went from 3 to 5 to 7 cm, then for students to be coded as noticing a quantitative pattern in the changes in distance, they would have needed to say (or write) something like “the changes in distance go up by 2 cm each second.” If students said, “it goes up 2 cm each second,” or “the changes in distance go up by 2” without either specifying that the changes in distance were what was increasing or that they were changing each second or without specifying what the units were, they were coded as having noticed a partially-quantitative pattern in the changes in distance. If students said, “it goes up 2,” without specifying what was
going up and without specifying what the units were, they were coded as having noticed a numeric pattern in the changes in distance. The fourth code, quantitative pattern in the speeds, was used for responses like “the speed increases by 2 cm per second each second.” Finally, numeric patterns not associated with changes in distance was used when students noticed patterns not directly associated with the changes in distance (or the speed).

In a similar manner to how CoF 1 was presented, the initial center of focus prior to CoF 2 being established will be presented first. Then, evidence will be presented that shows that CoF 2 was established. Finally, the focusing interactions, the features of the mathematical tasks and the nature of the mathematical activity that contributed to the emergence of this new center of focus will be discussed.

**Initial Focus Prior to Shift**

As shown in Figure 6.11, when students initially began exploring quadratic distance-time functions in the instructional intervention, they primarily noticed numeric patterns not directly associated with changes in distance. The pattern that all students noticed was a pattern that involved squaring the time values to produce the distance values. For example, during Lesson 8, students were drawing diagrams of the remote-control car quadratic distance-time data presented in 4-second intervals described above. Before beginning her drawing, Kendra noticed that, “Also, 4 times 4 is 16, 8 times 8 is 64,” as she pointed back and forth between times and distances on the remote-control car distance-time data table (see Figure 6.12). George noticed the same pattern because he said, “So like check it out, look, look, look, I found a pattern, 0 times 0 is 0, 4 times 4 is 16, 8 times 8 is 64.” He also pointed back and forth
between time and distance on the table as he explained the pattern.

![Figure 6.12. Kendra’s pointing gesture while explaining the pattern that multiplying the time by itself give the distance.](image)

During lesson 9, when students were exploring the remote-control car data presented in 2-second intervals (see Figure 6.13a), Brady noticed that, “It’s a square root. Watch 2 times 2, 4 times 4, 6 times 6, 8 times 8.” As he said, this he tapped his pen back and forth from the time column to the distance column like Kendra and George had done. During Lesson 10, when Armando was exploring tabular data for the quadratic distance-time function shown in Figure 6.13b, he made the following observation:

I’m figuring out that if you di--if you multiply these ones by itself [points at the time column] and then multiply it by 2, it gets you the distance, because if you multiply 1 by itself, and you multiply by 2 it’s 2 [i.e., $1^2 \times 2$], you multiply 2 by itself is 4, multiply by 2 it’s 8 [i.e., $2^2 \times 2$]. The same with 3, you multiply by itself it’s 9, by 2 it’s 18 [i.e., $3^2 \times 2$] and the rest [sweeps pen down table].
In each of these examples, students found numeric patterns not associated with changes in distance. Jenn and Peter made similar statements.

Nicholas was the lone student not to make a statement about multiplying the time by itself. However, for the Lesson 8 and 9 remote control car tasks, Nicholas divided distance values by time values to produce the same time value. Therefore, he was also coded as noticing numeric patterns not associated with changes in distance.

Interestingly, none of the students explicitly stated that they associated the pattern that they noticed with a particular quantities-based meaning. In fact, when the teacher asked Armando if the 4 in his pattern meant anything, he replied, “no.” This is important because, even though it is possible that a person could report a numerical pattern and at the same time understand the pattern’s quantitative significance, the fact

![Figure 6.13. (a) Brady’s data remote-control car data table for Lesson 9; (b) Armando’s data table for Lesson 10.](image-url)
that none of the students assigned meaning to the patterns they noticed is evidence that they did not associate quantitative meaning to the patterns.

**CoF 2: Noticing Changes in Distance Patterns**

As shown in Figure 6.11, all students exhibited a shift in their center of focus with respect to noticing patterns in quadratic function distance-time data. The general trend was that students shifted from more numeric patterns to more quantitative patterns. For some the shift emerged more quickly than for others.

Kendra, George and Nicholas were the first students to exhibit a shift in the patterns that they noticed. During Lesson 10, Kendra provided the first evidence of a shift toward noticing patterns in the changes in distance. She worked with the same quadratic function distance-time data for the remote-control car task from Lessons 8 and 9, except that the data were presented every 1 s, as opposed to every 4 s or every 2 s. After completing a drawing in which she recorded all changes in distance (see Figure 6.14), Kendra (K) had the following exchange with the teacher (T):

K: Umm, I want to use prime numbers but I forgot the exact definition.
T: Why do you want to use prime numbers?
K: Well like umm . . .
T: Oh you’re thinking about these numbers as prime numbers [points at the 3 yds, 5 yds, 7 yds, 9 yds, 11 yds, 13 yds, and 15 yds changes in distance labels in Kendra’s diagram]?  
K: Yeah, well I but I don’t know the exact definition.
T: OK so a prime number would be one that doesn’t have a divisor like you can’t divide it evenly unless you divide it by itself. So like 9, divides by 3 [points at the 9 yds change in distance in Kendra’s diagram].
K: OK.
T: So is there some other number you could say?
K: Odd numbers?
T: Odd numbers, OK.
In this transcript, Kendra indicated that she noticed a pattern in the changes in distance. At first she thought they were prime numbers. Then, she thought they were odd numbers. In the following written statement, Kendra made explicit the pattern she noticed: “I noticed a pattern in the change in distance, and what I noticed is that they are all odd numbers and you just add 2 to get to the next number. The speed changes every second.” This written statement indicates that she noticed a pattern in the changes in distance (and in the speed). Note that subsequent to the exchange with the teacher, she went back and changed the changes in distance labels to speeds by adding “/sec” under each label (see Figure 6.14). Kendra was coded as noticing a partially-quantitative pattern in the changes in distance because she explicitly stated that the pattern was in the changes in distance but did not include units when reporting that the changes went up by 2.

In Lesson 10, George also exhibited a shift toward noticing patterns in the changes in distance. However, his center of focus appeared to shift further toward quantitative patterns than Kendra’s did. The evidence for the shift came after he had completed a diagram for a quadratic distance-time function task (see Figure 6.13b for the data and Figure 6.15 for his diagram). In his diagram, he had recorded the following changes in distance with ft units: 8 ft, 24 ft, 40 ft, 56 ft, 72 ft, and 88 ft. When discussing his diagram with the teacher, he made the statement, “the distance between the changes in distance is 16 each second yeah” [points back and forth between 8 ft and 24 ft change in distance labels]. This response indicates that George noticed that the pattern was in the changes in distance. He did not use units in his verbal report of the pattern. However, he did consistently label, with ft units, the
In Lesson 10, the shift in Nicholas’s center of focus appeared to be greater than

changes in distance that he was pointing to during his verbal report and so was coded as noticing a *quantitative pattern* in the changes in distance (George was also coded as noticing a *partially-quantitative pattern* in the changes in distance already in Lesson 9).

In Lesson 10, the shift in Nicholas’s center of focus appeared to be greater than

<table>
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<th>Distance (yards)</th>
</tr>
</thead>
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</tr>
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<td>1</td>
</tr>
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</tr>
<tr>
<td>8</td>
<td>64</td>
</tr>
</tbody>
</table>

**Figure 6.14. Kendra’s drawing and subsequent written statement about the pattern in the changes in distance for the Lesson 10 remote-control car task.**

**Figure 6.15. George’s Lesson 10 diagram showing changes in distance.**
that exhibited by Kendra and George. Like Kendra, Nicholas drew a diagram for the remote-control car data which was presented in 1-second intervals. He included all changes in distance, changes in time and speeds (see Figure 6.16). He also included an additional quantity, the changes in speed, labeling each and every change in speed between adjacent speeds as 2 yards/second. Nicholas (N) and the teacher (T) had the following exchange about his diagram:

T: So how did you, are these the speeds down here [points at change in speed labels]?
N: The changes in speed.
T: Oh, the changes in speed, nice, so where are the speeds?
N: Right here [points at speed labels].

Figure 6.16. Nicholas’ Lesson 10 diagram which includes changes in speed labels.

In this excerpt, Nicholas differentiated between the speeds and the changes in speed. Unfortunately, the rest of the audio for this excerpt was cut off and so the subsequent conversation in which Nicholas might have explained further what pattern he was noticing was lost. Nevertheless, he was coded as noticing in this instance a quantitative pattern about speeds because he clearly identified the seven 2 ft/s changes in speed with labels and also distinguished between the speed labels and the changes in speed labels.
In subsequent lessons, students who had not yet exhibited a shift in the patterns that they noticed began to shift and student who had already exhibited a shift, shifted further toward quantitative patterns in the changes in distance or in the speeds. For example, Lesson 14 involved a task about a rocket in which students were presented with partial data and were asked to find additional data before drawing a picture (see Figure 6.17). Peter, who had not yet provided evidence of a shift, labeled all changes in distance in his diagram (Peter used COD as an abbreviation for change of distance; see Figure 6.18). He also recorded the speeds corresponding to the changes in distance in a speed table (i.e., 4 mi/1 min, 12 mi/1 min, 20 mi/1 min, 28 mi/1 min, 36 mi/1 min, 44 mi/1 min, and 52 mi/1 min). Then, he made the following written statement: “The speed of the rocket is 8 miles each minute, 8 miles/1 minute.” Peter did appear to notice a pattern here, although it is not clear if he noticed the pattern that the changes in distance were going up by 8 mi each min or the pattern that the speeds were going up by 8 mi/min each min. However, he incorrectly interpreted the pattern as a speed rather than a change in speed or a change in the change in distance. Therefore, he was coded as noticing a partially-quantitative pattern in the changes in distance.

Several other shifts occurred during Lessons 14 and 16. Kendra shifted further toward noticing quantitative patterns in Lesson 14. George shifted further toward noticing quantitative patterns in the speed in Lesson 14. Jenn also shifted toward quantitative patterns in the speed in Lesson 14. Brady and Armando shifted toward noticing numeric patterns in the changes in distance in Lesson 16 (see Figure 6.11).
Focusing Interactions Associated with CoF 2

CoF 2, like CoF 1, was about changes in quantities. Therefore, it is likely that the kinds of focusing interactions that contributed to the emergence of CoF 1, also indirectly contributed to the emergence of CoF 2. However, there were particular aspects of two of the four focusing interactions that were at play in shaping CoF 1, naming and quantitative dialogue, which appeared to contribute directly to the emergence of CoF 2 and which will be discussed next.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
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<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
</tr>
</tbody>
</table>

Figure 6.17. Data set and task prompt for Lesson 14 rocket activity.

Figure 6.18. Peter’s diagram for the Lesson 14 rocket activity.
Naming. As explained earlier, the quantities changes in distance and changes in time were named by the teacher, who subsequently encouraged students to use the names and who used them often himself. As the center of focus shifted toward noticing changes in distance and time (CoF 1) due to naming, there would have been a greater likelihood that students noticed patterns related to the changes in distance. Thus, at the very least naming changes in distance and time likely indirectly influenced the emergence of CoF 2.

However, there were also other two naming episodes that may have more directly contributed to the emergence of CoF 2. These episodes appeared to be influential because they occurred at the beginning of Lessons 10 and 14, which were two of the lessons in which most of the shifts in focus occurred (the other was Lesson 16). The first episode was when the teacher introduced a new name, changes in speed. The first time he mentioned this new name was during a class discussion in Lesson 10. Prior to the introduction of the names, the students had presented their drawings for the remote-control car and recorded the average speeds of the car over 2-second intervals on a chart (i.e., 2 yds/sec from 0-2 s, 6 yds/sec from 2-4 s, 10 yds/sec from 4-6 s and 14 yds/sec from 6-8 s; see Figure 6.19a).

Then, the teacher introduced the name changes in speed:

T: One question I have to ask and this is something we haven’t discussed before, does anybody see, this is a new quantity, in this picture, or this table, changes in speed, not speed, changes in speed? Peter does. Brady does. Anybody over there see what might I be referring to when I say what are the changes in speed? Can anybody see what I mean when I say changes in speed? Jenn does, how about Nicholas and Kendra. Nicholas does, Kendra do you? OK well now Nicholas you explain what
I mean when I say, not speed, that’s a different quantity but what about changes in speed?

N: Like how it goes from 2 yards a second to 6 yards a second. The change in speed would be like 4 yards a second.

T: The change in speed would be 4 yards per second. So it’s not, it’s not something that I’ve written up here yet but the change in speed from 2 yards per second to 6 yards per second [points to 2 yd/s and 6 yd/s on table] would be 4 yards per second.

T: So Kendra is this making sense? OK so could you pick another place where you see a change in speed?

K: 10 yards a second.

T: Where?

K: I mean 4 yards a second.

T: Where do you see it?

K: 10 and 14.

T: Oh, between the 10 and the 14, OK [points from 10 yd/s to 14 yd/s on table]. So does anybody see of umm 8 yards per second? Can anybody see a change in speed of 8 yards per second? Peter? You don’t? OK, Jenn.

J: From 2 yards per second to 10 yards per second.

T: So from 2 yards per second to 10 yards per second [points to speeds on table]. And you could look there as well so there’s many different changes in speed. Here this would be 8 yards per second, from 2 yards per second to 10 yards per second [points to table].

In this excerpt, the teacher introduced the new term and asked students to find examples of it in the speed-time table they had just created. This new term was quickly appropriated by Nicholas because right after this discussion, he used it. Specifically, he labeled it in his drawing of the remote-control car data presented in 1-second intervals (see Figure 6.16 for the diagram). Thus, naming the term appeared to influence what patterns Nicholas noticed.

However, naming changes in speed also likely influenced students, like Kendra and George, who noticed patterns in the changes in distance instead of patterns in the speeds. This is because by using this new name, the teacher was promoting the
The practice of looking for patterns in higher-order quantities. Since changes in distances are higher-order quantities than accumulated distances, naming changes in speed may have influenced students to look for patterns in the changes in distance. Kendra and George seemed to be influenced because their center of focus shifted toward partially-quantitative and quantitative patterns in the changes in distance immediately following the discussion on changes in speed as described above (see Figures 6.14 and 6.15 for their Lesson 10 diagrams).

The second naming event was when the teacher introduced the name \textit{acceleration}. This name was introduced during the beginning of Lesson 14. Prior to the introduction of this name, the class had just produced a table showing the exact
speeds for the remote-control car at 1-second intervals (see Figure 6.19b). The teacher then introduced acceleration:

T: Alright so we have all these ideas out, we haven’t really said what acceleration is other than to say that quadratic functions have to do with accelerating, so that’s the extra part of acceleration that we need to add in. So acceleration isn’t just about change in speed, it’s also about the change in time that goes with it. So when we’re talking about acceleration we want to include both the change in speed and the change in time. Kendra found that there was a change in speed of 4 yards per second and she found that there was a change in time of 2 seconds. So how would you say that now?

K: Umm well, 4 yards per second in 2 seconds.

T: Kendra noticed that a change of speed of 4 yards per second was associated with a change in time of 2 seconds. What would dividing now accomplish for us? Jenn?

J: That in 1 second, it would be 2 yards per second.

T: So in 1 second the speed would change 2 yards per second, at least in the example Kendra gave us. Well, what do you notice about other parts of this table and acceleration? What do you notice about the change in speed and the change in time? George?

G: It’s 2 yards every second.

T: It’s 2 yards every second. Nicholas actually said that a few minutes ago. It’s not just 2 yards a second for where Kendra found it, it’s 2 yards per second in every second, and how do you see that George?

G: Umm because every second, if you go down the table, every second it I mean every speed, the yard increases by 2.

T: . . . So on this table, every second the speed changes by 2 yards per second. And that is the third thing that defines quadratics. In every quadratic function, the speed changes by the same amount every second or every interval of time, it’s the same. The change in speed is the same for the whole table. In fact we say that the change in speed every second is constant. Or we say that the acceleration is constant.

Although the teacher introduced the term acceleration in this episode, the term had already been briefly mentioned in earlier lessons. However, the way that the teacher defined acceleration as comprised of change in speed and change in time was new to
the students. What seemed especially relevant for the emergence of CoF 2 was that
the teacher linked acceleration with the changes in speed and time for the entire table
(e.g., “Well, what do you notice about other parts of this table and acceleration? So
on this table, every second the speed changes by 2 yards per second”). In other words,
acceleration directed attention to the entire set of changes in speed.

In the subsequent Lesson 14 activity several students appeared to be influenced
because they began to look at the set of changes in distance or the set of speeds. For
example, when Jenn was working on the rocket task (see Figure 6.17 for the data), she
recorded the speeds, 4 mi/min, 12 mi/min, 20 mi/min, 28 mi/min, 36 mi/min, 44
mi/min and 52 mi/min in a speed-time table. Then, when Kendra asked her about the
pattern she said, “Yeah, adding 8 to all of them . . . a change in speed is 8 miles every
1 minute I think . . . 8 miles because when you add it, 8 miles to this [points at 4
mi/min in table] you, 8 miles to 4, you get 12 miles [points at 12 mi/min in table] and
so on [sweeps pen down table].” Jenn’s use of the words “all of them” and “every
minute” suggests that she was not just looking at a particular change in speed but at all
the changes in speed.

As stated above, these two lessons were important with respect to CoF 2
because in Lesson 10 Kendra’s, George’s and Nicholas’ center of focus with respect to
noticing patterns in the changes in distance shifted and in Lesson 14, Kendra’s,
George’s, Jenn’s and Peter’s center of focus shifted (see Figure 6.11). Thus, evidence
suggests that the act of naming changes in speed and naming acceleration contributed
to the emergence of CoF 2.
Quantitative dialogue. The second kind of focusing interaction that appeared to directly influence the emergence of CoF 2 was quantitative dialogue. As described above, quantitative dialogue was used by the teacher to direct students’ attention toward linking numeric values with the relevant quantities such as the changes in distance and the changes in time which would then also increase the likelihood that students noticed patterns in the changes in distance. Furthermore, in directing student attention to quantities, the use of quantitative dialogue would also increase the likelihood that students notice quantitative instead of numeric patterns. Therefore, the evidence presented earlier that showed that quantitative dialogue contributed to the emergence of CoF 1, also likely contributed to the emergence of CoF 2.

However, the quantitative dialogue was not only used to help students link numeric values to the relevant quantities in the situation but also to help students sustain a focus on talking in terms of quantities and on using appropriate units as the mathematical discussions shifted from lower to higher order quantities. Specifically, the students’ beginning attempts to make sense of the new terms changes in speed and acceleration resulted in quantitative dialogue that directed students’ attention from lower-order quantities to higher-order quantities.

The reason that focusing students’ attention from lower- to higher-order quantities became an issue was because students often abbreviated their talk about the new terms changes in speed and acceleration so that changes in speed sounded like changes in distance and acceleration sounded like speed. For example, in the introductory discussion about changes in speed (Lesson 10), the teacher asked:
T: Can anybody else see another place where there is a change in speed?
B: 6 and 10.
T: So, from 6 yards per second to 10 yards per second would be a change in speed, and what would it be?
B: By 4 yards.

In this excerpt, the teachers’ quantitative dialogue appeared to move Brady from numeric talk (i.e., “6 and 10”) to quantitative talk (i.e., “by 4 yards”). However, Brady abbreviated the change in speed of 4 yd/s to a change in distance of 4 yd. In response the teacher used quantitative dialogue to try to help students see that change in speed was a quantity composed of changes in distance and time:

T: It’d be umm OK. So this is an important point. Say it again what you said the change in speed is up here [points to Nicholas].
N: The change in speed is 4 yards a second.
T: He said it’s 4 yards a second. He didn’t say it’s 4 yards, because we’re talking about change in speed [emphasizes speed].
B: Oh, 4 yards per second.
T: So the change in speed would be 4 yards per second.

In this transcript, the teacher used quantitative dialogue to highlight how Nicholas’ report of changes in speed was a different quantity from that which Brady reported. Specifically, the teacher emphasized the use of speed units for change in speed (i.e., “it’s 4 yards a second”) and contrasted that with using distance units for change in speed (i.e., “he didn’t say it’s 4 yards, because we’re talking about change in speed”).

The fact that Nicholas, George and Kendra exhibited a shift during this lesson (i.e., Lesson 10) supports the claim that the quantitative dialogue present in this example contributed to the emergence of CoF 2.

A similar episode occurred during the introduction of the term acceleration.
(Lesson 14) after the teacher had written down the particular change in speed $6 \text{ yd/sec} - 2 \text{ yd/sec} = 4 \text{ yd/sec}$ under the heading Change in speed and had also written down the particular change in time interval $6.5 \text{ s} - 4.5 \text{ s} = 2 \text{ s}$ under the heading Change in time.

The teacher (T) and Kendra (K) then had the following exchange:

T: Anybody want to say what the acceleration is, it has to do with change in speed and the change in time as well . . . Kendra?

K: 4 yards in 2 seconds.

T: Yeah that’s pretty close. This is not 4 yards [points at 4 yd/s change in speed], because that would be a distance so then this would be a change in distance [points at heading “Change in speed”] but we’re talking about change in speed and change in time. So see if you can revise what you said.

K: The changes in speed is 4 yards per second and the change in time is 2 seconds.

T: Nice. So the change in speed was 4 yards per second and the change in time was 2 seconds.

In this transcript, the teacher emphasized that acceleration is composed of a change in speed and a change in time (i.e., “it has to do with change in speed and the change in time”). He then explained how Kendra’s report of acceleration would be a change in distance and change in time (i.e., “This is not 4 yards because that would be a distance”). The result was that Kendra changed her report of speed from “4 yards” to “the changes in speed is 4 yards per second and the change in time is 2 seconds.” This and other instances of quantitative talk, in which the teacher focused students toward higher-order quantities likely influenced the shift in noticing toward noticing patterns in the higher-order quantities (i.e., the changes in the changes in distance and the changes in speed). Once again, because Kendra, George and Jenn exhibited a shift in their center of focus with respect to quantitative patterns in Lesson 14, it supports the
claim that the quantitative dialogue around the term acceleration during Lesson 14 contributed to the emergence of CoF 2.

Sometimes however, instead of abbreviating higher-order quantities into lower-order quantities, students did the opposite, namely substituting the newly-learned-about higher-order quantities for lower-order quantities (e.g., interpreting a speed as a change in speed). When this happened, the teacher used quantitative talk to help students sort out the new terms from the old ones. For example, after Kendra (K) had written down the list of fractions made up of changes in distance and changes in time for the Lesson 14 rocket diagram (see Figure 6.20):

| T:  | So now you’ve written them, now these are no longer changes in distance these are, they’re written as s-- |
| K:  | S--change in speed. |
| T:  | Umm well you have miles per second so what would that be? It could be change in speed or it could be . . . |
| K:  | Change in . . . |
| T:  | Well let’s see where did you get the 12 from? |
| K:  | Here 1 to 2 [points from beginning to end of first interval on her diagram]. |
| T:  | And where did you get the per second from? |
| K:  | Umm because it’s 1 from here to here [sweeps finger back and forth between 4 mi and 16 mi on number line]. |
| T:  | OK if it was a speed then you would have a change in distance and a change in time put together. If it was a change in speed you would have subtracted two speeds. So did you subtract two speeds to get 12 miles per second? |
| K:  | No. |
| T:  | No. So then it’s just a speed. So these are speeds. Nice. So could you also find the changes in speed? |
| K:  | Well yeah it changes every second it changes also if . . . should I find a pattern between these [points at list of speeds] |
| T:  | If you see one, well you could just write what the changes in speed are like between here and here what’s the change in speed, between here and here [points from one speed to another]. |
In this excerpt, Kendra confused the speeds (e.g. 12 mi/min) with the new quantity changes in speed. The teacher used quantitative dialogue to help Kendra recall how she had found the speeds and how finding changes in speed is different from finding speeds. Subsequent to this discussion, Kendra wrote down the following statement about the speeds for the rocket: “I noticed that the patterns in the speed changes, so you just add 8 miles for every minute to get the next speed.” Thus, it appeared that the quantitative talk that the teacher had just used to help Kendra sort out the distinction between speeds and changes in speeds may have helped Kendra interpret the pattern in her list of speeds.

**Mathematical Tasks**

The features of the mathematical tasks, described in the section on CoF 1, which contributed to students noticing both the changes in distance and changes in time, likely also contributed to students noticing patterns in the changes in distance. In other words, the tasks that directly targeted changes in distance and time and the tasks that involved drawing diagrams of varying relationships between changing
quantities, at the very least, contributed indirectly to the emergence of CoF 2. However, there appeared to be three features of the mathematical tasks that contributed more directly to the emergence of CoF 2: (a) the gradual narrowing of the size of the spacing of the independent variable (i.e., the Lesson 8-10 remote-control car task sequence), (b) necessitating the generation of additional data points (e.g., the Lesson 14 rockets in space task), and (c) a complex relationship between distance and time (i.e., the Lesson 16 falling off a cliff task). As described above, each of these three features was associated with a task in which there was a shift in the patterns that students noticed. For the Lesson 8-10 sequence, it was Kendra, George and Nicholas who exhibited a shift, for the Lesson 14 task, it was Kendra, George, Jenn and Peter and for the Lesson 16 task it was Brady and Armando.

**Tasks that narrow the spacing of the time intervals.** Lessons 8-10 contained three tasks in which the spacing of the independent variable for the same quadratic distance-time function was gradually narrowed from 4- to 2- to 1-second intervals. Otherwise the three tasks were identical in that each was presented as tabular data and each included the following prompt: “The table shows the distance and time for a remote-controlled car. Draw a picture showing the car’s speed in terms of changes in distance and changes in time.” This sequence of tasks appeared to be influential in shaping CoF 2 because, in the culminating task (i.e., the Lesson 10 task), Kendra exhibited a shift toward noticing partially-quantitative patterns in the changes in distance, George shifted toward noticing quantitative patterns in the changes in distance and Nickolas shifted toward noticing quantitative patterns in the speeds.

Because the shifts occurred during Lesson 10, the reader might wonder if
perhaps the Lesson 10 task alone was the influential task, not the sequence of three
tasks. In other words, since Lesson 10 differed from Lessons 8 and 9 only in that the
times were spaced in 1-second intervals, perhaps it was that difference that influenced
what Kendra, George and Nicholas noticed. However, contrary evidence suggests that
this was not the case. This evidence comes from Lesson 2, in which students were
exploring the computer simulation of the swimming blue fish (see Figure 6.2 for a
screen shot). That task also involved times that were spaced 1-second apart because
students set the marks to drop every second. However, none of the students appeared
to notice patterns—numeric or quantitative—in the changes in distance for that task.
Therefore, evidence suggests that it wasn’t the presentation of the 1-second intervals
that prompted the shifts. However, the question that remains is why the sequence of
three tasks, which were identical except for the time intervals that gradually narrowed
from task to task, influenced the shift toward noticing patterns in the changes in
distance.

One hypothesis is that the gradual increase in the level of detail for the changes
in distance from task to task may have served to focus students’ attention toward the
pattern in the changes in distance. By *gradual increase in the level of detail*, I mean
that for each successive task more changes in distance were available, which also
meant that the total distance was more-finely partitioned. For the Lesson 8 task, there
were only two changes in distance, 16 yd and 48 yd. For the Lesson 9 task, the two
changes in distance transformed into the following four more detailed changes in
distance, 4 yd, 12 yd, 20 yd and 28 yd. For Lesson 10, the four changes in distance
transformed into the following eight even more detailed changes in distance, 1 yd, 3
yd, 5 yd, 7 yd, 9 yd, 11, yd, 13 yd and 15 yd. Perhaps this gradual progression of finer and finer partitions made the patterns in the changes in distance stand out. Evidence from Kendra, George and Nicholas suggests it did.

**Tasks that necessitate the generation of additional data points.** Lesson 14 contained a task in which students needed to generate additional data points. The task involved students drawing a diagram of the quadratic function distance-time data representing a rocket traveling in space (see Figure 6.17 for the data set). The prompt for this task stated the following:

A rocket in space turned on its rocket engines. The rocket’s engines caused the rocket’s speed to increase according to a quadratic distance-time function. Draw the first 7 minutes of rocket’s journey, using the pattern you noticed when you drew the first 3 seconds. Label all distances and times, changes in distances and times, speeds and changes in speed.

This task seemed to be important in shaping CoF 2 because several students exhibited shifts in what they noticed during this task. Specifically, Kendra exhibited a shift toward noticing quantitative patterns in the changes in distance. Peter shifted toward noticing partially-quantitative patterns in the changes in distance. George and Jenn shifted toward noticing quantitative patterns in the speeds.

This was the first task during the instructional intervention in which students were required to find additional data points. Students were given the distances the rocket travelled for the first three minutes of a flight and they were instructed to draw a diagram for the first seven minutes by extending the pattern that they noticed in the first three minutes of data.

George appeared to use the patterns in the changes in distance to determine the
additional points that he needed. For example, when George, who was still in the process of drawing his diagram, was asked by Peter what the pattern was and he said:

Well here look, the change in distance, it goes up by 8 every time, starts at 4 [points to 4 mi change in distance at start of his diagram] then it goes 8 [points at 12 mi change in distance in diagram] then it goes 8 [points at 20 mi change in distance in his diagram] then it goes 8 [sweeps pen over the rest of his uncompleted diagram].

Thus, George appeared to be using the 8 mi constant change in the changes in distance to find the other changes in distance so that he could draw the entire 7 min flight.

Jenn, Kendra and Peter initially did not appear to use patterns in the changes in distance to draw their diagrams. Instead, all three students articulated other patterns, not associated with the changes in distance when drawing their diagrams. For example, Jenn said:

I noticed that this one is like 4 and 1 [points at 4 mi and 1 min in her table], 8 times 2 [points at 16 mi and 2 min in her table], like 4, 8, 12, 16 you know it’s timsing, because 12 times 3 [points at 36 mi and 3 min in her table], 8 times 2 [points at 16 mi and 2 min in her table], 4 times 1 [points at 4 mi and 1 min in her table].

It was not until after their diagrams had been completed that these students provided evidence of noticing patterns in the changes in distance or in the speeds. Nevertheless, because generating extra data was the only significant difference between this task and the Lesson 10 remote-control car task, it suggests that generating data points influenced these four students with respect to the patterns that they noticed.

**Tasks with a complex quadratic distance and time relationship.** Lesson 16 contained a task in which the relationship between distance and time was complex. By complex, I mean a quadratic distance-time relationship that involved a dilation, rotation and translation of the $d=t^2$ relationship that students were most familiar with.
The task involved students again drawing a diagram of quadratic function distance-time data. For this task, the context was Wile E. Coyote free-falling off a cliff and at the same time a bee flying down the cliff at a constant speed (see Figure 6.21 for the data set). The prompt for this task stated the following:

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>Distance above ground (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>576</td>
</tr>
<tr>
<td>1</td>
<td>567</td>
</tr>
<tr>
<td>2</td>
<td>540</td>
</tr>
<tr>
<td>3</td>
<td>495</td>
</tr>
</tbody>
</table>

Suppose Wile E. Coyote jumped off a cliff that was 576 feet high and dropped according to a quadratic distance-time function shown in the table. Meanwhile, Bee flew straight down the cliff at a constant speed of 72 ft/s thinking he was racing Wile E. Draw a picture showing Wile E. and Bee’s motion side-by-side, including labels for distance, time, changes in distance, changes in time, speed, changes in speed and acceleration. Continue the diagram until they hit the ground.

Brady and Armando exhibited a shift toward noticing numerical patterns in the changes in distance on this task.

The quadratic function that represented this task ($d=576-9t^2$) was sufficiently complex (i.e., a dilation, rotation and translation of $d=t^2$) that generating additional data points by looking for a pattern in how the time was related to the distance was likely too difficult for the middle school students participating in this study to
determine. However, patterns in the distance and in the changes in distance were simpler and therefore more accessible. For example, Brady noticed the pattern that the numerical values of the changes in distance were the product of 9 and successive odd numbers (i.e., $9 \times 1$, $9 \times 3$, $9 \times 5$, etc) and Armando noticed that “it’s going down by 18 . . . 9 then minus 18 then minus 18 then minus 18” (i.e., that the numerical value of the change in the change in distance was 18).

It appeared that this task helped shift Brady’s and Armando’s center of focus toward noticing patterns in the changes in distance. This was important because up until that point, they had provided little evidence of noticing patterns in the changes in distance. In other words, these two students were lagging behind the other students with respect to shifting from noticing patterns not associated with changes in distance to noticing patterns associated with changes in distance. However, the new center of focus that emerged during this task was numeric instead of quantitative. This result shows that even though the task necessitated that students consider the patterns in the distances, students did not necessarily consider quantitative patterns. In order for Brady and Armando to notice quantitative patterns, they would have likely needed more activities like the remote-control car task and rocket task which seemed to promoted noticing quantitative patterns for the other students.

The results from the analysis of all three tasks suggests that perhaps the falling off a cliff task, which seemed to promote noticing numerical patterns, should have been done prior to the remote-control car and rocket tasks, which seemed to promote noticing quantitative patterns. Perhaps then more students would have noticed patterns in the changes in distance sooner and more students would have benefited
from the tasks that promoted noticing quantitative patterns.

**Nature of the Mathematical Activity**

The three classroom practices (Cobb & Yackel, 1996) that were shown to contribute to the emergence of CoF 1, at least indirectly also contributed to the emergence of CoF 2. These practices included students (a) presenting their diagrams in class, (b) noticing features of others’ diagrams, and (c) asking each other *Can you explain why?* questions. Because these practices promoted a focus on changes in distance and time (as argued previously), they would increase the likelihood that students noticed patterns in the changes in distance. However, when the particular participatory structures from Lesson 10, 14 and 16 were considered (i.e., the lessons in which the shifts toward noticing patterns in the changes in distance or in the speeds occurred), evidence was found that suggested that presenting work and noticing features in others’ diagrams more directly influenced the emergence of CoF 2.

**The practice of presenting student work.** As described above, a regular part of the class routine was for students to present their work. One aspect of this classroom practice that appeared to directly influence the emergence of CoF 2 was that the patterns that students noticed became public. This likely helped students who were not yet noticing patterns in the changes in distance to shift toward noticing them and it likely helped students who were noticing partially-quantitative patterns to shift to noticing quantitative patterns.

For example, during Lesson 15, when Jenn (J) was presenting her rocket diagram, she had an exchange with the teacher (T) in which she publically explained the pattern she noticed:
T: OK so umm my question focuses on the red pen that she [Jenn] wrote in there. What does the red pen . . . ?
J: Speed.
T: Those are speed. OK so if we look at those speeds, Jenn, you, did you notice a pattern in those speeds?
J: They’re all going up by 8.
T: 8 what?
J: 8 miles per minute.
T: They’re always going up 8 miles per minute. So if we think now to acceleration, what does that suggest to you about the acceleration? Say it again what happened with the speeds?
J: It goes up 8 miles per minute.
T: It goes 8, it goes up 8 miles per minute.

In this excerpt, Jenn made public the pattern that she noticed. Thus, other students were able to hear and see what she noticed and compare that to what they had noticed. This episode also provided the teacher with another opportunity to elucidate the quantities (i.e., speed and acceleration) and the appropriate units involved in the pattern Jenn was noticing (i.e., miles per minute).

The practice of noticing features of others’ diagrams. The practice of noticing features of others’ diagrams was closely related to the practice of presenting student work because much of the time noticing features of others’ work occurred when students were presenting their work publically. This practice likely contributed to the emergence of CoF 2 because students noticed patterns in other students’ work that they might not have noticed in their own work. Alternately, they may have noticed the same thing in someone else’s work that they noticed in their work which reinforced what they already noticed in their own work.

An example of a student noticing a pattern in someone else’s work came during Lesson 9 when Brady was presenting his diagram for the remote-control car task for data presented in 4-second intervals (see Figure 4.22 for diagram). Before
Brady presented his diagram, the teacher (T) asked Kendra (K) to describe what she saw in the diagram:

T: Alright, so before he explains his picture, uhh, Kendra can you explain what you see in his drawing?
K: Umm, he put the change in time like I think it’s like the box thing and then the change in distance on the bottom where 0 to 16 yards. And then over here he did the same.
T: OK anything else?
K: The loops, he they’re smaller in the beginning and bigger at the last 4 seconds to represent the how the time or the distance by time changes.

In this excerpt, Kendra appeared to compare the two sizes of changes in distance. In particular, she compared the 4 yd sections in the beginning of Brady’s diagram, which represented the first four changes in distance for each second, to the 12 yd sections at the end of the diagram, which represented the last four changes in distance for each second. Thus, despite Kendra not using the language of changes in distance and the units of yards, she compared two sizes of changes in distance, which is an important aspect of noticing patterns in the changes in distance. This comparison of two sizes of changes in distance was done publically for all students in the class to see. It was the very next lesson (Lesson 10) where Kendra’s, George’s and Nicholas’ center of focus shifted toward noticing patterns in the changes in distance or the speeds. It is possible that Kendra’s public comparison of two sizes of changes in distance contributed to those shifts.

Another example of noticing features in others’ work occurred in Lesson 10 when Jenn was presenting her remote-control diagram for data presented in 1-second intervals (see Figure 5.4). Before Jenn (J) presented, the teacher (t) asked Armando (A) to explain what he noticed in Jenn’s diagram:
T: Alright, so before she [Jenn] even says anything, Armando do you want to explain what you see in her drawing? She has a few scribbled out things, but what do you see there?
A: The yards are that every second it’s adding 2 yards and . . .
T: What are those, those things, what do we call those?
J: Change of distance.
T: Change in distance is adding 2 yards, OK, anything else?
A: That it’s speeding up all the time.

In this excerpt, Armando noticed that the yards in Jenn’s diagram were “every second . . . adding 2 yards.” The teacher and Jenn had to clarify that he was noticing the pattern that the changes in distance were adding by 2 yards each second (note that Armando was not coded as having a shift in focus here because in several subsequent episodes he focused on numeric patterns not associated with changes in distance).

Nevertheless, this public noticing of a pattern likely influenced what patterns other students noticed.

A third example of noticing in others’ work occurred during Lesson 15 when Jenn was presenting her diagram for the rocket task. The teacher asked Kendra what she saw in Jenn’s diagram:

T: OK and Kendra what do you, could you kind of repeat what George said or what do you see in the pattern of the speeds, those red umm numbers there?
K: Well, umm I also see that it goes 8 miles per minute.
T: And could you give an example of where you’re looking to see that?
K: Like around 28 miles per minute to 36 miles per minute, you add 8, well like from 28 you add 8 and you get 36 and from 36 miles per minute you add 8 miles and you get 44.
T: OK so it looks like we’re seeing the same pattern of this 8 miles per minute.

In this excerpt, Kendra noticed a pattern in the speeds, despite being somewhat unsure of how to report the pattern (i.e., she used both miles and miles per minute to report
changes in speed). This practice of publically noticing features in others’ diagrams likely influenced both the student who was doing the public noticing—Kendra, in this example—as well as the other students who were.

**CoF 3: Establishment and Origins**

In this section, I present evidence supporting the claim that a new center of focus emerged during the instructional intervention with respect to division. In particular, students went from noticing division as an arithmetic calculation (Clark et al., 2008; Slesnick, 1982) to noticing that division partitions continuous quantities into equal sections (Fischbein et al., 1985). Furthermore, I provide evidence that (a) the discursive practices of quantitative dialogue and highlighting working together, (b) the features of the mathematical tasks involving exploring division of continuous quantities and exploring division of composed units, and (c) the participatory structures of presenting diagrams and noticing features of others’ diagrams influenced the emergence of CoF 3.

There were three centers of focus involved in this shift (see Figure 6.22). The initial center of focus was on *division as an arithmetic calculation*. The next (transitional) center of focus was on *division as sharing* a set of discrete objects equally among a set number of groups. The final center of focus was on *division as partitioning* or sectioning a composed unit made up of continuous quantities into a set number of equal sections.

The *division as sharing* and the *division as partitioning* centers of focus were both based on a partitive model of division. A *partitive model* of division involves
conceiving of a dividend as being cut up into a set number of equal groups. For example, from a partitive model of division, $20 \div 4$ would mean sharing 20 cookies among 4 people to form 4 equal groups of 5 cookies each (whereas from a quotative model $20 \div 4$ could mean packing 20 cookies into packs of 4 to form 5 equal groups of 4 cookies each). The division as sharing and division as partitioning may appear to the reader as identical. However, evidence will show that there were important differences for students between thinking about the actions of sharing a discrete number of objects and cutting a continuous quantity. Therefore, I claim that the emergence of CoF 3 involved two shifts in focus, one shift toward noticing that dividing shares discrete objects equally among a set number of groups and a second shift toward noticing that dividing partitions a composed unit made up of continuous quantities into equal sections.

The way these centers of focus emerged separates the lessons into three clusters. During Lessons 1-4, the focus was on division as an arithmetic calculation, during Lessons 5 and 6, the focus was on division as sharing discrete objects and
during Lessons 7-16, the focus was on division as partitioning a composed unit made up of continuous quantities. In the next section, evidence will be provided that shows when each center of focus was at play and what was being noticed. Then, the focusing interactions, the features of the mathematical tasks and the nature of the mathematical activity that influenced the shifts in focus will be discussed in turn.

**Initial Focus: Division as Arithmetic Calculation**

As shown in Figure 6.22, during Lessons 1-4, students appeared to focus primarily on division as an arithmetic operation. Division was often used to find speeds (i.e., distance divided by time) or to find number patterns as students explored distance-time contexts. However, when students used dividing, they did not show evidence of understanding the meaning of the quotients they had produced or an understanding for why dividing produced the appropriate quotients.

During the beginning of Lesson 5, more definitive evidence emerged that showed that students noticed division as an arithmetic procedure. The activity in which this evidence was observed was when the teacher asked students to compare moving 10 cm in 4 s with moving 30 cm in 12 s, in the context of a race between a rabbit and a turtle. All students used division to show that these two distance-time pairs represented the same fastness (the word speed was defined in a later lesson). Some students divided 10 by 4 and 30 by 12 to show that both equaled 2.5 cm/s. Other students divided 30 cm and 12 s by 3 to show that it was the same as 10 cm in 4 s.

Each student presented their ideas in front of the class. When they had finished, the teacher asked them what division meant or why they used division.
Brady was asked if there was something in his drawing “that shows that 30 centimeters and 12 seconds divided by 3 does equal 10 centimeters and 4 seconds.” He said, “I didn’t write that down in my drawing,” and instead wrote down the long division arithmetic calculations for $30 \div 3$ and $12 \div 3$. Peter was asked, “So your motivation was to find how many centimeters each second? And so you divided 30 by 12. And anything you can say about why dividing 30 [cm] by 12 [sec] would give you how many centimeters each second?” Peter responded “I’m not sure . . . I think because dividing centimeters by the seconds will just remove it . . . give you a lower fraction.” Kendra was asked, “And what does dividing, what does dividing do for, when you take 10 centimeters and divide it by 4 seconds, what does it do?” She responded, “It like combines them, so 10 centimeters divided by 4 seconds would equal 2.5 centimeters per second.” George was asked, “Can you explain why 10 centimeters in 4 seconds is equal to 2.5 centimeters over 1 second?” He replied, “Uhh, well so you have uhh 10 over 4 and so you divide it by 2 over 2 equals 5/2 and if you divide 5/2 by 2/2 it gives 2.5 over 1.” Notice that in each of these examples, the students’ focus was either on the arithmetic calculation of dividing, like Brady who reproduced the long division calculations, or on procedural explanations, like Peter, who explained dividing as making a lower fraction, rather than on the context of rabbits and turtles running certain distances in certain times.

**Transitional Focus: Division as Sharing Discrete Objects**

During the second half of Lesson 5, the students’ center of focus appeared to shift toward noticing that division shares discrete objects equally among a given
number groups. The shift occurred immediately after the activity described above in which the teacher asked students to present diagrams that they had produced, showing that moving 10 cm in 4 s was the same fastness as moving 30 cm in 12 s. After all students had presented their ideas, the teacher said:

I noticed that in several of the drawings we have really good representations of the rabbit and turtle going the same fastness. But this question kind of keeps coming up about what division or what also what multiplication is kind of showing in the drawing and we have some good ideas, but I think we need to just talk about division just a little bit.

He then posed the question, “So let me take just for a second, a very simple problem like 6 cookies divided by 3 . . . Can anybody think of a real world situation that this could apply to?” George immediately proposed a story problem consistent with the partitive model of division: “You have 6 cookies and 3 people want to eat them.”

With the teacher’s help, a quotative model of division interpretation was also elicited (i.e., dividing as packing up one group at a time). However, even after the quotative model was introduced into the discussion, students appeared to focus on division sharing equally among groups (as opposed to packing one group at a time).

Students quickly provided an interpretation for the division problem from a partitive perspective. For example, Kendra said, “There was 6 cookies, and each person will eat 2.” Next, the teacher asked for volunteers to draw diagrams to show the meaning of division. George made a diagram showing the partitive interpretation. He explained his drawing as, “A circle is a cookie so [draws 6 circles] . . . and then this is a person so [draws 3 happy faces], so then each person gets 2 cookies [he groups 2 cookies with each face; see Figure 4.26a].” All the students appeared to
know this interpretation of division already because there was agreement among all seven students that this was correct. In other words, it did not appear that students were learning this for the first time.

At the end of this division discussion, students’ center of focus appeared to have shifted away from division as an arithmetic calculation because they no longer provided calculational or procedural explanations for division. For example, when they were asked what dividing 30 cm by 12 s meant, Brady explained that the 12 was like “12 groups” and the answer of 2.5 was like “2.5 cookies.” When the teacher then asked for a volunteer to come to the whiteboard to draw a representation of 30 ft divided by 12 s, Nicholas volunteered. He drew 30 tally marks horizontally across the board. The teacher asked him, “What do those represent?” Nicholas responded, “It’s the 30 feet . . . So, if you put it into 12 groups. You’d have . . . it’s 2.5.” Thus, both Brady and Nicholas appeared to be noticing that division shares the 30 ft among 12 groups and that in each group is 2.5 ft. This is consistent with a partitive model of division. Other students gave similar explanations.

One additional feature of the new center of focus emerged that has not yet been mentioned. This additional feature revealed itself during a subsequent discussion the next day (i.e., during Lesson 6). In particular, the teacher asked Brady to display his diagram showing that moving 30 cm in 12 s is the same fastness as moving 10 cm in 4 s (see Figure 6.23). Brady’s diagram was simply a number line with three tick marks, the left tick mark was labeled 10 cm and 4 sec, the center tick mark was labeled 20 cm and 8 s and the right tick mark was labeled 30 cm and 12 sec. The teacher raised a similar question that had been discussed the previous day, namely, “Can anybody see
the 30 centimeters and the 12 [seconds] divided by 3, just like 30 cookies . . . you
could divide 30 cookies by 3.” However, there were two subtle but important
differences between this question and the previous division questions the teacher had
asked about during Lesson 5. The first difference was that the problem he posed
asked about dividing a composed unit, namely dividing the unit composed of 30 cm
and 12 s. This was different than the questions asked about a single quantity (e.g.,
dividing 6 cookies among 3 people) or about dividing 30 cm by 12 s. The second
difference was that the diagram in which the students were supposed to respond to the
problem, displayed a continuous quantity rather than the discrete quantities displayed
in the George’s and Jenn’s cookie diagrams (i.e., 6 discrete cookies) and in Nicholas’
tally marks diagram (i.e., 30 discrete marks representing a continuous 30 cm).

Students responded with explanations that involved grouping, but they did not
seem to notice that dividing 30 cm and 12 s by 3 would partition the number line into
three equal sections. For example, Armando responded by saying, “So divided into
three sections . . . This is the first section, this is the second and this is the third.”
However, as he said spoke about each section, he underlined each of the three tick marks on Brady’s diagram as if to indicate that the tick marks were the three groups (see Figure 6.24a). Then the teacher asked, “Is there a way to circle the 30 centimeters and, 30 centimeters and the 12 seconds into 3 groups?” Instead of circling the three sections of the number line, Brady hesitantly circled the labels $30 \text{ cm}$ and $12 \text{ sec}$ with a big circle (see Figure 6.24b). Because there were no discrete objects to represent the centimeters or seconds in Brady’s diagram (i.e., he used a continuous line to represent 30 cm and 12 s), students seemed to either focus on other discrete objects like the tick marks (Armando underlined tick marks) or like the quantity labels (Brady circled labels). Finally, the teacher asked, “Is there anything on the number line that shows the 30 divided by 3, what that represents?” To this, Brady responded, “Can I draw bumps?” and he drew four bumps between each set of tick marks (see Figure 6.24c). This again appeared to be an attempt to create a discrete representation for continuous quantities.

In these examples, the students appear to be trying to group objects but didn’t seem to see that there were three 10 cm sections in the 30 cm length. Noticing that dividing the 30 cm and 12 s composed unit by 3 partitions the number line in Brady’s diagram into three sections, each with 10 cm and 4 s, appeared to be challenging for the students given their current center of focus.

CoF 3: Partitioning Continuous Quantities

Near the end of Lesson 6, the students’ center of focus with respect to division shifted a second time, this time to seeing division as partitioning a composed unit made up of continuous quantities into equal sections (CoF 3). The emergence of CoF
3 involved an episode in which students struggled to see division as partitioning continuous quantities. However the goal of this section is to show that CoF 3 did get
established. Therefore, a description of the struggle involved in the emergence of CoF 3 will be saved for the focusing interactions, the features of the tasks and the nature of the activities sections.

By Lesson 7, CoF 3 appeared to be well-established, because students referred to division in ways that are consistent with partitioning composed units made up of continuous quantities into equal sections. For example, during Lesson 7 the teacher showed students enlargements of some drawings they had made when they were exploring the orange fish’s computer animation (see Figure 6.2 for a screen shot of the computer animation and Figure 6.5b for a sample diagram). Recall that the fish swam in three sections: (a) a 15 ft in 3 s section, (b) a 2 ft in 2 s section, and (c) an 18 ft in 6 s section. The teacher first showed an enlargement of part of Nicholas’ drawing (see Figure 6.25a), in which the teacher had highlighted with arrows 2 ft and 2 s, and 1 ft and 1 s, and he asked, “Could someone explain why dividing would show that 2 feet in 2 seconds is the same as 1 foot in 1 second?” Armando, among others, responded saying, “You separate the 2 feet into two different groups . . . [of] 1 feet and 1 second.” Thus, Armando appeared to notice that he was dividing a composed unit of 2 ft and 2 s. Also, Armando did not seem to be bothered by the picture which showed that the feet and seconds were represented as continuous quantities in Nicholas’ diagram.

Next the teacher showed an enlargement of part of Armando’s diagram, in which the teacher had highlighted with arrows 18 ft and 6 s and 6 ft and 2 s. The teacher asked, “So again my question is could someone explain how 18 feet in 6 seconds is the same as 6 feet in 2 seconds either with multiplication or division, you
have a choice.” Nicholas responded saying, “It’s 6 groups of 3 feet per second because there’s 6 bumps and each of them represents 3 feet in one.” Jenn responded to the same question with, “If we divided the 6 seconds into 3 groups . . . You divide the umm seconds by 3 getting 2 seconds in each and 6 feet.” Jenn also came up to the projected image and pointed out where the two groups after the first group of 6 ft and 2 s would be (i.e., she pointed to the 6-12 ft section and the 12-18 ft section).

Nicholas and Jenn, like Armando, noticed that they were dividing a composed unit because the groups that they partitioned also contained both parts of the composed unit. Also, neither of them seemed to have difficulty grouping the quantities, even though the feet and seconds were being represented in Armando’s diagram as
continuous quantities.

This new focus on division persisted throughout the rest of the instructional intervention. For example, during Lesson 9, Nicholas explained why dividing 16 cm in 4 s by 4 gave the number of yards per second:

Because 16 over 4, you can simplify to the 4 over 1 by dividing 16 by 4 you get 4, 4 equal groups of . . . 4 yard . . . 4 divided by 4 would be 1 so that would be 4 groups of 1 [second].

In other words, Nicholas partitioned the 16 yd into 4 equal groups of 4 yd and the 4 s into 4 equal groups of 1 s. All students provided evidence of CoF 3 from Lesson 7 on.

There was an episode at the end of Lesson 6 where students reverted back to focusing on procedures which will be discussed in the section on features of mathematical tasks.

Thus, the evidence suggests that the students had shifted from noticing that division shares a set of discrete objects (like cookies) equally among a set number of groups to noticing that division partitions a composed unit made up of continuous quantities into a set number of equal sections.

**Focusing Interactions**

The kinds of focusing interactions that seemed to be most influential in the emergence of CoF 3 were *quantitative dialogue* and *highlighting*. Not only did both play a pivotal role, but they appeared to be working together during the critical episode in the emergence of this new center of focus (i.e., Lesson 6). This is consistent with Goodwin (1994), who argues that different discursive practices can “mutually elaborate on each other” (p. 613). Therefore, rather than present evidence for each kind of focusing interaction separately, evidence will be presented for both
quantitative dialogue and highlighting as they occurred within the critical episode.

**Quantitative dialogue and highlighting.** Recall that *quantitative dialogue* was defined as dialogue that focuses attention on quantities (Lobato et al., 2011a). In contrast, *numeric dialogue* is verbal communication that centers on numerical values to the exclusion of the quantities that the numeric values represent and *object dialogue* is defined as “verbal communication that emphasizes physical objects or events . . . without additionally focusing on relationships among measurable attributes of these objects” (Lobato et al., 2011a, p. 36). Evidence will show that during the critical episode in which CoF 3 emerged, the teacher used quantitative dialogue to focus students’ attention toward linking numerical values to the relevant quantities in the situation. Also recall that *highlighting* is defined as employing particular methods to foreground certain features of the perceptual field (e.g., using sticky-notes, enlarging parts of photographs; Goodwin, 1994). Two important categories of highlighting that were relevant to the emergence of CoF 3 were written and gestural highlighting.

The critical episode occurred during Lesson 6, when the teacher referred to Brady’s diagram (see Figure 6.23) and posed the question, “Can anybody see the 30 centimeters and the 12 seconds divided by 3?” Armando was the first to respond: “So is it the numbers there on the top, 3 divided by 30 is 10 and 12 divided by 3?” [points toward 30÷3 and 12÷3 long division calculations that Brady had recorded next to his diagram] The teacher said, “Ah so you’re looking right here? [points to long division calculations] . . . But what about 30 centimeters and 12 seconds? If we divided those by three, what is it, can we see it in the picture?” In this part of the episode the teacher
used quantitative dialogue to shift students’ attention away from numerical calculations to the relevant quantities, the 30 cm and the 12 s.

Next, Armando said, “This is the first section [underlines the tick mark below 30 cm], this is the second [underlines the tick mark below 20 cm] and this is the third [underlines the tick mark below 10 cm]. Like if you divided the 30 and 12 by three it will go back to the first one” (see Figure 6.24a). The teacher gave the following response:

Is there something you can circle? Like here [points at cookie drawings] there is 6 cookies and we divided by 3, so we have 1, 2, 3 all six cookies are circled into 3 groups [gestures toward picture of six cookies grouped with three circles]. Is there a way to circle the 30 centimeters and, 30 centimeters and the 12 seconds into 3 groups?

This part of the episode involved quantitative dialogue and highlighting working together. The teacher appeared to use quantitative dialogue to press students to think about how to separate the relevant quantities into three groups, instead of just looking for three of any quantity (e.g., three tick marks). At the same time, he used gestures to highlight the picture of the six cookies grouped into equal groups.

Then Brady said, “circle these?” and he drew a circle around the labels 30 cm and 12 s. The teacher responded:

So let’s say we have 6 cookies [writes the words 6 cookies] . . . I’m not so much interested in . . . I could have circled the numbers . . . I’m just wondering, instead of circling say the ‘6 cookies’ [draws a circle around the words 6 cookies] we circled the picture part of it [points to picture of cookies] rather than the number itself.

This was again an instance of quantitative dialogue and highlighting working together. The teacher used quantitative dialogue to explain that the label 6 cookies was not the quantity, the cookies were the quantity. He reinforced this quantitative dialogue with
written highlighting (i.e., circling the words ‘6 cookies’) to differentiate between circling a label and grouping quantities with circles.

After Brady (B) drew the four bumps on the number line, apparently in an attempt to turn the continuous number line into discrete objects, as described above, the teacher (T) simplified the problem of dividing 30 cm and 12 s by 3 to dividing 30 cm by 3, and compared it with the discrete case of dividing 6 cookies by 3:

T: OK, well so if we have 30 divided by 3, that’s just kind of like 30 cookies divided by 3 people . . . If we had 30 cookies and three people, then we would wanna circle . . . ?
B: 10 cookies per person?
T: 10 per. So is there a way that we could do 10 per on your picture?
B: So these are . . . ?
T: Where is there 10 of something?
B: This? 10 centimeters? [writes 10 cm between 10 cm and 20 cm labels]

In this excerpt, the teacher used *quantitative dialogue* to connect the numerical value 10 with the relevant quantity centimeters. Specifically, when he said, “Where is there 10 of something?” he was likely trying to get Brady to step out of the analogy of dividing cookies among people and find 10 of the relevant quantity, in the context of distance-time relationships.

This part of the episode also contained an instance of *written highlighting*, when Brady wrote 10 cm between the 10 and 20 cm labels that were already on the diagram (see Figure 6.26). By annotating the diagram with the extra 10 cm label between the 10 and 20 cm labels, Brady likely drew attention away from the tick marks as representing three discrete groups and drew attention to the line between the first two tick marks as representing a section.
Next, Brady said, “It goes by 10, right?” and moved his pen rapidly back and forth between the 10 and 20 cm labels (see Figure 6.26). The teacher noticed Brady’s gesture and said, “OK so why don’t you circle that whole part that you were motioning with your pen?” Then, Brady drew a circle grouping the 10 to 20 cm labels together. In response, the teacher said:

OK there’s a, there’s a group of 10 because he kept going back and forth between here and here as 10 centimeters [teacher points back and forth between 10 and 20 cm label] so that’s a 10 per, like 10 cookies. But when we divide by 3, that means that we need 3 groups of those 10, so are there 2 more groups that you see there?

Then, Brady drew two additional circles, one circle to group the 20 and 30 cm labels together and one circle to group the beginning of the diagram and first 10 cm label together (Brady had to extend the beginning of the line to the left, and add a starting tick mark and $0 \sec$ label to create a section from 0-10 cm but he forgot to put in a 0 cm label; see Figure 6.27).
This part of the exchange contained instances of gestural and written highlighting. Brady’s and the teacher’s gestures highlighted the section between the first two tick marks. The written inscriptions (i.e., the circles around each section) highlighted the three equal sections that the continuous 30 cm of distance would be partitioned into if divided by 3. Thus, Brady’s and the teacher’s gestures and written inscriptions highlighted the three equal partitions of one of the two quantities involved in the 30 cm/12 s composed unit (i.e., the 30 cm but not the 12 s).

These instances of highlighting likely had an influence on the class as they watched Brady and the teacher. However, it also appeared that Brady was influenced by his own highlighting. Specifically, drawing the first circle appeared to help him notice where to draw the second and third circles because he did so less hesitantly than the first circle. This is consistent with Goodwin’s (1994) example of an archeologist outlining a post-mold in the dirt as being an example of highlighting that “further
reif[ies] the object that the archaeologist proposes to be visible” (p. 611).

The teacher then asked, “Could someone finish Brady’s picture and circle what does 12 divided by 3, the 12 seconds?” Jenn volunteered. She drew three circles, one around 0 sec and 4 sec, one around 4 sec and 8 sec and one around 8 sec and 12 sec (see Figure 6.28). Then she said:

So from 0 seconds to 4 seconds, it’s gonna be 4 whole seconds, and from 4 seconds to 8 seconds it’s gonna be 4 seconds and from 8 seconds to 12 seconds, it’s gonna be 4 seconds, making 3 groups of 4 seconds.

This part of the episode contained more written highlighting. However, Jenn’s highlighting partitioned the other quantity in the 30 cm/12 s composed unit, namely the 12 s, into three 4-second sections. Thus, together Brady’s and Jenn’s highlighting partitioned the composed unit made up of continuous quantities into equal sections.

**Mathematical Tasks**

Because much of the “same fastness” task from Lesson 6, in which CoF 3 emerged, was discussed as part of the presentation of the focusing interactions, the

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**Figure 6.28.** Jenn’s highlighting of pairs of time labels with circles.
details already mentioned will not be repeated in this section. However, there were
two important features of this task that warrant explicit mention, because these
features were likely influential in bringing about the shift in focus. The first feature
was that the teacher chose a diagram in which time and distance were represented as
continuous quantities. The second feature was that the task prompt involved dividing
a composed unit.

**Tasks about dividing in the context of continuous quantities.** As shown
above, the task during Lesson 6 centered on Brady’s diagram. His diagram was sparse
(i.e., a horizontal line and three tick marks with distance and time labels). One of the
consequences of the diagram being sparse was that there were few discrete objects.
The only discrete marks were the three tick marks indicating the 10, 20 and 30 cm
marks, while the continuous number line represented all the values from 0 to 30 cm.
As shown in the focusing interactions section, students seemed to have difficulty
partitioning the continuous line into 3 sections.

Consider the alternative, namely that the teacher could have used a student
diagram in which there were many discrete marks that could be grouped like the
cookies were grouped. Perhaps CoF 3 would not have emerged. In fact, there was
such a student diagram and the teacher did present it to the students, the day before he
presented Brady’s diagram. Not surprisingly, students had much less difficulty
thinking about grouping. In particular, near the end of Lesson 5, the teacher displayed
Peter’s diagram showing that moving 30 cm in 12 s was the same fastness as moving
10 cm in 4 s (see Figure 6.29). Rather than just having a few tick marks on a number
line, Peter’s diagram contained a tick mark (and cm label) for each whole number
from 1 to 30. Peter’s diagram also contained 12 little squares and 12 little triangles, which were situated above and below the number line and spaced 2.5 cm apart according to the tick marks. Each little square and triangle represented the position of rabbit and turtle each second respectively. Also, Peter’s diagram contained 11 arches connecting the rabbit’s positions (i.e., connecting the little squares) and 11 loops connecting turtle’s positions (i.e., connecting the little triangles; there were only 11 arches and loops because they started at 2.5 cm instead of at 0 cm).

![Peter’s diagram showing that moving 30 cm in 12 s is the same fastness as moving 10 cm in 4 s.](image)

The teacher asked the students, “So is there anything on Peter’s picture that can show us that 30 divided by 12, 30 centimeters divided by 12 seconds is 2.5 centimeters for each?” Armando replied:

I see the . . . squares or triangles for every 2.5 centimeters . . . There’s 12 groups for the seconds . . . and then for every centimeter, for every 2.5 centimeters it’s like a group too and the number line represents 30.

Then, the teacher said, “OK, I see you pointing at 2.5 and 5 and 7.5 and 10. Does anybody else see where the 2.5, the groups of 12, the 12 groups of 2.5 are?” George responded with, “umm the bumps” and Peter said, “I have 12 bumps.” In each of
these responses it appears that the students were focusing on the discrete objects in the diagram (i.e., the squares and triangles, the bumps). Furthermore, students did not appear to refer to equal sections of the number line. Therefore, it seems that if the teacher had used only tasks with these kinds of diagrams, in which there were many discrete objects to focus on, students may not have shifted to noticing an important feature of division, namely that division partitions continuous quantities into equal sections.

**Tasks about dividing composed units.** As explained above, during the Lesson 6 task, the teacher changed the task prompt from a division question that involved dividing a unitary quantity (e.g., dividing 6 cookies by 3, dividing 30 cm by 12 s) to a question that involved dividing a composed unit.

Interestingly, this question originated with Brady because when he was explaining his diagram back in Lesson 5, he said:

> The turtle and the rabbit have the same fast, the same fastness because the drawing of the turtle. I started off with the time and length of the rabbit and divided 30 centimeters and 12 seconds by 3 umm right here I divided 30, by 30 and 12 by 3 and I got 10 centimeters and 4 seconds. And now with the rabbit I was doing my calculations and I was showing the time and length too. Though it, 10 centimeters in 4 seconds, I multiplied it by 3, which equals 30 centimeters in 12 seconds. So it’s going the same fastness.

The important observation about this response was that Brady spoke about dividing both the 30 cm and the 12 s by 3. In other words, he was referring to dividing the composed unit 30 cm/12 s instead of dividing unitary quantities. The teacher used Brady’s idea of dividing a composed unit briefly in Lesson 5 before abandoning it in favor of the discussion about cookies. However, during Lesson 6, the teacher returned
to dividing a composed unit and so it was fitting that he used Brady’s diagram. It was
the discussion around dividing composed units in which students’ center of focus
began to shift.

**Nature of the Mathematical Activity**

The emergence of CoF 3 was likely influenced by the following two classroom
practices: (a) the practice of presenting student work, and (b) the practice of noticing
features of others’ diagrams. For the emergence of this new focus, these two practices
appeared to be working together and so will also be discussed together.

**The practices of presenting student work and noticing features of others’

*diagrams.* The context in which CoF 3 emerged was the discussion of Brady’s
diagram from the previous lesson. As discussed above, the use of Brady’s diagram
was central to the shift in focus because his diagram had few discrete objects on the
continuous number line. However, there were two additional aspects of presenting
student diagrams and noticing features of others’ diagrams that likely contributed to
the eventual shift in what students were noticing with respect to division. First, the
practice of presenting one student’s diagram right after the other revealed that students
were noticing different features in Peter’s diagram—a diagram populated with discrete
objects—then they were in Brady’s sparse diagram. Had student work not been
juxtaposed in this public way, it might never have come to light that sharing discrete
objects, rather than partitioning continuous quantities, was the center of focus when
students looked at Peter’s diagram.

Second, the practice of presenting student work was also likely important in
the emergence of CoF 3 because it created an ideal context for the entire class to begin
to notice particular features of division in a powerful and potentially lasting way. In particular, using Brady’s somewhat impoverished diagram as the context meant that the class, with the teacher’s guidance, could decide together what important features of the problem they should be focusing on. It appeared that this deliberate, public effort resulted in all students sharing a common language about division and noticing similar features of division.

**Discussion**

The discussion of the results presented in this chapter is organized into 3 sections. In the first section, the conceptual connections between the findings regarding the centers of focus and the backward transfer findings are presented. In the second and third sections, features of the centers of focus and the insights that these results offer into backward transfer will be discussed.

**Conceptual Connections between Noticing and Backward Transfer**

Based on the evidence presented above, I claim that there were three conceptual connections between what students noticed in the instructional intervention (the three centers of focus) and the findings of backward transfer (see Figure 6.30). Specifically, here was a connection between, (a) CoF 1 and the first part of BTF 1 (i.e., that in the post-interview students reasoned proportionally with changes in quantities), (b) CoF 1, CoF 2 and the second part of BTF 1 (i.e., that students coordinated the non-proportional and proportional relationships in a $y=mx+b$, $b\neq0$ context), and (c) CoF 3 and BTF 3 (i.e., that students conceived of division using a partitive model of division).
Recall, that in chapter 2, centers of focus were described as the observable product of the individual (mental) component of noticing. Furthermore, noticing was described as an umbrella process, to which both individual and social sub-processes belong, where the individual component is conceived of as part of beginning of reflective abstraction. Thus, if reflective abstraction refers to an entire collection of mental processes, including (a) creating and re-presenting mental images of one’s activities, (b) comparing images and abstracted regularities based on one’s present goals, (c) forming mental actions from those abstracted regularities, and (d) operating on and coordinating mental actions (Campbell, 2001; Lobato et al., 2011a; Simon, Tzur, Heinz, & Kinzel, 2004; Thompson, 1985; von Glasersfeld, 1995), then noticing would be fall under creating and re-presenting of mental images. More specifically, according to this conception of noticing, when a perceptual (or conceptual) object is noticed, the observer doesn’t just perceive that object, they also re-present the object.

<table>
<thead>
<tr>
<th>Noticing Findings</th>
<th>Backward Transfer</th>
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<tbody>
<tr>
<td>CoF 1: Noticing changes in distance and time</td>
<td>Relationship 1: Finding 1: (a) Reasoning proportionally with changes in quantities</td>
</tr>
<tr>
<td>CoF 2: Noticing quantitative patterns in the changes in distance</td>
<td>Relationship 2: (b) Coordinating non-proportional and proportional relationships</td>
</tr>
<tr>
<td>CoF 3: Noticing that division partitions composed units made up of continuous quantities into equal sections</td>
<td>Relationship 3: Finding 3: Conceiving of division with a partitive model of division</td>
</tr>
</tbody>
</table>

Figure 6.30. Relationships between noticing and backward transfer.
to themselves (von Glasersfeld, 1995). von Glasersfeld (1995) describes re-presenting as an observer experiencing “a visualized image of the object” (p. 59) and as recognizing the object. In the context of an example from the results, when students noticed changes in distance and time, they perceived and re-presented to themselves as mental images those changes in distance and time. One might say that when students recognized the changes in distance and time, then the changes in distance and time had been noticed.

Of course, this kind of individual component of noticing is not directly observable. Thus, the dissertation study examined the centers of focus—the observable products of the mental processes involved in noticing.

**Connection between CoF 1 and BTF 1.** I provide two arguments to support the claim that there is a conceptual connection between noticing the changes in distance and the changes in time in quadratic function distance-time data and reasoning proportionally with changes in quantities about linear functions. First, noticing changes in distance and time in a quadratic context (CoF 1) and reasoning with changes in quantities in linear function contexts (first part of BTF 1) both involve the same center of focus on changes in quantities. In other words, the center of focus for changes in distance and time that emerged during instruction on quadratics (CoF 3) persisted into the post-interviews when students reasoned proportionally with changes in quantities in linear contexts (first part of BTF 1). For example, on the post-interview water pump #2 task, all but one student recorded both the changes in water volume and the corresponding changes in time. Furthermore, as reported above, all but one student reasoned with the changes in both quantities to a greater extent in the
post-interview than in the pre-interview. Thus, changes in quantities, which were established as a center of focus in the instructional intervention, appeared to also be a center of focus during the post-interview.

Second, the students that appeared to most quickly establish a focus on changes in distance and time in the intervention, provided the greatest increase in proportional reasoning with changes in quantities in the post-interview. In the instructional intervention, Jenn and Nicholas were the first students of their own accord to attend to the changes in distance and changes in time. George, Armando, Brady and Peter required an extra lesson before they also provided such evidence. Kendra required three additional lessons before she also attended to both the changes in distance and changes in time (see Figure 6.1). In the post-interview, Jenn, Nicholas and Brady’s reasoning changed more substantially than the other students reasoning did, from non-proportional reasoning to reasoning proportionally with changes in quantities, while Peter, George and Armando’s reasoning with respect to changes in quantities changed less, and Kendra did not provide evidence of reasoning with changes in quantities. Therefore, the quicker the establishment of CoF 1, the greater the BTF 1 effect seemed to be. Brady was the exception because he exhibited similarly substantial changes in reasoning on the post-interview as Jenn and Nicholas did, despite not attending as quickly to changes in distance and time in the instructional intervention.

**Connection between CoF 1, CoF 2 and BTF 1 (part 2).** I provide two arguments to support the two-part claim that there was a conceptual connection between CoF 1 and the second part of BTF 1 (i.e., that students coordinated non-
proportional and proportional relationships in linear function $y=mx+b$, $b\neq0$ contexts) and a conceptual connection between CoF 2 and the second part of BTF 1. CoF 1 and the second part of BTF 1 are connected because coordinating non-proportional and proportional relationships in a $y=mx+b$, $b\neq0$ context (BFT 1) involves noticing changes in both quantities. Consider evidence presented earlier from Jenn working on the post-interview business cost task (see Figure 4.5 for the data). On that task, Jenn attended to both the changes in cost and the changes in number of employees as she coordinated non-proportional and proportional relationships (see Table 6.3 for Jenn’s reasoning). Specifically, as can be seen in Table 6.3, Jenn attended to both the changes in cost and the number of employees in step 2 when she reasoned that if the rate was $625/employee$, then a change in cost of $3125$ would produce a change in number of employees of $5$. Thus, coordinating non-proportional and proportional relationships in $y=mx+b$, $b\neq0$ contexts, involves noticing changes in both quantities.

Second, CoF 2 and the second part of BTF 1 are connected because both involved coordinating multiple levels of quantities. Evidence that shows that CoF 2 involved coordinating multiple levels of quantities comes from Lesson 10, during the remote-control car task when, as reported earlier, George said “the distance between the changes in distance is 16.” In noticing this pattern, George coordinated the second-order changes in distance and the third-order 16 yd changes in the changes in distance. Evidence showing that coordinating non-proportional and proportional relationships also involved coordinating multiple levels of quantities comes from Jenn’s response to the post-interview business cost task (see Table 6.3). In step 1 of the task, she worked with two first-order quantities (i.e., the $12875$ and $16000$ cost),
Table 6.3. Jenn’s transcript and reasoning showing that she coordinated non-proportional and proportional relationships.

<table>
<thead>
<tr>
<th>Transcript</th>
<th>Non-proportional reasoning</th>
<th>Proportional reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>16000-$12875=$3125</td>
<td>1 employee costs $625</td>
</tr>
<tr>
<td>J: So 16000 minus 12875 [points at top point on graph (19 emp, $12875); uses calculator, writes 3125]</td>
<td>(non-proportional reasoning)</td>
<td>However many times $625 goes into $3125 is how many employees cost $3125</td>
</tr>
<tr>
<td>Step 2</td>
<td>• $3125÷$625/emp= 5 emp</td>
<td>• $3125÷$625/emp= 5 emp (proportional reasoning)</td>
</tr>
<tr>
<td>J: Now I'm gonna do 3125 divided by 625 . . . I was gonna see how much money can you get into here [points to 3125] which is gonna give you the amount of people, so however many times this [points to 625/1] can go into there [uses calculator, writes 5].</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Step 3</td>
<td>19 emp+5 emp=24 emp</td>
<td></td>
</tr>
<tr>
<td>J: Yeah, so this is 19 people [points at upper most point on graph], so umm if you were to add 5 more people because 625 goes into there [points to 3125], then you get 24 people. So she can only hire 24 people.</td>
<td>(non-proportional reasoning)</td>
<td></td>
</tr>
</tbody>
</table>

to find a second-order quantity (i.e., the $3125 change in cost). In step 2, she reasoned proportionally with the second-order quantities, change in cost and change in numbers of employees (i.e., the $3125 change in cost and the $625 change in cost per change in 1 employee). In step 3, she worked with a first-order quantity (19 employee) and a second-order quantity (a change in employees of 5) to find another first-order quantity (24 employees). Thus, both CoF 1 and CoF 2 are conceptually connected to the
second part of BTF 1.

Connection between CoF 3 and BTF 3. I provide two arguments to support the claim that there was a conceptual connection between noticing that division partitions composed units made up of continuous quantities into equal sections in quadratic function contexts and explaining division using a partitive model of division in linear function contexts. First, students partitioned or split composed units of changes in distance and changes in time into equal parts, in the quadratic function contexts during the instructional intervention, which is consistent with treating the relationship between change in distance and change in time for that particular interval as a linear function (i.e., as a linear approximation of the quadratic function). For example, as described above, during the Lesson 8 remote-control car task presented in 4-second intervals (see Figure 4.21), Nicholas explained dividing a change in distance of 16 yd (from 0 to 16 yd) by the change in time of 4 s (from 0 to 4 s) as, “Because 16 over 4, you can simplify to the 4 over 1 by dividing 16 by 4 you get 4, 4 equal groups of . . . . 4 divided by 4 would be 1 so that would be 4 groups of 1 [second].” Thus, Nicholas treated the quadratic function within that 4 s interval as a linear distance-time relationship and used a partitive model of division to separate the 16 yd into 4 equal groups of 4 yd and the 4 s into 4 equal groups of 1 s.

Second, when students divided in the post-interview, their explanations sometimes involved partitioning composed units made up of continuous quantities into equal sections. For example, during the post-interview water pump #2 task, Armando said:

I put 20 gallons over 8 minutes [writes 20/8]. Then I kept dividing so it’s 10
over 4 [writes =10/4], and it’s 5 over 2 [writes =5/2] . . . it breaks it into groups . . . separates it into 4 groups of 5 gallons in 2 minutes [draws 4 circles and labels each with a 5 g and a 2 m].

In this excerpt, Armando partitioned a continuous quantities composed unit of 20 gal and 8 min into 4 groups of 5 gal and 2 min. Thus, he exhibited a similar center of focus as CoF 3.

**Features of Noticing**

There were two interesting features of the process of noticing that came to light as a result of comparing how the three centers of focus that were presented above emerged. First, some centers of focus emerged gradually over time while others emerged more suddenly at a particular point in time. For example, CoF 1 emerged gradually. As described earlier, all students noticed changes in distance and time in Lesson 3 when the teacher introduced the terms and gave students a worksheet about finding changes in distance and time. However, it took all but one student several more lessons before changes in distance and time became a center of focus. For some students, this CoF 1 emerged more gradually than for others (i.e., two students appeared to have CoF 1 by Lesson 6, four more students by Lesson 8 and the remaining student by Lesson 11). CoF 2 also emerged gradually.

In contrast CoF 3 emerged suddenly. As described earlier, during Lesson 6 the entire group of students appeared to have the transitional focus on division as sharing discrete objects. Then, when Brady highlighted a section of his number line diagram, the center of focus appeared to emerge for multiple students simultaneously. I do not mean to imply here that this center of focus emerged without substantial time and
effort. On the contrary, a significant amount of time and effort (by the teacher and the students) was dedicated to seeing division in the student representations of continuous quantities. However, when the students did eventually see division as partitioning a continuous quantity into equal sections, it happened rather suddenly.

These two examples suggest a second interesting feature of noticing, namely that some perceptual/conceptual objects are easier to notice than others. On the one hand, division as partitioning a composed unit made up of continuous quantities into equal sections seemed to be difficult for students to notice and seemed to require some insight that at first the students did not possess before it suddenly emerged. This may explain why division as sharing discrete objects among equal groups emerged as a transition from division as an arithmetic calculation to CoF 3.

On the other hand, changes in distance and time (CoF 1) seemed to be easy for students to notice. The difficult part about noticing changes in distance and time was maintaining focus (i.e., students stopped noticing changes in distance and time when not explicitly asked to attend to them). A possible conjecture about why students at first did not maintain focus is that they might not have understood why it would be useful to attend to changes in distance and time. Later, as they had more experiences, particularly with functions in which the changes in distance changed over equal intervals of time (i.e., quadratic functions), they may have begun to appreciate that attending to the changes in distance and time is an important part of making sense of quadratic functions.

A metaphor that seems to fit with these features of noticing is the chemical reaction. Some chemical reactions between reactants happen easily given that there is
sufficient external energy to drive the reaction. When the energy is removed, the reaction dies out. Other chemical reactions between reactants do not happen even if external energy to drive the reaction is provided. For these kinds of reactions, a catalyst may be needed to get the reaction started. Once the catalyst is introduced, the reaction happens suddenly and completely. The first kind of chemical reaction compares to the case of noticing the easy-to-see changes in distance and time. When an external source, like the teacher, directed students’ attention, noticing occurred (e.g., students noticed changes in distance and time). However, when the external source of attention-directing was removed, the noticing stopped. The second kind of chemical reaction compares to the example of noticing the hard-to-see equal sections of continuous quantities created by partitioning. Despite the teacher directing students’ attention toward the appropriate conceptual and perceptual objects (i.e., the number line on which the continuous quantities were represented), noticing did not happen automatically (i.e., students did not notice that dividing partitions a continuous quantity into equal sections). However, a student-action (i.e., Brady’s highlighting of a section of the number line) seemed to act as a catalyst because suddenly students began to see something they had not seen before.

These two features of noticing—that noticing can happen gradually or suddenly and that some objects are harder to notice than others—suggest that noticing is a complex process that involves more than just having one’s attention directed to a particular object. It may involve some catalyzing action or some sudden insight. Alternately, it might involve some gradually-acquired awareness about the importance or relevance of the object.
Insights into Backward Transfer Processes

As reported earlier, several conceptual connections between what students noticed in the classroom and the findings of productive backward transfer were identified. These conceptual connections led to new insights into the phenomenon of backward transfer. The first insight is that noticing seems to operate as a backward transfer process. This insight addresses the goal of the third research question, which was to identify a potential process that could help to account for the backward transfer effects that were produced in the dissertation study.

To fully appreciate what it means that conceptual connections were identified between mathematical noticing and backward transfer, recall that in looking for processes in the dissertation study (i.e., in using a process-orientation to causality; Maxwell, 2004) there were two parts to the logic of argumentation with respect to the focusing framework (Lobato et al., 2011a). The first part was to identify conceptual connections among the centers of focus, the focusing interactions, the features of the mathematical tasks and the nature of the mathematical activity. These connections helped to account for how the individual part of noticing (i.e., the emergence of centers of focus) was socially organized. The second part was to identify the conceptual connections between centers of focus and the observed backward transfer effects (i.e., the changes in students’ prior knowledge due to the intervention). These connections showed that what students noticed mathematically, together with the socially situated features of the environment that helped bring about what students noticed, provided an explanatory frame for the backward transfer effects.
The second insight into backward transfer is that the same processes—individual and social—that helped to account for forward transfer also helped to account for backward transfer. In a previous study, Lobato et al. (2011a) identified conceptual links between noticing and forward transfer effects. Thus, this dissertation study extends noticing as a transfer mechanism to include backward transfer. Furthermore, it also supports the claim made in the introduction that backward transfer is part of transfer rather than some other unrelated phenomenon.

One implication of noticing being both a forward and backward transfer process is that shifts in centers of focus simultaneously imply both forward and backward transfer effects. Thus, if a shift in a center of focus is found to be conceptually connected to a transfer finding in one direction, it may be reasonable to expect that the same center of focus is also conceptually connected to a transfer effect in the other direction.
CHAPTER 7:
CONCLUSION

In this chapter, I compare the goals that were set for the dissertation study with the major findings that came out in the results. Then, I present the reasons why these findings are significant, the limitations of the study, future research directions and final remarks.

Summary of Findings

The seed from which this study grew was the surprising finding in pilot data that several students’ prior foundational knowledge changed as they learned something new. Specifically, two students, who displayed productive linear functions reasoning prior to an instructional unit on quadratic functions, reasoned in unproductive ways on linear functions tasks after the instructional unit. In the pre-interview, they were able to coordinate the two quantities in a linear function so as to preserve the multiplicative relationship, despite the challenge of data that was presented with uneven intervals. The students were also able to reason in context to reach correct conclusions about whether or not the rate of change was constant. However, in the post-interview, these students seemed to focus primarily on the dependent variable and thus arrived at incorrect conclusions about the rate of change. Furthermore, the unproductive changes in reasoning appeared to be linked to the instructional intervention on quadratic functions. This is despite the intervention being situated in what could be considered ideal circumstances—an after school
program, small class size, a researcher as the teacher, reform-oriented innovative activities, use of technology, and freedom from the constraints that usually mark school practice.

In response to these surprising observations, a major goal of this dissertation study was to reverse these findings by creating an instructional environment for quadratic functions that would help students deepen their linear functions knowledge. Now that the study has been completed, it is possible to report that this goal was achieved. Specifically, in the post-interviews students reasoned more productively in linear function contexts than they did in the pre-interviews in the following three ways, by: (a) reasoning proportionally with the changes in the quantities involved in linear functions and coordinating non-proportional and proportional relationships in $y=mx+b$, $b\neq0$ contexts, (b) drawing mathematized diagrams of linear function contexts, and (c) providing meaning-based explanations of division when reasoning about linear functions. What was particularly encouraging about these results was that the students with average to below-average understanding of linearity seemed to benefit the most.

Another goal of the study was to assess whether or not there was a connection between the phenomenon of interest (i.e., that prior knowledge was reorganized and enriched because of learning something new) and the transfer of learning. In other words, could the phenomenon of interest be conceived as a case of backward transfer? Theoretically, this connection was achieved by treating forward transfer, from an actor-oriented transfer perspective, as the influence of prior experiences on activity in new situations (broadly, the generalization of learning experiences) and backward
transfer as the influence on prior knowledge by the acquisition and subsequent
generalization of new knowledge. Empirically, there were two major connections
between the findings related to backward transfer and those related to forward transfer.
First, students reasoned more productively with changes in quantities during the post-
interviews, both on a novel quadratic function task (forward transfer) and on linear
function tasks (backward transfer) than they did on similar tasks during the pre-
interviews. Second, students better coordinated multiple levels of quantities during
the post-interviews, both on the novel quadratic function task (coordinating second-
and third-order quantities; forward transfer) and on a $y=mx+b, b\neq 0$ task (coordinating
first- and second-order quantities; backward transfer) than they did on similar tasks in
the pre-interviews.

A third goal of this study was to investigate whether or not noticing, which has
been shown to offer explanatory power as a forward transfer process in emerging
exploratory work (Lobato et al., 2011a), could also serve as a backward transfer
process. This goal was also achieved because the three major findings of noticing that
were identified in this study were conceptually connected to the three major backward
transfer findings summarized above. However, before the connections can be related,
the findings of noticing must be summarized.

First, students shifted toward noticing both changes in distance and changes in
time (Center of Focus 1) through the influence of four kinds of focusing interactions
(i.e., naming, quantitative dialogue, producing graphical representations and
highlighting), two features of mathematical tasks (tasks that specifically target an
exploration of changes in quantities and tasks that draw attention to changes in
quantities by varying the relationship between changes in quantities) and three participatory structures (presenting diagrams, noticing features of others’ diagrams and asking *Can you explain why?* questions). Second, students shifted toward noticing quantitative patterns in the changes in distance (CoF 2) through the influence of two kinds of focusing interactions (naming and quantitative dialogue), three features of mathematical tasks (tasks that gradually narrow the spacing of the time intervals, tasks that necessitate the generation of additional data points and tasks for which the time-distance relationship is more complex than simply squaring time values to make distance values) and two participatory structures (presenting student work and noticing features of others’ diagrams). Finally, students shifted to noticing that dividing partitions a composed unit made up of continuous quantities into equal sections (CoF 3) through the influence of two focusing interactions (quantitative dialogue and highlighting), two features of mathematical tasks (tasks that explore division of continuous quantities and tasks that explore division of composed units) and two participatory structures (presenting student work and noticing features of others’ diagrams).

As stated above, there were three conceptual connections between these noticing findings and the backward transfer findings summarized above. First, there was a connection between the first center of focus finding (CoF 1; noticing the changes in distance and time in the quadratics function distance-time instruction) and part 1 of the first backward transfer finding (BTF 1; students reasoning proportionally with the changes in quantities involved in linear functions in the post-interview). Second, there was a conceptual connection between CoF 2 (noticing patterns in the
changes in distance in the quadratic function distance-time instruction) and part 2 of BTF 1 (coordinating non-proportional and proportional relationships in $y=mx+b$, $b \neq 0$ contexts in the post-interview). Third, there was a conceptual connection between CoF 3 (noticing that division partitions a composed unit made up of continuous quantities into equal sections in the quadratic function distance-time instruction) and BTF 3 (explaining division using a partitive model of division in the post-interview).

In summary, these findings suggest that the three main goals to produce productive influences on prior knowledge through the acquisition of new knowledge, to examine whether productive influences on prior knowledge were cases of backward transfer and to identify a backward transfer process were reached. The significance of these findings is discussed next.

**Significance**

There are at least four ways in which this study contributes to educational and social science research in general and to mathematics education research specifically. First, this study expands on the conceptualization of transfer according to the actor-oriented approach to transfer. Prior to the dissertation study, the conceptualization of transfer, from the AOT perspective had been directed toward forward transfer. However, the conceptual connections between forward and backward transfer identified in this study suggest that forward and backward transfer can be productively conceived as part of the same phenomenon, and that the AOT perspective can be expanded to include backward transfer. This is significant theoretically, because by including backward transfer in the conceptualization of transfer, the AOT approach
further addresses the criticism that traditional approaches to transfer often underestimate the amount of transfer that is present in a give situation. In other words, by accounting for how the acquisition of knowledge influences thinking forward onto novel contexts and backward onto prior knowledge contexts, the AOT approach allows researchers to more accurately account for the generalization of learning.

Second, this study contributes to the ongoing discussion about transfer processes across disciplines and theoretical orientations. Many researchers, from diverse theoretical perspectives, have identified a variety of transfer processes (e.g., Engle, 2006; Gentner & Markman 1997; Lobato et al., 2011a; Nemirovsky, 2010; Nokes, 2009; Singley & Anderson, 1989; Wagner, 2006, 2010). The dissertation study contributes to that effort by offering noticing as a backward transfer process. This contribution is significant because not only can noticing as a backward transfer process be added to the growing list of transfer processes, but in so doing, the dissertation study answers a recent call for research to examine how multiple transfer processes interact (Nokes, 2009). In a recent critique of transfer research, Nokes (2009) argues that past research has neglected to adequately investigate the relationships between transfer processes and advocates for a “multiple mechanisms perspective” (p. 32). The dissertation study addresses this critique because the focusing framework (Lobato et al., 2011a), which was employed in the investigation of noticing, is a multiple-mechanisms approach, in which several sub-processes of noticing—both individual and social—are coordinated. Specifically, the individual sub-processes of creating and re-presenting mental images (which are identified empirically via the products of individual noticing—centers of focus) are coordinated
with social sub-processes of particular discourse practices (namely the focusing interactions of highlighting, naming/renaming, quantitative dialogue, and producing graphical representations), which in turn, are situated within the use of particular mathematical tasks with “attention-focusing” features and within particular normed activities and practices.

Furthermore, this study contributes to the growing body of research on noticing (Ainley & Luntley, 2007; Mason, 2002; Scherer & Steinbring, 2006; Sherin, Jacobs, and Philipp, 2010; van Es & Sherin, 2002, 2006). Much of the current research on noticing focuses on teacher noticing (Sherin, Jacobs, and Philipp, 2010). Significantly less research focuses on student noticing (e.g., Lobato, 2011b; Marton, 2006; Schwartz & Bransford, 1998). While the dissertation study offers a look into student noticing as it pertains to backward transfer, it adds to the underdeveloped knowledge-base on noticing and student learning.

Third, this study contributes more specifically to the transdisciplinary research that already exists on backward transfer (see Chapter 2 for a review) by examining backward transfer in a context not previously examined (prior research on backward transfer has mostly been conducted in linguistics). The dissertation study also contributes to prior research because it offers an existence proof of productive backward transfer and transdisciplinary research on backward transfer has primarily focused on unproductive effects. Finally, in offering noticing as a backward transfer mechanism, this study contributes the new idea that, across content areas and contexts, noticing could be leveraged in the learning of new knowledge to enrich foundational prior knowledge.
Fourth, this study suggests a general approach for supporting the deepening of prior knowledge during instruction on new content (for which the prior knowledge is conceived as foundational). Prior efforts to maintain students’ prior knowledge as new content is being taught have largely relied upon the principle of a spiraled curriculum (Bruner, 1960). A refinement of this approach involves the identification of the particular conceptual understandings that overlap between the prior and targeted content domains. In this study, it was not enough that linear functions knowledge and quadratic functions knowledge are closely related (as argued in Chapter 2). Otherwise the pilot data would have also contained productive backward transfer. What distinguished the dissertation study from the study associated with the pilot data was a much better understanding of the conceptual understandings that are crucial for students to develop that cut across linear and quadratic functions.

This study suggests a two-fold approach. First, identify conceptual areas in which the foundational knowledge is usually still developing as the new content is already being taught. Producing productive backward transfer in the context of this subset of instances would likely have the biggest payoffs for students. For example, in the dissertation study, productively influencing students’ proportional reasoning was significant because proportional reasoning is usually a concept that middle school students have not fully worked out yet. Second, identify cross-cutting themes that can form the bases for the lessons that address new content and simultaneously influence prior knowledge in productive ways. In the dissertation study, one of the cross-cutting themes that was identified was covariation. As a result, lessons were designed to promote covariational reasoning in quadratic function contexts with the added goal of
promoting covariational reasoning in linear function contexts. Besides the contributions outlined above, this study also offers several new potentially rich research directions, which will be discussed next.

**Future Research Directions**

There are four research directions suggested by this dissertation study. The first direction will be to continue the design-based study started here with new iterations so that the intervention, which was a successful first attempt, can be further refined and developed to the point where the intervention materials can be shared with teachers. As part of future iterations of this study, the plan is to test several other conjectures about what might promote productive backward transfer that were not tested in this study. Two of these conjectures are: (a) the use of overlapping intervals of the independent variable could draw students’ attention to changes in both quantities, and (b) having students compare a quadratic function to the function of its tangent line could help students better understand how quadratic functions are different from linear functions and at the same time further deepen students’ understanding of linearity.

The second direction for future research will be to scale up the quadratic function/linear function study. The eventual plan is to give a practicing teacher the materials developed in the multiple iterations of the first design-based study to use with an existing mathematics class. The goal would be to obtain results about backward transfer that are more generalizable (i.e., to see if a practicing teacher could obtain similar results using the materials developed in the more idealized setting).
The third direction will be to start a new design-based study that examines backward transfer for a different mathematical topic. Two possible topics of study are an examination of how learning about trigonometric functions influences students’ prior understanding of ratios and an investigation of how learning about rational expressions influences students’ understanding of fractions.

Finally, even though several guidelines for pedagogy have already emerged out of the results of this study, it would be interesting to look more systematically at general instructional principles that could promote productive backward transfer. The dissertation study suggests that students may significantly benefit from teachers leveraging backward transfer to influence student understanding in efficient and meaningful ways. General principles for promoting productive backward transfer could eventually come to be used, in a similar way to how general principles for building new knowledge on prior knowledge are currently being used (e.g., principles of the zone of proximal development; Vygotsky, 1987; principles of scaffolding, Hmelo-Silver, 2006). I would be interested to look at principles for promoting productive backward transfer across the pedagogy spectrum, from curriculum design to classroom activity design.

Thus far, this chapter has focused on the dissertation study’s successful results and promising contributions. However, as with any study, the dissertation study had limitations, which will be discussed next.

**Limitations**

There were at least three limitations to this study. In each case, the limitation
reflects a trade-off between having the results be more widely generalizable and realizing an important benefit of using the approach that was used in the study. The first limitation was that the learning environment in which backward transfer was produced was an idealized setting in which the class size was small (seven students), there was a minimum of distraction (i.e., the school in which the study was conducted was on summer break), and the students were mentally fresh (i.e., they only attended the sessions 2 hours per day). The benefit of using a small class size was that it facilitated producing the phenomenon of backward transfer. In other words, because the teacher was able to give students individual attention, engage the class in meaningful discussion and rich activities with minimal distractions, and assess how each student was thinking, he was better able to fully implement the features of the lessons that promoted productive backward transfer. The trade-off for using a small class in an idealized setting was that the claims about backward transfer and noticing that emerged from this study may not readily generalize to a more realistic classroom setting.

A second limitation was that the researcher served as the teacher. The benefit of taking this approach, which was consistent with a design-based research methodology (Hoadley, 2004), was that it helped ensure that a backward transfer effect would be produced. The teacher/researcher could be responsive to the developments within the classroom and make in-the-moment changes to the intervention as needed. This was seen as particularly important because little was known about backward transfer in mathematics education and the real possibility existed that productive backward transfer would not be produced by the intervention.
With the researcher as teacher, the chances were better that it would be produced. However, the trade-off was that the researcher’s intended and unintended actions, as well as his agenda, became part of the data. The results need to be interpreted with this in mind.

A third limitation was the difficulty that I experienced in getting analytical traction on how the features of the mathematical tasks and the nature of the mathematical activity contributed to what students noticed. The benefit of considering features of tasks and the nature of the mathematical activity was that it more accurately reflected the assumption underlying this part of the study, that noticing emerges through multiple factors “distributed across mental, material, social, and cultural planes” (Lobato, 2008a, p.174). The approach that was taken to track influences from the features of tasks and the participatory structures of the class, was to look at the tasks and the nature of the activity around the time when students shifted from one center of focus to another. However, the trade-off in using this approach was that other features of tasks and other participatory structures that came earlier in the intervention that might also have influenced what got noticed, did not get the same kind of consideration. Thus, the influences from the features of tasks and the nature of the mathematical activity that led to the emergence of the centers of focus were likely under-specified.

**Final Remarks**

In this dissertation study, I have built a case for why an investigation of backward transfer might be relevant to the field of mathematics education. First with
several surprising observations from pilot data and later with data collected for the study, backward transfer has been unintentionally and intentionally produced, with either unproductive or productive consequences. These various findings suggest that, as some have said about forward transfer, backward transfer may be ubiquitous (Barnett & Ceci, 2002; Detterman, 1993; Lobato, 1996). If this is the case, then backward transfer may play an important role in how students’ mathematical knowledge develops.

Considering, on the one hand, that backward transfer potentially plays an important role in the development of mathematical knowledge and, on the other hand, that this single dissertation study appears to be the only mathematics education study to have explicitly looking at backward transfer, a more extensive examination of backward transfer in mathematics education seems warranted. The results reported in this study suggest that such a line of inquiry could be potentially rich and could provide new ways to connect mathematics education research to other fields, new ways to approach pedagogy and new ways to think about transfer.
Appendix A

Sample Lesson Plan
Content Goals: Students find speeds for successively smaller nested intervals and compare. Students make conclusions about the speed at a point based on comparisons of speeds for nested intervals. Conclude that for a quadratic distance-time function, the speed over an interval is the same as the speed at the center of the point.

Activity: Zooming in on speed – Students explore finding speed over short intervals of time (e.g., .5 s, .25 s etc).

Watch the remote-control car movie which is on your computer’s desktop. The car moves according to a quadratic distance-time function. The goal of this activity is to find the speed for the car at exactly 2.5 seconds by ZOOMING in on 2.5 seconds.

Step 1 - Using the SimCalc Table, find the speed for the Remote Control Car from 2-3 seconds (Go to Table Properties and set the START, END and INCREMENT). Explain what you found.

Step 2 - Find the speed of the car from 2-2.5 seconds and from 2.5-3 seconds. Compare these speeds to the speeds you found for the car in Step 1. Now, what do you predict is the speed of the car at 2.5 seconds?

Step 3 - Zoom in closer on 2.5 seconds. Find the speed of the car from 2.25-2.75 seconds. Compare this to the speeds you found for the car in Steps 1 and 2.

Step 4 - Find the speed of the car from 2.25-2.5 seconds and from 2.5-2.75 seconds. Compare these speeds to the speeds you found in Step 3.

Based on your observations, explain what you think the speed is for the remote-control car at 2.5 sec?

Discussion Question: The goal was to find a more exact value for speed at a particular point for a quadratic function. What did you find out about the speed of the car at 2.5 seconds?
Appendix B

Lesson Goals Table
**LESSON GOAL TABLE**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Math Goals</th>
</tr>
</thead>
</table>
| 1-2    | ♦ For the first few lessons, use the word fastness instead of speed to avoid prior competing ideas that speed is distance divided by time.  
♦ Associate constant fastness with the same distance traveled per unit of time and emphasize that the unit does not have to be 1 sec.  
♦ Associate increasing fastness with increasing distance traveled per unit of time.  
♦ Link fastness visually to the *dropmarks* feature of SimCalc.  
♦ Introduce terminology of linear functions, quadratic functions, and piecewise linear functions.  
♦ Link a linear function to constant fastness. |
| 3-4    | ♦ Construct a ratio (as a composed unit) of distance to time as a measure of fastness.  
♦ Begin using the term *speed* to replace fastness.  
♦ Explain why various ratios represent the same speed (e.g., why 10 cm in 4 s is the same speed as 20 cm in 8 sec) by iterating and partitioning and through the use of pictures.  
♦ Start to build an equivalence class of ratios that all represent the same speed.  
♦ Link multiplication with grouping quantities in diagrams. Link division with *splitting into groups* and with partitioning in diagram.  
♦ Construct unit ratios and understand what division accomplishes (i.e., link partitioning with the arithmetic calculation of division).  
♦ Draw diagrams to represent constant speed. Track accumulated distances and time as well as changes in distances and changes in time.  
♦ Contrast Linear with Piecewise linear relationships. |
| 5-6    | ♦ Emphasize the difference between accumulated quantities and elapsed quantities.  
♦ Call the elapsed distance the change in distance and elapsed time the change in time.  
♦ Focus on change in distance and change in time in speeding up contexts.  
♦ Draw pictures showing accumulated distances and times and changes in distances and times for speeding up (try to establish equal emphasis on both quantities; CLG 1 and 2).  
♦ Associate constant speed with the same change in distance traveled per given change in time.  
♦ Associate a faster speed with increasing changes in distance traveled per given change in time.  
♦ Link larger and smaller changes in distance per given changes in time to visual display in *drop marks* mode of SimCalc. |
<p>| 7-8    | ♦ Examine the set of changes in distance for consecutive equal-sized |</p>
<table>
<thead>
<tr>
<th>409</th>
<th>changes in time for quadratic functions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>♦</td>
<td>Define this as the approximation of the speeds of a quadratic function.</td>
</tr>
<tr>
<td>♦</td>
<td>Start with larger than unit intervals and gradually move down to smaller unit intervals.</td>
</tr>
<tr>
<td>♦</td>
<td>Compare with finding the speeds for gradually smaller intervals for linear functions.</td>
</tr>
<tr>
<td>♦</td>
<td>Begin definition of a quadratic as the change in distance increasing at a constant rate.</td>
</tr>
<tr>
<td>♦</td>
<td>Compare changes in distance and time as well as speed for quadratic distance-time function to embedded linear function (same starting and ending point).</td>
</tr>
<tr>
<td>♦</td>
<td>Emphasize that speeding up for a quadratic is “constant”. Don’t (necessarily) talk about the units for speeding up in this lesson but emphasize that the speed is changing at a constant rate (more distance in same amount of time).</td>
</tr>
<tr>
<td>♦</td>
<td>Draw diagram that captures increasing speeds.</td>
</tr>
<tr>
<td>♦</td>
<td>Introduce acceleration.</td>
</tr>
<tr>
<td>♦</td>
<td>Introducing attending to changes in distance and time in tables.</td>
</tr>
<tr>
<td>9-10</td>
<td>♦ Examine the set of constantly increasing speeds with intervals larger than and equal to 1 unit to discover that no matter the interval size, the increase in speed is a constant pattern. Add to definition of quadratic.</td>
</tr>
<tr>
<td></td>
<td>♦ Introduce speed/time tables and how increasing speed is recorded in intervals.</td>
</tr>
<tr>
<td></td>
<td>♦ Link different sized intervals to drop marks in SimCalc.</td>
</tr>
<tr>
<td></td>
<td>♦ Draw diagrams to capture increasing speed for different sized intervals.</td>
</tr>
<tr>
<td></td>
<td>♦ Define acceleration as the change in speed for a certain change in time.</td>
</tr>
<tr>
<td>11-12</td>
<td>♦ Explore very short motion and what it means as far as speed is concerned (what it means for a speed to be a certain value even though the movement does not occur for the length of time indicated in the units).</td>
</tr>
<tr>
<td></td>
<td>♦ Students find speeds for successively smaller nested intervals and compare. Students make conclusions about the speed at a point based on comparisons of speeds for nested intervals. Conclude that the speed over an interval is the same as the speed at the center of the point.</td>
</tr>
<tr>
<td></td>
<td>♦ Examine what a given difference between two speeds means (CLG 4).</td>
</tr>
<tr>
<td></td>
<td>♦ Examine the difference between successive speeds at points for a quadratic function (unit interval).</td>
</tr>
<tr>
<td></td>
<td>♦ Draw a diagram that captures the difference between successive speeds.</td>
</tr>
<tr>
<td></td>
<td>♦ Examine the difference between successive speeds for quadratic functions (non-unit intervals, greater than 1).</td>
</tr>
<tr>
<td></td>
<td>♦ Add to definition of quadratic functions that the change in speed for a quadratic function is constant for any given change in time.</td>
</tr>
</tbody>
</table>
|     | ♦ Discuss constantly increasing speed in SimCalc movie and constant
| 13-14 | • Construct a ratio of change in speed and corresponding change in time (CLG 5).  
• Label the ratio acceleration.  
• Examine the acceleration for a quadratic function and add to definition of quadratic functions.  
• Draw a diagram of quadratic motion that captures the acceleration.  
• Examine constant change in speed for different time intervals (greater or less than 1 unit; CLG 5). |
| 15-16 | • Examine the same change in speed for different time intervals and different changes in speed for the same time interval as a way to necessitate the need to coordinate both.  
• Contrast constantly increasing speed with secant line embedded linear function. Make point-by-point comparisons, comparing accumulated distances, changes in distances, speeds and accelerations.  
• Draw diagrams for both quadratic and embedded linear functions. |
Appendix C

Selected Interview Tasks
Another pump was used to fill a different pool.

After 4 minutes, there were 10 gallons of water in the pool.  
After 6 minutes, there were 15 gallons in the pool.  
After 11 minutes, there were 27.5 gallons in the pool.  
After 14 minutes, there were 35 gallons in the pool.

Draw a picture or diagram that shows whether or not the pump is pumping equally fast.

- Emphasize that the data shows how much is contained in the pool at a given time, rather than how much is going out of the hose at a given time.

1. Explain where in your drawing you can see that the pump is either pumping equally fast or not pumping equally fast.

- If they say that is being pumped faster at certain times, ask where it’s the fastest and why?
  - Where on the diagram are they looking?
  - If they find changes in gallon amounts and/or minutes, ask what that tells them? Do they need to find changes for both quantities or just one, in order to tell?
If they reason across the data additively, ask them what the differences mean (e.g., 10-4=6, what does the 6 mean?). What does it tell them in terms of the pool situation? Does it matter that 4 is smaller than 10, whereas in the first pump, 8 was bigger than 5?

Ask them if they can create data for a third pump that would be pumping equally fast?

If they say it is pumping equally fast, ask how the diagram shows this?

If their explanation involves reasoning with .4 or 2.5 (i.e., reasoning across), ask how they found the .4 or 2.5 and why that means it's pumping equally fast over time? How did they use .4 or 2.5 to determine that the pump was constant? What does the .4 or 2.5 mean? What does it tell you in terms of the pool situation? Could they see the .4 or 2.5 in their drawing? If they use 2.5, suggest that someone else used .4 and vice versa and see if they can explain the difference.

If their explanation involves reasoning multiplicatively with 4 and 10 (i.e., reasoning down), ask them how they can see multiplication in their drawing. Ask them to demonstrate how multiplication of or division by 4 and/or 10 verifies that the pump is constant.

Ask them to explain why it's pumping equally fast in particular places on the table such as between 6 and 11 minutes.

*If students struggle with pump 2, do these next questions more quickly and then move to pump 3.*
Ask students if they can explain their answers with the picture they drew earlier.


- If students don't understand what rate of change is then ask what the rate of the pump is, ask how fast the pump is pumping or how much is the pump pumping every second (in that order).
- Ask about the units for the value they produced (if giving examples, say "is it gallons, minutes or something else").
- Ask how they know it’s 10/4=2.5 and not 4/10. What does division do?
- Ask how the rate could be used to find other values of the table.
- Ask the student if they can find the "changes in gallons" and "changes in time" in the data and what those changes mean (e.g., "if I asked you to look at the table and find the changes in the gallons and the changes in the time, could you tell me what they are...if you can, what do those changes tell you about the pumping rate?").
- If students can't find the rate of pumping, ask them to make up a rate and to explain it.

6. Is there anything in this task that reminds you about things you did in class?
Water Pump 3
*Do Pump 3 with students that struggled with pump 2.

Water is being pumped through a hose into a large swimming pool. The table shows the amount of water that is in the pool over time. The amount of water is measured in gallons. The time is measured in minutes.

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>Water in the Pool (gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>24</td>
</tr>
<tr>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>24</td>
<td>72</td>
</tr>
<tr>
<td>32</td>
<td>96</td>
</tr>
</tbody>
</table>

Emphasize that the gallons column is the water IN the pool, e.g., after 16 minutes, 48 gallons are IN the pool. This could be misinterpreted as over the next 16 minutes, 48 more gallons are pumped in.

1. Do you think the water is being pumped equally fast over time or is it being pumped faster at certain times? What did you look at in the table to make your decision?

Responses:
- If faster, ask where it’s the fastest and why?
- If faster, ask them to make up a table where it would be pumping equally fast.
- If same, how do you know? If they say, because it’s “times 3”, ask why that means it’s pumping equally fast over time? What does the 3 (in “times 3”) mean? What does it tell you in terms of the pool situation?
- If they find differences 8 or 24, ask what that tells them? Suppose you worked with (8, 24) and (24, 72) - what would the differences be and what do they mean? How is this different from working with the table entries? Why do you want to find the differences?
2. What do you think the next entry in the table will be? (if incorrect, ask them to add another row to the table)
3. How much water is in the pool after 22 minutes? How did you figure that out? (if incorrect, ask about 20 minutes)
4. Can you add a row to the table that will before 8 min and 24 gal?

5. Only for kids who see the “TIMES 3 PATTERN”, ask:
   - How many minutes does it take to pump 43 gallons?
   - How much water is in the pool after 6.5 minutes?

6. How much water is in the pool after 1 minute?

7. Draw a picture to show that pumping____ gal in _____ min
   is equally fast as pumping ____ gal in _____ min.

   - Use numbers that have come up and that are multiples. Example:
     6 gal in 2 min and 24 gal in 8 min or
     3 gal in 1 min and 24 gal in 8 min
   - Insist that they draw a PICTURE not a graph.
   - If they have trouble, ask if they could start by drawing a pool or some water.


Responses:
   - How do they know it’s 3 and not 1/3?
• If they’re mentioned 3 a lot, e.g., the 3 in $y = 3x$, the 3 in the row 1 min, 3 gal, and maybe it’s come up in working with differences, ask if the 3’s are connected in some way or if it’s just a coincidence.

9. Only for kids who look really good up to this point, ask:
   How much time does it take to pump 1 gallon into the pool?
On this computer screen, a person can click on the corner of the square and drag it to make the square *grow* bigger. When the length of the square is 1 cm, then the area is 1 cm$^2$, when the length of the square is 3 cm the area is 9 cm$^2$ (green square), when the length is 5 cm the area is 25 cm$^2$ (orange square) and when the length is 7 cm the area is 49 cm$^2$ (pink square).

- Ask the student to explain what is going on in this context.
1. **Does the square’s area grow** at a constant rate, does its rate increase or decrease or something else or How fast does the area grow? Explain how you know?

- If the student say that the square does not grow at a constant rate, ask if there is a way to characterize or describe how it is growing.
- If the student says that the square does grow at a constant rate, ask them what in particular they see as constant in the data.

2. **Is there one rate of growth or are there more than one?**

   What is the rate (or rates) at which the area of the square grows? What are the units for the rate(s) of growth (cm, cm², or something else)?

- If the student does not understand rate, ask how fast the area grows.
- Ask what the units for the rate are.
- If they find rates of growth, ask them to explain a particular rate of growth (e.g., what does 8 cm² growth of area/2 cm growth of length mean, etc). Also ask if the rate can be seen in the pictures of the square.
- If they find one rate of growth, ask them to explain that growth and whether it can be seen in the squares.
- If they don’t find the rates of growth ask if they know what the rates of growth would mean if they could find them. Also, ask what they would need to know in order to find the rates of growth.

**One student said the following:**
4. Does what this student noticed matter when you think about how the area of the squares is growing? If so, how? If not, why not?

A third student said:
I noticed that if you subtract rates of growth of area, it’s always 8... for example, if you subtract $8 \text{ cm}^2$ growth of area per 2 cm increase in length from $16 \text{ cm}^2$ growth of area per 2 cm increase in length, you get 8 and if you subtract $16 \text{ cm}^2$ per 2 cm from $24 \text{ cm}^2$ per 2 cm you also get 8...

5. What might this tell you about the growth of the area?

- Ask what the units might be for that quantity, if it reminds them of something they learned in class and if anything in the picture relates to this.
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